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THE EXTENSION OF THE KREIN-ŠMULIAN THEOREM FOR ORDER-CONTINUOUS BANACH LATTICES

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Abstract. If X is a Banach space and $C \subset X$ a convex subset, for $x^{**} \in X^{**}$ and $A \subset X^{**}$ let $d(x^{**}, C) = \inf\{\|x^{**} - x\| : x \in C\}$ be the distance from x^{**} to C and $\hat{d}(A, C) = \sup\{d(a, C) : a \in A\}$. Among other things, we prove that if X is an order-continuous Banach lattice and K is a w*-compact subset of X^{**} we have: (i) $\hat{d}(\overline{\text{co}}^{w^*}(K), X) \leq 2\hat{d}(K, X)$ and, if $K \cap X$ is w*-dense in K, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$; (ii) if X fails to have a copy of $\ell_1(\aleph_1)$, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$; (iii) if X has a 1-symmetric basis, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$.

1. Introduction. If X is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. The weak-topology of X is denoted by w and the weak*-topology of X^* by w*. If C is a convex subset of X^{**}, for $x^{**} \in X^{**}$ and $A \subset X^{**}$, let $d(x^{**}, C) = \inf\{\|x^{**} - x\| : x \in C\}$ be the distance from x^{**} to C and $\hat{d}(A, C) = \sup\{d(a, C) : a \in A\}$. Observe that: (i) $\hat{d}(\overline{\text{co}}(A), C) = \hat{d}(\text{co}(A), C) = \hat{d}(A, C)$ where $\text{co}(A)$ is the convex hull of A; (ii) if $X^\perp = \{z \in X^{***} : z(x) = 0,\ \forall x \in X\}$ and $Q : X^{**} \to \frac{X^{**}}{X}$ is the canonical quotient mapping, then:

$$d(x^{**}, X) = \sup\{z(x^{**}) : z \in S(X^\perp)\} = \|Qx^{**}\|.$$

With this terminology, the Krein-Šmulian Theorem (see [3, p. 51]) states the following: if X is a Banach space and $K \subset X^{**}$ a w*-compact subset such that $\hat{d}(K, X) = 0$ (thus, K is a weakly compact subset of X), then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = 0$, that is, $\overline{\text{co}}^{w^*}(K) \subset X$ and $\overline{\text{co}}^{w^*}(K) = \overline{\text{co}}(K)$ is also a weakly compact subset of X ($\overline{\text{co}}(K) = \|\cdot\|$-closure of $\text{co}(K)$ and $\overline{\text{co}}^{w^*}(K) =$ w*-closure of $\text{co}(K)$). So, in view of this situation, we can pose two natural questions:

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(A) If $K \subset X^{**}$ is a w*-compact subset, does the equality $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$ always hold?

The answer to this question is negative. In fact, we constructed (see [5], [7]) a Banach space X such that: (i) there exists a w*-compact subset $H \subset B(X^{**})$ such that $H \cap X$ is w*-dense in H, $\hat{d}(H, X) = \frac{1}{2}$ and $\hat{d}(\overline{\text{co}}^{w^*}(H), X) = 1$; (ii) there exists a w*-compact subset $K \subset B(X^{**})$ such that $\hat{d}(K, X) = \frac{1}{3}$ and $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = 1$.

(B) Does there exist a universal constant $1 \leq M < \infty$ such that always X has M-*control inside* its bidual X^{**}, that is, $\hat{d}(\overline{\text{co}}^{w^*}(K), X) \leq M\hat{d}(K, X)$ for every w*-compact subset of X^{**}?

The answer to this question is affirmative. In [5] we proved the following result, which extends the Krein-Šmulian Theorem: if $K \subset X^{**}$ is a w*-compact subset and $Z \subset X$ a subspace of X, then $\hat{d}(\overline{\text{co}}^{w^*}(K), Z) \leq 5\hat{d}(K, Z)$ and, if $Z \cap K$ is w*-dense in K, then $\hat{d}(\overline{\text{co}}^{w^*}(K), Z) \leq 2\hat{d}(K, Z)$. So, in view of these results we have: (i) the universal constant M of our extension of the Krein-Šmulian Theorem satisfies $3 \leq M \leq 5$; (ii) for the category of w*-compact subsets $K \subset X^{**}$ such that $X \cap K$ is w*-dense in K, the constant M is exactly 2.

