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IS A^{-1} AN INFINITESIMAL GENERATOR?

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Abstract. In this paper we study the question whether A^{-1} is the infinitesimal generator of a bounded C_0-semigroup if A generates a bounded C_0-semigroup. If the semigroup generated by A is analytic and sectorially bounded, then the same holds for the semigroup generated by A^{-1}. However, we construct a contraction semigroup with growth bound minus infinity for which A^{-1} does not generate a bounded semigroup. Using this example we construct an infinitesimal generator of a bounded semigroup for which its inverse does not generate a semigroup. Hence we show that the question posed by deLaubenfels in [13] must be answered negatively. All these examples are on Banach spaces. On a Hilbert space the question whether the inverse of a generator of a bounded semigroup also generates a bounded semigroup still remains open.

1. Introduction. Before we begin with giving background information on the problem, we first formulate the problem. Note that this is the Hilbert space version of the question posed by deLaubenfels at the end of [13].

PROBLEM 1.1. *Let H be a Hilbert space and let A be the infinitesimal generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on H, i.e., $\|T(t)\| \leq M$ for all $t \geq 0$. Furthermore, assume that A^{-1} exists as a closed and densely defined operator.*

Are the above conditions sufficient for A^{-1} to be the infinitesimal generator of a bounded C_0-semigroup?

In Section 2 we show that the answer to our question is positive if (under the conditions of the problem) any A^{-1} is the infinitesimal generator of a C_0-semigroup. Thus we get the boundedness for "free". Since the question is posed on a Hilbert space, one may wonder what is the solution to this problem on a Banach space. In Section 3 we show that for a Banach space the answer to the problem is negative. More precisely, we construct a

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contraction semigroup with growth bound minus infinity for which the inverse generator does not generate a bounded semigroup. Hence the question posed by deLaubenfels in [13] should be answered negatively. However, on general Banach spaces, there is a positive result as well. Namely, the inverse of the generator of a sectorially bounded semigroup is again a generator of a sectorially bounded analytic semigroup. This result originates from deLaubenfels, [13], but we present a new proof. In Section 4 we summarize some results on Problem 1.1. On Hilbert spaces it is not hard to show that the answer to our central problem is positive for the class of contraction semigroups.

In the remaining of this section we discuss the motivation for our problem. The first motivation comes from systems theory. Within infinite-dimensional systems theory one studies the following set of equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup on the Hilbert space H, B is a bounded linear operator from the Hilbert space U to the dual of the domain of A^*, i.e., $B \in \mathcal{L}(U, D(A^*)')$, $C \in \mathcal{L}(D(A), Y)$, where Y is a third Hilbert space, and $D \in \mathcal{L}(U, Y)$. In [2], Ruth Curtain introduced the following related system:

$$\begin{aligned} \dot{x}_1(t) &= A^{-1}x_1(t) + A^{-1}Bu_1(t), \\ y_1(t) &= -CA^{-1}x_1(t) + (D - CA^{-1}B)u_1(t). \end{aligned} \tag{2}$$

This system has the nice property that $A^{-1}B \in \mathcal{L}(U, H), CA^{-1} \in \mathcal{L}(H, Y)$. Hence this system is a bounded linear systems as studied in Curtain and Zwart [3]. Furthermore, the systems (1) and (2) share many properties. For instance, (1) is input-state stable if and only if (2) is. Here input-state stability means that for all inputs $u \in L^2((0, \infty); U)$ the solution of (1) exists and is (uniformly) bounded on $[0, \infty)$.

The only stability property which is not known is the state-state stability, i.e., if the semigroup generated by A is (strongly) stable if and only if the semigroup generated by A^{-1} (strongly) stable. It is not hard to show that this property holds if the answer to our problem is positive, see also Grabowski and Callier [9].

