FUNCTION SPACES VIII
BANACH CENTER PUBLICATIONS, VOLUME 79
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

AN OPERATOR CHARACTERIZATION OF L^p-SPACES IN A CLASS OF ORLICZ SPACES

MACIEJ BURNECKI

Institute of Mathematics and Computer Science, Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
E-mail: mburn@im.pwr.wroc.pl

Abstract. We consider an embedding of the group of invertible transformations of $[0,1]$ into the algebra of bounded linear operators on an Orlicz space. We show that if this embedding preserves the group action then the Orlicz space is an L^p-space for some $1 \le p < \infty$.

1. Introduction. Let us denote the Lebesgue measure on the σ-algebra of Borel subsets of $[0,1]$ by m. We call a Borel mesurable function $\tau : [0,1] \to [0,1]$ a transformation if it is nonsingular, that is, if $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$. Transformations equal almost everywhere are identified. A transformation is called invertible if it is injective, onto $[0,1]$ and its inverse, which is Borel measurable (see [4], § 39, V, Theorem 1), is nonsingular. We denote by G the group of all invertible transformations.

Let $\phi : [0,\infty) \to [0,\infty)$ be convex, $\phi(x) = 0$ iff $x = 0$, and let ϕ satisfy the condition Δ' globally (which means that there exists $c > 0$ such that $\phi(xy) \le c\phi(x)\phi(y)$ for $x, y > 0$). It is shown in [2], Proposition 2.1, that if $\tau \in G$ and ω_τ stands for the Radon–Nikodym derivative of the measure $m \circ \tau^{-1}$ with respect to m then the formula

$$T_\tau^{(\phi)}(f) = (f \circ \tau^{-1})(\phi^{-1} \circ \omega_\tau)$$

defines a bounded linear operator $T_\tau^{(\phi)}$ on the Orlicz space $L^\phi(m)$ (for information on Orlicz spaces see, for example, [5]). This is a generalization of a similar formula for L^p-spaces (compare [3], § 5).

If $\phi(x) = \lambda x^p$ for $x \in [0,\infty)$, where $\lambda > 0$ and $p \ge 1$ (the case of $(L^p(m), \lambda||\cdot||_p)$), then the embedding $\tau \mapsto T_\tau^{(\phi)}$ of G into $\mathcal{L}(L^\phi(m))$ preserves the group action with constant λ. This is a consequence of the formulas given at the beginning of the proof below. The embedding in question need not preserve the group action for Orlicz spaces in general

2000 *Mathematics Subject Classification*: Primary 46E30; Secondary 47B37.

Key words and phrases: measurable transformation, L^p-space, Orlicz space, bounded linear operator.

The paper is in final form and no version of it will be published elsewhere.

(for an example see [2], Remark 2.3.4). We are going to show that the preservation of the group action characterizes, in fact, L^p-spaces.

2. The characterization. We write χ_X for the characteristic function of a subset X of $[0,1]$. As usual, the set of non-negative integers is denoted by $\mathbb{N}$. The symbol $|\overline{PQ}|$ stands for the length of the interval $\overline{PQ}$, where $P,Q \in \mathbb{R}^2$.

Let $p_n, q_n \in (0,\infty)$ be chosen so that the set $\{(p_n,q_n) : n \in \mathbb{N}\}$ is dense in $(0,\infty)^2$. Choose $P_n = (x_n,y_n)$, $Q_n = (y_n,z_n)$ in $[0,1]^2$ with the following properties:

(a) $P_0 = Q_0 = (0,0)$,
(b) $0 < |\overline{P_nP_{n+1}}| \le 1/2^{n+1}$ and $0 < |\overline{Q_nQ_{n+1}}| \le 1/2^{n+1}$ for $n \in \mathbb{N}$,
(c) $\tan\sphericalangle(OX, \overline{P_nP_{n+1}}) = p_n$ and $\tan\sphericalangle(OX, \overline{Q_nQ_{n+1}}) = q_n$ for $n \in \mathbb{N}$.

Set $\alpha_\infty = \lim_{n\to\infty}\alpha_n$ for a convergent sequence (α_n) in $\mathbb{R}$ or in $\mathbb{R}^2$.

Define $\pi,\rho : [0,1] \to [0,1]$ by the following conditions:

1. the graph of π^{-1} consists of the intervals $\overline{P_nP_{n+1}}$ and of the interval $\overline{P_\infty(1,1)}$,
2. the graph of ρ^{-1} consists of the intervals $\overline{Q_nQ_{n+1}}$ and of the interval $\overline{Q_\infty(1,1)}$.

Clearly, $\pi,\rho \in G$.