Although the answer to question (A) is, in general, negative there are many Banach spaces X for which $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$. This is the case (see [5]), for instance, if $\ell_1 \not\subseteq X^*$, if the unit ball $B(X^*)$ of the dual X^* is w*-angelic (for example, if X is weakly compactly generated (WCG) or weakly Lindelöf determined (WLD)), if $X = \ell_1(I)$, if K is fragmented by the norm of X^{**}, etc.

In order to find classes of Banach space with 1-control in its bidual or, at least, a better control than in the above general case, we examine in this paper the class of Banach lattices. First, we have the following remarks:

(a) In [7, Prop. 10] we have constructed a Banach lattice X and a w*-compact subset $K \subset B(X^{**})$ such that $\hat{d}(K, X) = \frac{1}{3}$ and $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = 1$. So, concerning the control inside the bidual, the class of Banach lattices behaves as in the general case.

(b) In [6] we proved that if φ is an Orlicz function, I an infinite set and $X = \ell_\varphi(I)$ the corresponding Orlicz space, equipped with either the Luxemburg or the Orlicz norm, then for every w*-compact subset $K \subset X^{**}$ we have $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$ if and only if φ satisfies the Δ_2 condition at 0, that is, if and only if $\ell_\varphi(I)$ has a 1-symmetric basis.

In view of these results it is natural to ask the following questions. If X is a Banach space with a 1-symmetric basis, does X have 1-control inside X^{**}? What happens if X is an order-continuous Banach lattice? The purpose of this note is to consider these problems. Among other things, we prove that if X is an order-continuous Banach lattice and K is a w*-compact subset of X^{**} then: (i) $\hat{d}(\overline{\text{co}}^{w^*}(K), X) \leq 2\hat{d}(K, X)$ and, if $K \cap X$ is w*-dense in K, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$; (ii) if X fails to have a copy of $\ell_1(\aleph_1)$, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$; (iii) if X has a 1-symmetric basis, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$.

Our notation is standard as in the books [9],[10]. $|A|$ denotes the cardinality of a set A, ω_1 the first uncountable ordinal and $\aleph_1 = |\omega_1|$. Let X be a Banach space. If A is a subset of X, then $[A]$ is the subspace generated by A and $\overline{[A]}$ the closure of $[A]$. The Banach space X is said to be weakly compactly generated (for short, WCG) if there exists a w-compact subset W of X such that $X = \overline{[W]}$. The Banach space X is said to have *M-control* inside its bidual X^{**} if $\hat{d}(\overline{\text{co}}^{w^*}(K), X) \leq M\hat{d}(K, X)$ for every w*-compact subset K of X^{**}. The notions (countable or uncountable) of 1-unconditional decomposition, 1-unconditional basis, 1-symmetric basis and transfinite basis can be found in [9] and [12]. Concerning the notions of Banach lattices, order-continuity (for short, o-continuity), etc., let us refer the reader to the books [10] and [11].

2. The structure of X, X^* and X^{**} when $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$. Let us consider the structure of X, X^* and X^{**} when X is a 1-unconditional direct sum $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ of a family of Banach subspaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ of X.

Definition 1. A Banach space X is said to be an 1-unconditional direct sum of a family of Banach subspaces $\{X_\alpha : \alpha \in \mathcal{A}\}$ of X, for short, $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ 1-unconditional, when $X = \overline{[\cup_{\alpha\in\mathcal{A}} X_\alpha]}$ and, if $x_\alpha \in X_\alpha, \epsilon_\alpha = \pm 1$, $\alpha \in \mathcal{A}$, and A is a finite subset of $\mathcal{A}$, then $\| \sum_{\alpha\in A} \epsilon_\alpha x_\alpha \| \leq \| \sum_{\alpha\in A} x_\alpha \|$.

If $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ is a 1-unconditional direct sum, then

(i) For each subset $A \subset \mathcal{A}$ there exists a projection $P_A : X \to X$ such that $\|P_A\| = 1$ and $P_A(X) = \sum_{\alpha\in A} \oplus X_\alpha$.