The second motivation for our problem comes from numerical analysis. Consider the (abstract) differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{3}$$

A standard method for solving this differential equation is by using Crank-Nicolson method. In this method the differential equation (3) is replaced by the difference equation

$$x_d(n+1) = (I + \Delta A/2)(I - \Delta A/2)^{-1}x_d(n), \qquad x_d(0) = x(0), \tag{4}$$

where Δ is the time step.

If H is finite-dimensional, and thus A is a matrix, then it is easy to show that the solutions of (3) are bounded if and only if the solutions of (4) are bounded. Or equivalently,

$$\sup_{t \geq 0} \|e^{At}\| =: M_c < \infty$$

if and only if

$$\sup_{n\in\mathbb{N}} \|A_d^n\| =: M_d < \infty,$$

where $A_d = (I + \Delta A/2)(I - \Delta A/2)^{-1}$. However, the best estimates for M_d are depending on M_c and the dimension of H, see [5]. In Guo and Zwart [10] the following estimate was obtained,

$$M_d = \sup_{n\in N} \|A_d^n\| \le 2 \cdot e(M_c^2 + M_{c,-1}^2), \tag{5}$$

where $M_{c,-1} = \sup_{t\ge 0} \|e^{A^{-1}t}\|$.

It is not hard to show that if the answer to our problem is positive, then $M_{c,-1}$ is a function of M_c only, and thus by (5) M_d becomes a function of M_c only. We close this section with the remark that (5) holds for any Hilbert space, and so if A generates a bounded semigroup and if the answer to our problem is positive, then the solutions of the difference equation (4) are bounded. This observation can also be found in [1, 6].

2. Equivalent formulation of the problem. In this section we consider the following weaker formulation of Problem 1.1.

PROBLEM 2.1. *Let H be a Hilbert space and let A be the infinitesimal generator of a bounded C_0-semigroup $(T(t))_{t\ge 0}$ on H, i.e., $\|T(t)\| \le M$ for all $t \ge 0$. Furthermore, assume that A^{-1} exists as a closed and densely defined operator.*

Are the above conditions sufficient for A^{-1} to be the infinitesimal generator of a C_0-semigroup?

Hence the difference between Problem 1.1 and Problem 2.1 is that we do not require that the semigroup generated by A^{-1} is bounded. It is clear that if Problem 1.1 has a positive answer, then so does Problem 2.1. In the following theorem we show that both problems are equivalent. Note that this does not mean that for any A for which A^{-1} generates a C_0-semigroup, this semigroup will be bounded. It merely means that if the answer to Problem 2.1 is positive for <u>all</u> A's and <u>all</u> Hilbert spaces, then the answer to Problem 1.1 is positive as well.

THEOREM 2.2. *If the answer to Problem 2.1 is positive for all A's and all Hilbert spaces, then the answer to Problem 1.1 is positive as well.*

Proof. Assume that A generates a bounded semigroup $(T(t))_{t\ge 0}$, and that A^{-1} exists as a closed and densely defined operator. Since the answer to Problem 2.1 is by assumption positive, we conclude that A^{-1} generates a C_0-semigroup $(S(t))_{t\ge 0}$. It remains to show that this semigroup is bounded. In order to do this, we introduce the Hilbert space $\ell_2(\mathbb{N}, H)$ as

$$\ell_2(\mathbb{N}, H) = H \oplus H \oplus \dots$$

with the norm

$$\left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \right\|_{\ell_2(\mathbb{N},H)} = \sqrt{\sum_{n\in\mathbb{N}} \|x_n\|_H^2}. \tag{6}$$