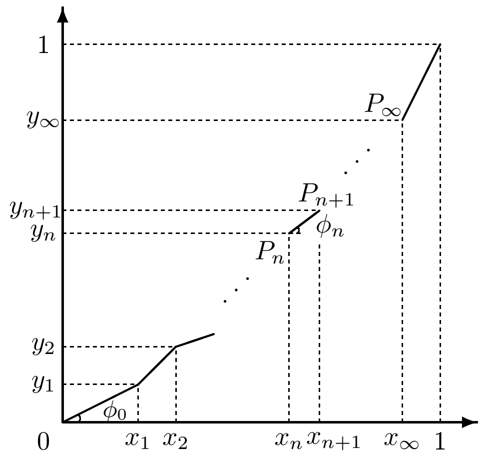

Fig. 1. Transformation π^{-1}: $\tan\phi_n = p_n$

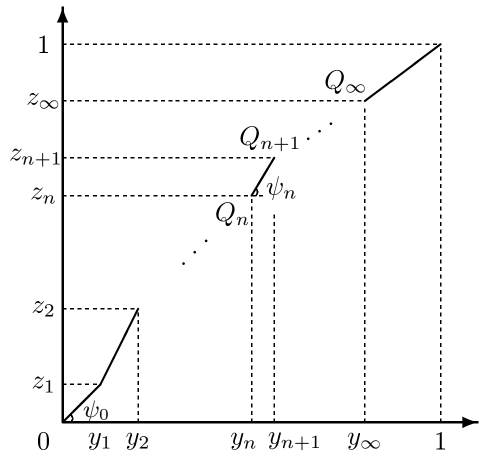

Fig. 2. Transformation ρ^{-1}: $\tan\psi_n = q_n$

LEMMA. *Let π,ρ be as defined above and $0 < \lambda < \infty$. If $\phi : [0,\infty) \to [0,\infty)$ is convex, $\phi(x) = 0$ iff $x = 0$, and ϕ satisfies the condition Δ' globally then the following two conditions are equivalent*:

1. $T^{(\phi)}_{\pi\circ\rho}(\chi_{[0,1]}) = \lambda T^{(\phi)}_\pi \circ T^{(\phi)}_\rho(\chi_{[0,1]})$;
2. *there exists $1 \le p < \infty$ such that $\phi(x) = \lambda x^p$ for $x \in [0,\infty)$.*

Proof. We show that (1) implies (2). For $f \in L^\phi(m)$ and arbitrary $\tau,\sigma \in G$ we have

$$T^{(\phi)}_{\tau\circ\sigma}(f) = (f\circ\sigma^{-1}\circ\tau^{-1})\phi^{-1}\circ((\omega_\sigma\circ\tau^{-1})\omega_\tau),$$
$$T^{(\phi)}_\tau \circ T^{(\phi)}_\sigma(f) = (f\circ\sigma^{-1}\circ\tau^{-1})(\phi^{-1}\circ\omega_\sigma\circ\tau^{-1})(\phi^{-1}\circ\omega_\tau).$$

Hence

$$T^{(\phi)}_{\tau\circ\sigma}(\chi_{[0,1]}) = \phi^{-1}\circ((\omega_\sigma\circ\tau^{-1})\omega_\tau)$$

and

$$T^{(\phi)}_{\tau}\circ T^{(\phi)}_{\sigma}(\chi_{[0,1]}) = (\phi^{-1}\circ\omega_\sigma\circ\tau^{-1})(\phi^{-1}\circ\omega_\tau).$$

Let now $\tau=\pi$ and $\sigma=\rho$. We obtain

$$T^{(\phi)}_{\pi\circ\rho}\left(\chi_{[0,1]}\right) = \sum_{n=0}^{\infty}\chi_{(x_n,x_{n+1})}\phi^{-1}(p_nq_n) + a\chi_{(x_\infty,1)}$$

and

$$\lambda T^{(\phi)}_{\pi}\circ T^{(\phi)}_{\rho}\left(\chi_{[0,1]}\right) = \lambda\sum_{n=0}^{\infty}\chi_{(x_n,x_{n+1})}\phi^{-1}(p_n)\phi^{-1}(q_n) + b\chi_{(x_\infty,1)}$$

for some $a,b\in\mathbb{R}$. This gives

$$\phi^{-1}(p_nq_n) = \lambda\phi^{-1}(p_n)\phi^{-1}(q_n)$$

for every $n\in\mathbb{N}$.

Since $\{(p_n,q_n): n\in\mathbb{N}\}$ is dense in $(0,\infty)^2$ and ϕ is continuous and increasing, there exists $0<p<\infty$ such that $\phi^{-1}(t)=t^{1/p}/\lambda$ for $t\in(0,\infty)$ (compare [1], 2.1.2). By the convexity of ϕ, we have $1\le p<\infty$. ■

In particular, when $\lambda=1$ and all invertible transformations are considered, we obtain the following theorem.

THEOREM. *Let $\phi:[0,\infty)\to[0,\infty)$ be convex, $\phi(x)=0$ iff $x=0$, and let ϕ satisfy the condition Δ' globally. Then the following two conditions are equivalent*:

1. *the embedding $\tau\mapsto T^{(\phi)}_\tau$ of G into $\mathcal{L}(L^\phi(m))$ preserves the group action* (*with constant* 1);
2. *there exists $1\le p<\infty$ such that $\phi(x)=x^p$ for $x\in[0,\infty)$.* ■

Acknowledgments. The author whould like to thank Professor Zbigniew Lipecki for many helpful remarks which have improved the text significantly.

References

[1] J. Aczel, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Berlin, 1961.

[2] M. Burnecki, *Invertible transformations acting on Orlicz spaces*, Arch. Math. (Basel) 70 (1998), 319–330.

[3] J. R. Choksi and S. Kakutani, *Residuality of ergodic measurable transformations and of ergodic transformations which preserve an infinite measure*, Indiana Univ. Math. J. 28 (1979), 453–469.

[4] K. Kuratowski, *Topology*, Vol. 1, PWN, Warszawa, 1966.

[5] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.