(ii) Every $x \in X$ has a unique representation of the form $x = \sum_{\alpha\in\mathcal{A}} x_\alpha$ with $x_\alpha \in X_\alpha$ such that the subset $\{\alpha \in \mathcal{A} : x_\alpha \neq 0\}$ is countable, the above series converges unconditionally and $\| \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha x_\alpha \| = \|x\|$, where $\epsilon_\alpha = \pm 1$, $\forall \alpha \in \mathcal{A}$.

(iii) If $u \in X^*$, the α-th coordinate u_α of u will be the restriction $u_\alpha := u \restriction X_\alpha \in X_\alpha^*$ of u to X_α. We will identify u with the family $(u_\alpha)_{\alpha\in\mathcal{A}}$ of its coordinates.

We consider each dual X_α^* canonically and isometrically embedded into X^* as follows. If $P_\alpha : X \to X_\alpha$ is the projection associated to X_α, then $P_\alpha^*(X_\alpha^*)$ is a subspace of X^* isometric to X_α^*. We identify X_α^* with $P_\alpha^*(X_\alpha^*)$. Consider in X^* the closed subspace $Y_0 := \overline{[\bigcup_{\alpha\in\mathcal{A}} X_\alpha^*]}$, which is actually the 1-unconditional direct sum of the closed subspaces $\{X_\alpha^* : \alpha \in \mathcal{A}\}$, that is, $Y_0 = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha^*$ 1-unconditional. Let Y_0^* be the dual of Y_0.

Fact 1. *Y_0^* can be embedded canonically and isometrically into X^{**}.*

Indeed, if $z \in Y_0^*$, for each $\alpha \in \mathcal{A}$ let $z_\alpha := z \restriction X_\alpha^*$ be the α-th coordinate of z and identify z with the family $(z_\alpha)_{\alpha\in\mathcal{A}}$ of its coordinates. In order to embed Y_0^* into X^{**}, define the mapping $h : Y_0^* \to X^{**}$ as follows:

$$\forall z \in Y_0^*,\ \forall u \in X^*,\ h(z)(u) = \sum_{\alpha\in\mathcal{A}} z_\alpha(u_\alpha).$$

Concerning the definition of h we have to check several things, namely:

(A) First, we need to be sure that the series $\sum_{\alpha\in\mathcal{A}} z_\alpha(u_\alpha)$ converges. If $\mathcal{A}_0 \subset \mathcal{A}$ is a finite subset, then $\sum_{\alpha\in\mathcal{A}_0} u_\alpha \in Y_0$ and, if $\epsilon_\alpha = \pm 1$, $\forall \alpha \in \mathcal{A}$, then

$$\sum_{\alpha\in\mathcal{A}_0} z_\alpha(\epsilon_\alpha u_\alpha) = z\Big(\sum_{\alpha\in\mathcal{A}_0} \epsilon_\alpha u_\alpha\Big) \leq \|z\| \cdot \Big\| \sum_{\alpha\in\mathcal{A}_0} \epsilon_\alpha u_\alpha \Big\| \leq \|z\| \cdot \|u\|.$$

Thus we get: (i) $|\{\alpha \in \mathcal{A} : z_\alpha(u_\alpha) \neq 0\}| \leq \aleph_0$; (ii) the series $\sum_{\alpha\in\mathcal{A}} z_\alpha(u_\alpha)$ converges absolutely and, moreover, $\sum_{\alpha\in\mathcal{A}} z_\alpha(u_\alpha) \leq \|z\|\cdot\|u\|$. Therefore $h(z) \in X^{**}$ with $\|h(z)\| \leq \|z\|$, $\forall z \in Y_0^*$, and h is a continuous linear mapping such that $\|h\| \leq 1$.

(B) Let us see that h is an isometry. As $h(z) \restriction Y_0 = z$, $\forall z \in Y_0^*$, we have

$$\|h(z)\| = \sup\{\langle h(z), u\rangle : u \in B(X^*)\} \geq \sup\{\langle z, u\rangle : u \in B(Y_0)\} = \|z\|.$$

On the other hand, $\|h(z)\| \leq \|z\|$. So, h is an isometry.

We know that the subspace $Y_0^\perp := \{z \in X^{**} : \langle z, y\rangle = 0,\ \forall y \in Y_0\}$ of X^{**} is isometrically isomorphic to the dual $(\frac{X^*}{Y_0})^*$.