On $\ell_2(\mathbb{N}, H)$ we consider the operator $A_{\rm ext} = \mathrm{diag}\,(n^{-1}A)$. It is easy to see that this generates the C_0-semigroup $(\mathrm{diag}\,(T(n^{-1}t)))_{t\geq 0}$. Since $(T(t))_{t\geq 0}$ is bounded, this semigroup is bounded. The operator $A_{\rm ext}$ has as inverse $\mathrm{diag}\,(nA^{-1})$. By the conclusion of Problem 2.1 this generates a C_0-semigroup. It is not hard to see that this semigroup is given by $(\mathrm{diag}\,(S(nt)))_{t\geq 0}$. Since a C_0-semigroup is uniformly bounded on every compact interval, and since for all $t \geq 0$

$$\|\mathrm{diag}\ (S(nt))\| = \sup_n \|S(nt)\|,$$

we conclude that

$$\sup_{t\geq 0} \|S(t)\| = \sup_{t\in[0,1]} \sup_n \|S(nt)\| = \sup_{t\in[0,1]} \|\mathrm{diag}\ (S(nt))\| < \infty.$$

Thus we have proved the theorem. ■

REMARK 2.3. Note that if the space H in the above theorem is a Banach space, then the space $\ell_2(\mathbb{N}, H)$ is a Banach space as well. Thus the same theorem holds on a Banach space.

3. $e^{A^{-1}t}$ on a Banach space. In this section we show that the answer to our problem is negative if we would relax our assumption on the space. That is, if we would consider the problem on a Banach space. However, before we present these negative results, we show that the answer is positive if A generates a (sectorially) bounded analytic semigroup. From Theorem 2.5.2 of [14] we recall that A generates a (sectorially) bounded analytic semigroup, ($\|T(t)\| \leq M$ for all $t \in \mathbb{C}$ with $|\arg(t)| < \theta, \theta > 0$) if and only if

$$\|(sI - A)^{-1}\| \leq \frac{M}{|s|} \text{ for } \arg(s) < \pi/2 + \theta. \tag{7}$$

Using this characterization it is easy to prove that A^{-1} generates a sectorially bounded analytic semigroup if and only if A does.

THEOREM 3.1. *Let A be the generator of a sectorially bounded analytic semigroup and let A^{-1} exist as a closed, densely defined operator. Then A^{-1} generates a sectorially bounded analytic semigroup.*

Proof. Choose $s \in \mathbb{C} \setminus \{0\}$ with argument less than $\frac{\pi}{2} + \theta$. Then

$$(sI - A^{-1}) = \left(A - \frac{1}{s}I\right) \cdot s \cdot A^{-1}$$

Taking the inverse, we get

$$\begin{aligned}(sI - A^{-1})^{-1} &= A\left(A - \frac{1}{s}I\right)^{-1} \cdot \frac{1}{s} = \left(A - \frac{1}{s}I + \frac{1}{s}I\right)\left(A - \frac{1}{s}I\right)^{-1} \cdot \frac{1}{s} \\ &= \frac{1}{s} + \frac{1}{s^2}\left(A - \frac{1}{s}I\right)^{-1}.\end{aligned} \tag{8}$$

Since $\arg(s^{-1}) = -\arg(s)$, we may use our estimate for the resolvent of A, and we obtain for all $s \in \mathbb{C} \setminus \{0\}$ with argument less than $\frac{\pi}{2} + \theta$

$$\|(sI - A^{-1})^{-1}\| \leq \frac{1}{|s|} + \frac{1}{|s|^2}\left\|\left(A - \frac{1}{s}I\right)^{-1}\right\| \leq \frac{1}{|s|} + \frac{1}{|s|^2} \cdot \frac{M}{|\frac{1}{s}|} = \frac{M+1}{|s|}.$$

Using Theorem 2.5.2 of [14], we conclude that A^{-1} generates a sectorially bounded analytic semigroup. ■

In the rest of this section the following function will play an important role.

$$h_{\mathrm{ac}}(t) = \frac{1}{\sqrt{t}} J_1(2\sqrt{t}), \tag{9}$$

where $J_1(\cdot)$ is the Bessel function of the first kind and of the first order. The following properties can be found in the literature (see e.g. Watson [15]):

a. $h_{\mathrm{ac}}(\cdot)$ is continuous on $(0,\infty)$, and continuous from the right at zero with limit one.
b. $|h_{\mathrm{ac}}(\cdot)|$ is bounded by one on $[0,\infty)$.
c. For large t, we have

$$h_{\mathrm{ac}}(t) \approx \frac{1}{\sqrt{\pi}} t^{-\frac{3}{4}} \cos\left(2\sqrt{t} - \frac{3}{4}\pi\right).$$

d. The function $h_{\mathrm{ac}}(\cdot)$ is the derivative of $-J_0(2\sqrt{t})$, where J_0 is the Bessel function of the first kind and of the zeroth order.

