A REDUCIBILITY PROBLEM FOR THE CLASSICAL RESIDUE FORMULA

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Let $z_1, \ldots, z_n$ be $n$ distinct points in $\mathbb{C}$ and let

$$G(z) = \prod_{k=1}^{n} (z - z_k).$$

Denote by $\Gamma$ a simple contour surrounding $\{z_k\}_{k=1}^{n}$. The residue formula

$$\frac{1}{2\pi i} \int_{\Gamma} f(z) \frac{G'(z)}{G(z)} \, dz = \sum_{k=1}^{n} f(z_k)$$

is valid in a class of analytic functions, in particular, it is true for all polynomials of degree $\leq 2n - 1$. In this sense (1) is a Gauss type quadratic formula of order $n$.

DEFINITION 1. Let $m$ be an integer, $2 \leq m \leq n$. A configuration $\{z_k\}_{k=1}^{n}$ is called $m$-reducible if there exists another configuration $\{w_j\}_{j=1}^{m}$ such that

$$\frac{1}{n} \sum_{k=1}^{n} f(z_k) = \sum_{j=1}^{m} \alpha_j f(w_j), \quad f \in \mathcal{P}(\mathbb{C}), \quad \deg f \leq 2m - 1,$$

with some complex coefficients $\alpha_1, \ldots, \alpha_m$. Obviously, (2) implies that $\sum_{j=1}^{m} \alpha_j = 1$.

REMARK. It does not make sense to extend Definition 1 to $m = 1$ since in this case the barycenter $w_1$ of the system $\{z_k\}$ satisfies (2) with $\alpha_1 = 1$. Thus one can say that every configuration is $1$-reducible.

DEFINITION 2. A configuration $\{z_k\}_{k=1}^{n}$ is called irreducible if for each $m \in [2, n)$ it is not $m$-reducible.

Note that these properties are affine invariant, i.e. they are invariant with respect to transformations $z \mapsto az + b$.

It is shown in [1] that a triangle $\{z_k\}_{k=1}^{3}$ is irreducible if and only if it is either equilateral or isosceles with the angle between the equal sides which is equal to

$$\alpha = \frac{\pi}{2} + \arctan \frac{\eta}{\sqrt{4 - \eta^2}}$$
where \( \eta \) is the unique real root of the cubic equation
\[
4\eta^3 - 12\eta^2 + 9\eta + 2 = 0
\]
(so that \( \eta \approx 0.5283\pi \)).

Also it turns out that for every \( n \in \mathbb{N}, n \geq 3 \) the regular \( n \)-gon is irreducible. It would be interesting to find other examples for \( n \geq 4 \) and, maybe, to describe explicitly all of them for small \( n \). In general there is a characterization of irreducibility by a union of systems of algebraic equations. (This can be easily extracted from [1, Theorem 6].)

**Conjecture.** For every \( n \) the set of irreducible configurations is finite up to affine equivalence.

To support formally this conjecture let me indicate that each system mentioned above consists of \( n - 2 \) equations. On the other hand, the affine class of \( n \)-configuration depends exactly on \( n - 2 \) complex parameters.

**References**