Fact 2. *$X^{**} = h(Y_0^*) \overset{m}{\oplus} Y_0^\perp$, that is, X^{**} is the monotone direct sum of $h(Y_0^*)$ and $Y_0^\perp$. Observe that this means that every $z \in X^{**}$ has a unique decomposition $z = z_1 + z_2$ with $z_1 \in h(Y_0^*)$ and $z_2 \in Y_0^\perp$ so that $\|z\| \geq \|z_1\| \vee \|z_2\|$.*

Indeed, it is clear that $h(Y_0^*) \cap Y_0^\perp = \{0\}$. Let $z \in X^{**}$ and put $w_1 := z \restriction Y_0$. Let us see that $z - h(w_1) \in Y_0^\perp$. For every $\alpha \in \mathcal{A}$ and every $v \in X_\alpha^*$ we have

$$\langle z - h(w_1), v\rangle = \langle z, v\rangle - \langle h(w_1), v\rangle$$
$$= \langle z \restriction X_\alpha^*, v\rangle - \langle w_{1\alpha}, v\rangle = \langle z \restriction X_\alpha^*, v\rangle - \langle z \restriction X_\alpha^*, v\rangle = 0.$$

Thus, $X^{**} = h(Y_0^*)\oplus Y_0^\perp$. Moreover the above direct sum is monotone because, if $z = z_1 + z_2 \in X^{**}$ with $z_1 \in h(Y_0^*)$ and $z_2 \in Y_0^\perp$, we have on the one hand

$$\|z\| \geq \sup\{\langle z_1 + z_2, u\rangle : u \in B(Y_0)\} = \sup\{\langle z_1, u\rangle : u \in B(Y_0)\} = \|z_1\|.$$

On the other hand, given $\epsilon > 0$, choose $v \in B(X^*)$ such that $\|z_2\| - \frac{\epsilon}{2} \leq \langle z_2, v\rangle$. We know that $\langle z_1, v\rangle = \sum_{\alpha\in\mathcal{A}} z_{1\alpha}(v_\alpha)$ (where $z_{1\alpha} := z \restriction X_\alpha^*$) so that there exists a finite subset $\mathcal{A}_0 \subset \mathcal{A}$ such that $|\sum_{\alpha\in\mathcal{A}\setminus\mathcal{A}_0} z_{1\alpha}(v_\alpha)| \leq \frac{\epsilon}{2}$. Thus, if $u = v - \sum_{\alpha\in\mathcal{A}_0} v_\alpha$, then $u \in B(X^*)$, $\langle z_2, u\rangle = \langle z_2, v\rangle$ and

$$|\langle z_1, u\rangle| = \Big|\sum_{\alpha\in\mathcal{A}} z_{1\alpha}(u_\alpha)\Big| = \Big|\sum_{\alpha\in\mathcal{A}\setminus\mathcal{A}_0} z_{1\alpha}(v_\alpha)\Big| \leq \frac{\epsilon}{2}.$$

Hence

$$\|z_2\| - \tfrac{\epsilon}{2} \leq \langle z_2, v\rangle = \langle z_2, u\rangle \leq \langle z_1 + z_2, u\rangle + \tfrac{\epsilon}{2} = \langle z, u\rangle + \tfrac{\epsilon}{2} \leq \|z\| + \tfrac{\epsilon}{2}.$$

As $\epsilon > 0$ is arbitrary, we get $\|z_2\| \leq \|z\|$ and so the direct sum $X^{**} = h(Y_0^*)\oplus Y_0^\perp$ is monotone.

Finally observe that the canonical copy $J(X)$ of X in X^{**} is inside $h(Y_0^*)$ although $J(X) \neq h(Y_0^*)$ in general.

3. 1-unconditional direct sums of WCG subspaces. Let us investigate the control inside its bidual of a Banach space which is a 1-unconditional direct sum of WCG subspaces. First, we need the following lemma.