Furthermore, using property d. and [4], we have that

$$\int_0^\infty e^{-st} h_{\mathrm{ac}}(t)dt = 1 - e^{-1/s}. \tag{10}$$

Using this equation we can prove the following lemma.

LEMMA 3.2. *Let A generate an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Then the following equality holds:*

$$e^{A^{-1}\tau} x_0 = x_0 - \int_0^\infty \tau h_{\mathrm{ac}}(t\tau) T(t) x_0 dt. \tag{11}$$

Proof. Since $T(t)$ is exponentially stable, A^{-1} exists as a bounded operator. Thus it generates the C_0-semigroup $(e^{A^{-1}t})_{t\geq 0}$. Furthermore, combining the exponential stability with property b., we see that the integral in (11) is well-defined.

To show the equality in (11) we take the Laplace transform on both sides. The Laplace transform of $e^{A^{-1}\tau}x_0$ equals $(sI - A^{-1})^{-1}x_0$, whereas the Laplace transform of x_0 equals x_0/s. Thus it remains to calculate the Laplace transform of the integral term in (11):

$$\begin{aligned}
&\int_0^\infty e^{-s\tau}\left[\int_0^\infty \tau h_{\mathrm{ac}}(t\tau)T(t)x_0 dt\right] d\tau \\
&\qquad = \int_0^\infty \left[\int_0^\infty e^{-s\tau}\tau h_{\mathrm{ac}}(t\tau)d\tau\right] T(t)x_0 dt, \qquad \text{using Fubini} \\
&\qquad = \int_0^\infty \left[\int_0^\infty e^{-\frac{s}{t}\widetilde{\tau}} \frac{1}{t^2}\widetilde{\tau} h_{\mathrm{ac}}(\widetilde{\tau})d\widetilde{\tau}\right] T(t)x_0 dt \qquad \widetilde{\tau} = t\tau \\
&\qquad = -\int_0^\infty \left[\frac{d}{ds}\int_0^\infty e^{-\frac{s}{t}\widetilde{\tau}} h_{\mathrm{ac}}(\widetilde{\tau})d\widetilde{\tau}\right] \frac{1}{t} T(t)x_0 dt
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^\infty \left[\frac{d}{ds}(1-e^{-t/s})\right]\frac{1}{t}T(t)x_0dt \qquad \text{by (10)}\\
&= \int_0^\infty \left[e^{-t/s}\frac{1}{s^2}\right]T(t)x_0dt\\
&= \frac{1}{s^2}\left(\frac{1}{s}I-A\right)^{-1}x_0.
\end{aligned}
$$

Using equation (8) we have that the Laplace transform of the right-hand side of (11) is equal to

$$\frac{1}{s}x_0 - \frac{1}{s^2}\left(\frac{1}{s}I-A\right)^{-1}x_0 = (sI-A^{-1})^{-1}x_0,$$

which proves the lemma. ■

Under the assumption that A generates an exponentially stable semigroup, the above lemma gives the relation between the semigroups generated by A^{-1} and A. Since the semigroup generated by A is exponentially stable, A^{-1} is a bounded operator, and thus it is the infinitesimal generator of a C_0-semigroup. Based on convergence results of semigroups, we assert the following. Let A be the infinitesimal generator of a bounded semigroup. Then A^{-1} is the infinitesimal generator of a C_0-semigroup if and only if the limit of $\int_0^\infty \tau h_{\mathrm{ac}}(t\tau)e^{-\varepsilon t}T(t)x_0dt$ for $\varepsilon \downarrow 0$ exists for all $x_0 \in X$.