Lemma 2. *Let X be a Banach space and K a w-compact subset of X^*. Given $z \in B(X^{**})$ and $\epsilon > 0$, there exists $x \in X$ such that $\|x\| \leq 1 + \epsilon$ and*

$$\forall k \in K,\ z(k) - \epsilon \leq x(k) \leq z(k) + \epsilon.$$

Proof. Without loss of generality, we suppose that K is convex and symmetric with respect to 0 (otherwise, pick $\overline{\text{co}}(K \cup -K)$ instead of K). Consider the Banach space $Z = X \oplus_1 \mathbb{R}$. Then $Z^* = X^* \oplus_\infty \mathbb{R}$ and $Z^{**} = X^{**} \oplus_1 \mathbb{R}$. Let $H_1 := \{(k, z(k) - \frac{\epsilon}{2}) : k \in K\}$ and $H_2 := \{(k, z(k) + \frac{\epsilon}{2}) : k \in K\}$ be two w-compact convex disjoint subsets of Z^* such that, if $H = H_2 - H_1$, then $H \subset Z^*$ is a w-compact convex subset (and so a w*-compact subset) of Z^* fulfilling that $H \cap \overset{\circ}{B}(0; \frac{\epsilon}{2}) = \emptyset$. Thus, if we pick $\rho > 0$ with $\frac{2}{2+\epsilon} \leq \rho < 1$, then $H \cap B(0; \frac{\rho\epsilon}{2}) = \emptyset$. By the Hahn-Banach Theorem there exists a vector $\varphi \in B(Z)$ such that $\langle h, \varphi \rangle \geq \frac{\rho\epsilon}{2}$, $\forall h \in H$. If $\varphi = x_0 + t_0$, with $x_0 \in X,\ t_0 \in \mathbb{R}$ and $\|\varphi\| = \|x_0\| + |t_0| \leq 1$, then for every $(k_1, z(k_1) - \frac{\epsilon}{2}) \in H_1$ and every $(k_2, z(k_2) + \frac{\epsilon}{2}) \in H_2$ we have

$$\varphi((k_2, z(k_2) + \tfrac{\epsilon}{2})) - \varphi((k_1, z(k_1) - \tfrac{\epsilon}{2})) \geq \frac{\rho\epsilon}{2}.$$

Thus

$$x_0(k_2) + t_0 z(k_2) + t_0\frac{\epsilon}{2} \geq x_0(k_1) + t_0 z(k_1) - t_0\frac{\epsilon}{2} + \frac{\rho\epsilon}{2}, \tag{3.1}$$

whence choosing $k_1 = k_2$ in (3.1), we get $t_0\epsilon \geq \frac{\rho\epsilon}{2}$, that is, $\frac{\rho}{2} \leq t_0 \leq 1$. So, $\|x_0\| \leq 1 - \frac{\rho}{2}$. Putting $k_1 = 0$ in (3.1) we get

$$\forall k \in K,\ x_0(k) + t_0 z(k) + t_0\frac{\epsilon}{2} \geq -t_0\frac{\epsilon}{2} + \frac{\rho\epsilon}{2}.$$

Thus

$$\forall k \in K,\ -\frac{1}{t_0}x_0(k) \leq z(k) + \frac{\epsilon}{2}\frac{2t_0 - \rho}{t_0} \leq z(k) + \epsilon.$$

On the other hand, putting $k_2 = 0$ in (3.1) we obtain

$$\forall k \in K,\ \frac{t_0}{2}\epsilon \geq x_0(k) + t_0 z(k) - t_0\frac{\epsilon}{2} + \frac{\rho\epsilon}{2}.$$

Thus

$$\forall k \in K,\ z(k) - \epsilon \leq z(k) - \frac{\epsilon}{2}\frac{2t_0 - \rho}{t_0} \leq -\frac{1}{t_0}x_0(k).$$

Therefore, if $x = -\frac{1}{t_0}x_0$, then x satisfies the statement of the Lemma. ∎

Proposition 3. *Let X be a Banach space, which is a 1-unconditional direct sum of a family $\{X_\alpha : \alpha \in \mathcal{A}\}$ of WCG Banach spaces, we say, $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$. Then*

(A) *X has 2-control inside the bidual X^{**}.*

(B) *If the spaces X_α are reflexive and $X := \sum_{\alpha\in\mathcal{A}} \oplus_{\ell_1} X_\alpha$ (that is, X is the direct ℓ_1-sum of the family $\{X_\alpha : \alpha \in \mathcal{A}\}$), then X has 1-control in the bidual X^{**}.*

Proof. We adopt the notation of the above paragraphs. So, let $Y_0 = \sum_{\alpha\in\mathcal{A}} \oplus X^*_\alpha$, $X^{**} = h(Y_0^*) \overset{m}{\oplus} Y_0^\perp$, etc. Observe that in the case (B) we have $Y_0 = \sum_{\alpha\in\mathcal{A}} \oplus_{c_0} X^*_\alpha$, that is, Y_0 is the direct c_0-sum of the subspaces $\{X^*_\alpha : \alpha \in \mathcal{A}\}$. Let K_α be a w-compact subset of X_α such that $0 \in K_\alpha$ and $X_\alpha = \overline{[K_\alpha]}$, $\alpha \in \mathcal{A}$. In the case (B) we pick $K_\alpha := B(X_\alpha)$. Suppose that there exist a w*-compact subset $K \subset B(X^{**})$ and some real numbers $a, b > 0$ such that

(1) $\hat{d}(\overline{\text{co}}^{w^*}(K), X) > b > 2a > 2\hat{d}(K, X) > 0$ in the case (A).
(2) $\hat{d}(\overline{\text{co}}^{w^*}(K), X) > b > a > \hat{d}(K, X) > 0$ in the case (B).

By Lemma 12 of [5] we have

FACT. *There exist* $\psi \in S(X^{***}) \cap X^{\perp}$ *and a* w^**-compact subset* $\emptyset \neq H \subset K$ *such that for every* w^**-open subset* V *of* X^{**} *with* $V \cap H \neq \emptyset$ *there exists* $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ *such that* $\langle \psi, \xi \rangle > b$.

Now we proceed step by step:

STEP 1. By the Fact there exists a vector $\xi_1 \in \overline{\mathrm{co}}^{w^*}(H)$ such that $\langle \psi, \xi_1 \rangle > b$. Since $B(X^*)$ is w*-dense in $B(X^{***})$, we can find a vector $x_1^* \in B(X^*)$ such that $\langle \xi_1, x_1^* \rangle > b$ and another vector $\eta_1 \in H$ so that $\langle \eta_1, x_1^* \rangle > b$. Let $\eta_1 = v_1 + w_1$ with $v_1 \in h(Y_0^*)$ and $w_1 \in Y_0^{\perp}$. Then $a > d(\eta_1, X) \geq d(\eta_1, h(Y_0^*)) = \|w_1\|$, whence

$$\langle v_1, x_1^* \rangle = \langle \eta_1, x_1^* \rangle - \langle w_1, x_1^* \rangle > b - a.$$

As $\langle v_1, x_1^* \rangle = \sum_{\alpha \in \mathcal{A}} v_{1\alpha}(x_{1\alpha}^*) > b - a$, we can find a finite subset $\mathcal{A}_1 \subset \mathcal{A}$ such that, if y_1 is the restriction of x_1^* to $\sum_{\alpha \in \mathcal{A}_1} \oplus X_\alpha$ (so $y_1 = \sum_{\alpha \in \mathcal{A}_1} x_{1\alpha}^* \in B(\sum_{\alpha \in \mathcal{A}_1} \oplus X_\alpha^*) \subset B(Y_0)$), then $\langle \eta_1, y_1 \rangle = \langle v_1, y_1 \rangle > b - a$.

STEP 2. Let $V_1 = \{u \in X^{**} : \langle u, y_1 \rangle > b - a\}$, which is a w*-open subset of X^{**} with $V_1 \cap H \neq \emptyset$, because $\eta_1 \in V_1 \cap H$. By the Fact there exists $\xi_2 \in \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$ with $\langle \psi, \xi_2 \rangle > b$. Let $0 < 2\epsilon_1 < 2^{-1} \wedge (\langle \psi, \xi_2 \rangle - b) \wedge (a(\hat{d}(K, X))^{-1} - 1)$. Consider in X^{**} the subset $L_1 := \{\xi_2\} \cup (\sum_{\alpha \in \mathcal{A}_1} K_\alpha)$. Clearly L_1 is a w-compact subset of X^{**}. Moreover, in the case (B), we have $B(\sum_{\alpha \in \mathcal{A}_1} \oplus_1 X_\alpha) \subset L_1$. By Lemma 2 there exists a vector $x_2^* \in X^*$ such that $\|x_2^*\| \leq 1 + \epsilon_1$ and

$$\forall k \in L_1,\ \langle \psi, k \rangle - \epsilon_1 < \langle k, x_2^* \rangle < \langle \psi, k \rangle + \epsilon_1.$$