Using the formula (11), we can show that in general $(e^{A^{-1}t})_{t\geq 0}$ is not (uniformly) bounded. However, on the domain A, we have that this semigroup converges to zero.

Theorem 3.3. *Let A be the infinitesimal generator of an exponentially stable semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Then the following holds:*

a. If $\|T(t)\| \leq Me^{-\omega t}$, with $M, \omega > 0$, then for $x_0 \in D(A)$ we have

$$\|e^{A^{-1}t}x_0\| \leq \frac{M}{\omega}\|Ax_0\|. \tag{12}$$

b. For $x_0 \in D(A)$, we have that for large t

$$\|e^{A^{-1}t}x_0\| \approx O(t^{-1/4}). \tag{13}$$

So for $x_0 \in D(A)$ we have that $\lim_{t\to 0} e^{A^{-1}t}x_0 = 0$.

Proof. We begin by studying the integral term in (11). Substituting $\widetilde{t} = t\tau$ in this term gives

$$
\begin{aligned}
\int_0^\infty \tau h_{\mathrm{ac}}(t\tau)T(t)x_0dt &= \int_0^\infty h_{\mathrm{ac}}(\widetilde{t})T\left(\frac{\widetilde{t}}{\tau}\right)x_0d\widetilde{t}\\
&= -J_0(2\sqrt{\widetilde{t}})T\left(\frac{\widetilde{t}}{\tau}\right)x_0\Big|_0^\infty + \int_0^\infty J_0(2\sqrt{\widetilde{t}})\frac{1}{\tau}T\left(\frac{\widetilde{t}}{\tau}\right)Ax_0d\widetilde{t}\\
&= x_0 + \int_0^\infty J_0(2\sqrt{\widetilde{t}})\frac{1}{\tau}T\left(\frac{\widetilde{t}}{\tau}\right)Ax_0d\widetilde{t},
\end{aligned}
$$

where we have used property d. and the fact that $J_0(0) = 1, J_0(\infty) = 0$. So we have that

$$e^{A^{-1}\tau}x_0 = -\int_0^\infty J_0(2\sqrt{\widetilde{t}})\frac{1}{\tau}T\left(\frac{\widetilde{t}}{\tau}\right)Ax_0d\widetilde{t} \tag{14}$$

Since the Bessel function J_0 is bounded by one on the positive real line, we can bound the semigroup generated by A^{-1} as

$$\|e^{A^{-1}\tau}x_0\| \leq \int_0^\infty \frac{1}{\tau} M e^{-\frac{\omega}{\tau}\tilde{t}} \|Ax_0\| d\tilde{t} = \frac{M}{\omega}\|Ax_0\|,$$

which proves (12).

In order to prove part b., we use that fact that there exists a $t_1 > 0$ such that $|J_0(2\sqrt{\tilde{t}})| \leq \frac{1}{\pi}\tilde{t}^{-1/4}$ for $\tilde{t} > t_1$, see Watson [15, p. 199]. Using (14) we have

$$\begin{aligned}
\|e^{A^{-1}\tau}x_0\| &\leq \int_0^{t_1} \frac{1}{\tau} M \|Ax_0\| d\tilde{t} + \int_{t_1}^\infty \frac{1}{\pi}\tilde{t}^{-1/4}\frac{1}{\tau} M e^{-\frac{\omega}{\tau}\tilde{t}} \|Ax_0\| d\tilde{t} \\
&= \frac{Mt_1}{\tau}\|Ax_0\| + \int_{t_1}^\infty \frac{1}{\pi}\tilde{t}^{-1/4}\frac{1}{\tau} M e^{-\frac{\omega}{\tau}\tilde{t}} \|Ax_0\| d\tilde{t} \\
&= \frac{Mt_1}{\tau}\|Ax_0\| + \int_{t_1/\tau}^\infty \frac{M}{\pi} t^{-1/4}\tau^{-1/4} e^{-\omega t}\|Ax_0\| dt \\
&\leq \frac{Mt_1}{\tau}\|Ax_0\| + \frac{M}{\pi}\frac{1}{\tau^{1/4}}\|Ax_0\| \int_0^\infty t^{-1/4} e^{-\omega t} dt.
\end{aligned}$$

From this it is clear that $\|e^{A^{-1}\tau}x_0\|$ converges to zero for τ going to infinity. ∎

So we see that the semigroup generated by A^{-1} is uniformly bounded on the domain of A. In the next theorem we estimate this semigroup on the Banach space, and find it is bounded by a constant times $t^{1/4}$. In Example 3.5 we show that this bound is the best possible. Thus the semigroup $(e^{A^{-1}t})_{t\geq 0}$ needs not to be bounded, even when $(T(t))_{t\geq 0}$ is exponentially stable.

Theorem 3.4. *Let A be the infinitesimal generator of a exponentially stable semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Then there exists an $M_0 > 0$ such that for $t \geq 0$*

$$\|e^{A^{-1}t}\| \leq 1 + M_0 t^{\frac{1}{4}}. \tag{15}$$

Proof. We again concentrate on the integral term in (11). By using property c. and a. we see that there exists an $M_2 > 0$ such that for all $t > 0$ $|h_{\mathrm{ac}}(t)| \leq M_2 t^{-\frac{3}{4}}$. Hence

$$\begin{aligned}
\int_0^\infty \|\tau h_{\mathrm{ac}}(t\tau)T(t)x_0\| dt &\leq \int_0^\infty \tau |h_{\mathrm{ac}}(t\tau)| M e^{-\omega t}\|x_0\| dt \\
&\leq M \int_0^\infty \tau \frac{M_2}{(t\tau)^{3/4}} e^{-\omega t}\|x_0\| dt \\
&= MM_2\|x_0\|\tau^{\frac{1}{4}} \int_0^\infty \frac{1}{t^{3/4}} e^{-\omega t} dt \\
&= MM_2\|x_0\|\tau^{\frac{1}{4}} \frac{\pi\sqrt{2}}{\omega^{\frac{1}{4}}\Gamma(\frac{3}{4})}.
\end{aligned}$$

Combining this inequality with equation (11), we see that (15) holds. ∎

Example 3.5. In our example we take $X = C_0([0,1))$, i.e., the space of all continuous functions on $[0,1)$, which have limit zero at one. This is a Banach space under the supremum norm. On this state space we choose as semigroup

$$(T(t)f)(\eta) = \begin{cases} f(t+\eta), & t+\eta \in [0,1), \\ 0, & t+\eta \geq 1. \end{cases} \tag{16}$$

It is not hard to see that this is a strongly continuous semigroup on X with $\|T(t)\| \leq 1$, and $T(t) = 0$ for $t \geq 1$. So it is a contraction semigroup with growth bound minus infinity. In particular, the semigroup is exponentially stable.

From Lemma 3.2 we have that

$$\begin{aligned}(e^{A^{-1}\tau} f)(\eta) &= f(\eta) - \int_0^{1-\eta} \tau h_{\rm ac}(t\tau) f(t+\eta) dt \\ &= f(\eta) - \int_0^{(1-\eta)\tau} h_{\rm ac}(\tilde{t}) f\left(\frac{\tilde{t}}{\tau} + \eta\right) d\tilde{t}. \end{aligned} \tag{17}$$

By constructing a suitable sequence of f's, we show that $\|e^{A^{-1}\tau}\|$ behaves like $\sqrt[4]{\tau}$. Therefore we fix $\tau > 1$.

Let ϕ_ε be a continuous function on $\mathbb{R}$ which is positive, bounded by one, $\phi_\varepsilon(0) = 0$, and $\phi_\varepsilon(t) = 1$ for $|t| \geq \varepsilon$. Furthermore, define ψ_τ to be a continuous function on $[0, \infty)$ which is positive, bounded by one, $\psi_\tau(t) = 0$ for $t \geq \tau$, and $\psi_\tau(t) = 1$ for $t < \tau - 1$.