In particular, $\langle \xi_2, x_2^* \rangle > b + \epsilon_1$ and $|\langle x_2^*, k \rangle| \leq \epsilon_1 \leq 2^{-2}$, $\forall k \in \sum_{\alpha \in \mathcal{A}_1} K_\alpha$. Since $\langle \xi_2, x_2^* \rangle > b + \epsilon_1$, we can choose $\eta_2 \in V_1 \cap H$ such that $\langle \eta_2, x_2^* \rangle > b + \epsilon_1$ and also $\langle \eta_2, y_1 \rangle > b - a$ because $\eta_2 \in V_1$. Let $\eta_2 = v_2 + w_2$ with $v_2 \in h(Y_0^*)$ and $w_2 \in Y_0^{\perp}$. Observe that $\|w_2\| = d(\eta_2, h(Y_0^*)) \leq d(\eta_2, X) \leq \hat{d}(K, X) < a$ and $|\langle w_2, x_2^* \rangle| \leq (1 + \epsilon_1)\hat{d}(K, X) \leq a$. Now we choose y_2 and $\mathcal{A}_2$ in the cases (A) and (B):

CASE A. We have

$$\langle v_2, x_2^* \rangle = \langle \eta_2, x_2^* \rangle - \langle w_2, x_2^* \rangle \geq \langle \eta_2, x_2^* \rangle - |\langle w_2, x_2^* \rangle| > b - a.$$

Thus, as $\langle v_2, x_2^* \rangle = \sum_{\alpha \in \mathcal{A}} \langle v_{2\alpha}, x_{2\alpha}^* \rangle > b - a$, we can find a finite subset $\mathcal{A}_2$ of $\mathcal{A}$ satisfying $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}$ such that, if y_2 is the restriction of x_2^* to $\sum_{\alpha \in \mathcal{A}_2} \oplus X_\alpha$ (so $y_2 = \sum_{\alpha \in \mathcal{A}_2} x_{2\alpha}^* \in \sum_{\alpha \in \mathcal{A}_2} \oplus X_\alpha^* \subset Y_0$ with $\|y_2\| \leq 1 + \epsilon_1$), then $\langle \eta_2, y_2 \rangle = \langle v_2, y_2 \rangle > b - a$. Observe that for every $k \in \bigcup_{\alpha \in \mathcal{A}_1} K_\alpha$ we have $\psi(k) = 0$, whence

$$|\langle y_2, k \rangle| = |\langle x_2^*, k \rangle| \leq \epsilon_1 \leq 2^{-2}.$$

CASE B. Let $\gamma_{21} := x_2^* \restriction \sum_{\alpha \in \mathcal{A}_1} \oplus_1 X_\alpha$ (that is, $\gamma_{21} = \sum_{\alpha \in \mathcal{A}_1} x_{2\alpha}^*$) and $\gamma_{22} = x_2^* - \gamma_{21}$. Since $|\langle x_2^*, k \rangle| \leq \epsilon_1$, $\forall k \in \sum_{\alpha \in \mathcal{A}_1} K_\alpha$, and $B(\sum_{\alpha \in \mathcal{A}_1} \oplus_1 X_\alpha) \subset \sum_{\alpha \in \mathcal{A}_1} K_\alpha$, then $\|\gamma_{21}\| \leq \epsilon_1$. So

$$\langle v_2, \gamma_{22} \rangle = \langle \eta_2, x_2^* \rangle - \langle w_2, x_2^* \rangle - \langle v_2, \gamma_{21} \rangle \geq \langle \eta_2, x_2^* \rangle - \epsilon_1 - a > b - a.$$

Since $\langle v_2, \gamma_{22} \rangle = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}_1} \langle v_{2\alpha}, x_{2\alpha}^* \rangle > b - a$, we can find a finite subset $\mathcal{A}_2 \subset \mathcal{A} \setminus \mathcal{A}_1$ such that, if y_2 is the restriction of x_2^* to $\sum_{\alpha \in \mathcal{A}_2} \oplus X_\alpha$ (so $y_2 = \sum_{\alpha \in \mathcal{A}_2} x_{2\alpha}^* \in \sum_{\alpha \in \mathcal{A}_2} \oplus X_\alpha^* \subset Y_0$ with $\|y_2\| \leq 1 + \epsilon_1$), then $\langle \eta_2, y_2 \rangle = \langle v_2, y_2 \rangle > b - a$.