Let $t_k, k = 1, \dots, K$ be the zeros of $h_{\rm ac}$ in $[0, \tau]$.

$$g_\varepsilon(t) = \Big[\prod_{k=1}^{K} \phi_\varepsilon(t - t_k)\Big] \operatorname{sign}(h_{\rm ac}(t)) \psi_\tau(t), \qquad t \geq 0.$$

It is not hard to see that g_ε is continuous on $[0, \infty)$ with $g_\varepsilon(t) = 0$ for $t \geq \tau$ and $\sup_{t \geq 0} |g_\varepsilon(t)| = 1$. Hence $f(t) := -g_\varepsilon(t/\tau)$ is an element of X with norm one.

Next choose in equation (17) $\eta = 0$ and $f(t) = -g_\varepsilon(\tau t)$, then

$$\begin{aligned}(e^{A^{-1}\tau} f)(0) &= f(0) - \int_0^{\tau} h_{\rm ac}(\tilde{t}) f\left(\frac{\tilde{t}}{\tau} + 0\right) d\tilde{t} \\ &\geq f(0) + \int_{\{\tilde{t} \in [0, \tau-1] \mid |\tilde{t} - t_k| \geq \epsilon, \text{ for } k=1,\dots,K\}} |h_{\rm ac}(\tilde{t})| d\tilde{t}. \end{aligned}$$

Since f has norm one and $f(0) = 1$, we see that

$$\begin{aligned}\|e^{A^{-1}\tau}\| &\geq 1 + \sup_{\varepsilon > 0} \int_{\{\tilde{t} \in [0, \tau-1] \mid |\tilde{t} - t_k| \geq \epsilon, \text{ for } k=1,\dots,K\}} |h_{\rm ac}(\tilde{t})| d\tilde{t} \\ &= 1 + \int_0^{\tau - 1} |h_{\rm ac}(\tilde{t})| d\tilde{t}. \end{aligned} \tag{18}$$

By property c. we find that this is of the order $\tau^{1/4}$. So for the exponentially stable semigroup (16) the semigroup generated by A^{-1} is not uniformly bounded.

As one can see from the above proof, it would simplify a lot if we could choose $f(t) = \operatorname{sign}(h_{\rm ac}(t\tau))$. However, since the sign function is not continuous, we have to approximate it with continuous functions.

REMARK 3.6. The technique of Theorem 3.4 and Example 3.5 can be used to prove a more general result. Let $h : [0, \infty) \to \mathbb{R}$ be locally integrable. Then the transformation

$$\mathcal{H}(A, \tau)x_0 = \int_0^\infty \tau h(t\tau)T(t)x_0 dt$$

gives for every exponentially stable C_0-semigroup $T(t)$ a uniformly bounded operator-valued function if and only if h is absolutely integrable. Furthermore, we have that

$$\|\mathcal{H}(A, \tau)\| \le \int_0^\infty |h(t)|dt \cdot \sup_{t\ge 0} \|T(t)\|.$$

For more information, see also Chapter XV of [11].

Using Example 3.5 and Remark 2.3 it is easy to construct an example of a bounded C_0-semigroup for which A^{-1} does not generate a C_0-semigroup. This example shows that the question posed by deLaubenfels in [13] must be answered negatively.

EXAMPLE 3.7. Let $X = C_0([0, 1))$ be the Banach space of Example 3.5 and let $(T(t))_{t\ge 0}$ be the shift semigroup of the same example and let A be its infinitesimal generator. By $(S(t))_{t\ge 0}$ we denote the semigroup generated by A^{-1}. The Banach space $\ell_2(\mathbb{N}, X)$ is defined as

$$\ell_2(\mathbb{N}, X) = X \oplus X \oplus \dots$$

with the norm

$$\left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \right\|_{\ell_2(\mathbb{N},X)} = \sqrt{\sum_{n\in\mathbb{N}} \|x_n\|_X^2}.$$

On $\ell_2(\mathbb{N}, X)$ we consider the operator $A_{\text{ext}} = \operatorname{diag}(n^{-1}A)$. It is easy to see that this generates the C_0-semigroup $(T_{\text{ext}}(t))_{t\ge 0} := (\operatorname{diag}(T(n^{-1}t)))_{t\ge 0}$. Using the fact that $(T(t))_{t\ge 0}$ is a contraction semigroup, it is easy to show that $(T_{\text{ext}}(t))_{t\ge 0}$ is a contraction semigroup on $\ell_2(\mathbb{N}, X)$. If A_{ext}^{-1} were the generator of a C_0-semigroup, then this semigroup would be given by $(\operatorname{diag}(S(nt)))_{t\ge 0}$. Since a semigroup is bounded on every compact interval, we find

$$\sup_{t\ge 0} \|S(t)\| = \sup_{t\in[0,1]} \sup_n \|S(nt)\| = \sup_{t\in[0,1]} \|\operatorname{diag}(S(nt))\| < \infty.$$

However, we know from Example 3.5 that $(S(t))_{t\ge 0}$ is not uniformly bounded. Hence A_{ext}^{-1} is not the infinitesimal generator of a C_0-semigroup.

4. $e^{A^{-1}t}$ on a Hilbert space.

On Hilbert spaces there is, apart from the class of (sectorially) bounded analytic semigroups, a second class of generators for which our problem has been solved. This is the class of infinitesimal generators of contraction semigroups. It is well known that A is the infinitesimal generator of a contraction semigroup on the Hilbert space H if and only if $\operatorname{Re}\langle x, Ax\rangle \le 0$ for all $x \in D(A)$ and $(\lambda I - A)$ is invertible for some $\lambda > 0$. Using this, the following lemma follows almost directly.

LEMMA 4.1. *Let A be the infinitesimal generator of a contraction semigroup on the Hilbert space H and assume that A^{-1} exists as a closed, densely defined operator. Then A^{-1} is also the infinitesimal generator of a contraction semigroup.*

Proof. Since the domain of A^{-1} equals the range of A, we have to study the inner product of $A^{-1}x$ with x for x in the range of A. Since these elements can be written as Ay, with $y \in D(A)$, we find that

$$\operatorname{Re}\langle x, A^{-1}x\rangle = \operatorname{Re}\langle Ay, y\rangle.$$

Since by assumption this last expression is non-positive, we see that A^{-1} satisfies the first condition for being an infinitesimal generator of a contraction semigroup. Using e.g. (8), it is easy to show that $(\lambda I - A^{-1})$ is boundedly invertible for some $\lambda > 0$. Hence A^{-1} is the infinitesimal generator of a contraction semigroup on H. ■

Since on a Hilbert space, many bounded semigroups are similar to a contraction semigroup, we see that on Hilbert spaces the answer to our problem is yes in many cases. One case of particular interest is the class of bounded groups, see [17]. However, it is unknown whether the problem holds for any semigroup on a Hilbert space.

5. Closing remarks. While the paper was under review, I learned that A. M. Gomilko has recently found similar results to those presented here. For instance, he also found the relation (11). Furthermore, for every $p \in [1, \frac{4}{3}) \cup (4, \infty)$ he constructed a bounded generator of a contraction semigroup on ℓ^p, such that the inverse does not generate a C_0-semigroup, see [8], thus proving that the question by deLaubenfels should be answered negatively on reflexive Banach spaces and for bounded generators. Furthermore, in [7] he gives some sufficient conditions under which A^{-1} is also a infinitesimal generator of a bounded C_0-semigroup.

Yuri Tomilov pointed out that on pages 343–344 of [12], H. Komatsu gives (already in 1966) an example of an infinitesimal generator of a contraction semigroup on c_0 for which the inverse is not an infinitesimal generator.

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