Further we proceed by iteration. We obtain the sequences $\{y_k : k \geq 1\} \subset Y_0$, $\{\eta_k : k \geq 1\} \subset K$ and $\{\mathcal{A}_k : k \geq 1\}$, $\mathcal{A}_k \subset \mathcal{A}$, fulfilling the following conditions:

Case A. In this case we have:

(i) The finite subsets $\mathcal{A}_k$ of $\mathcal{A}$ satisfy $\mathcal{A}_k \subset \mathcal{A}_{k+1}$ for $k \geq 1$.

(ii) $y_k \in \sum_{\alpha\in\mathcal{A}_k} \oplus X^*_\alpha \subset Y_0$, $\|y_k\| \leq 1 + \epsilon_{k-1}$, $k \geq 2$, and $\langle \eta_j, y_k\rangle > b - a$ for $j \geq k$ with $j, k \in \mathbb{N}$.

(iii) For every $h \in \bigcup_{\alpha\in\mathcal{A}_k} K_\alpha$ we have $|\langle y_{k+1}, h\rangle| \leq 2^{-k-1}$, $\forall k \geq 1$.

Let $\mathcal{A}_0 := \cup_{n\geq 1}\mathcal{A}_n$, $X_0 := \sum_{\alpha\in\mathcal{A}_0} \oplus X_\alpha$ and let $P_0 : X \to X_0$ be the canonical projection on X_0, with norm $\|P_0\| = 1$. The space X admits the monotone decomposition

$$X = X_0 \overset{m}{\oplus} X_1 \text{ where } X_1 := \sum_{\alpha\in\mathcal{A}\setminus\mathcal{A}_0} X_\alpha.$$

Therefore we get the following monotone decompositions

$$X^* = X^*_0 \overset{m}{\oplus} X^*_1,\ X^{**} = X^{**}_0 \overset{m}{\oplus} X^{**}_1,\ X^{***} = X^{***}_0 \overset{m}{\oplus} X^{***}_1, \text{ etc.},$$

with projections $P_0 : X \to X_0$, $P^*_0 : X^* \to X^*_0$, $P^{**}_0 : X^{**} \to X^{**}_0$, $P^{***}_0 : X^{***} \to X^{***}_0$, etc. Observe that $P^*_0(y_k) = y_k$, $\forall k \geq 1$, that is, $y_k \in X^*_0 = P^*_0(X^*)$, $\forall k \geq 1$. Let η_0 be a w*-cluster point of the sequence $\{\eta_k : k \geq 1\}$ in X^{**}. Obviously $\eta_0 \in K$. Moreover, since $\langle \eta_j, y_k\rangle > b - a$, $\forall j \geq k$, we get $\langle \eta_0, y_k\rangle \geq b - a$, $\forall k \geq 1$. Let φ_0 be a w*-cluster point of $\{y_k : k \geq 1\}$ in X^{***}. Then

(i) $\varphi_0 \in B(X^{***})$. Actually φ_0 is in $P^{***}_0(X^{***}) = X^{***}_0$, that is, $P^{***}_0(\varphi_0) = \varphi_0$.

(ii) By construction $\varphi_{0\restriction K_\alpha} = 0$, $\forall \alpha \in \mathcal{A}_0$. Thus $\varphi_0 \in X^\perp_0$, because $\bigcup_{\alpha\in\mathcal{A}_0} K_\alpha$ generates X_0.

(iii) $\langle \varphi_0, \eta_0\rangle \geq b - a$ because $\langle \eta_0, y_k\rangle \geq b - a$, $\forall k \geq 1$.

Let $W := P^{**}_0(K) \subset B(X^{**}_0)$, which is a w*-compact subset of X^{**}_0, and $w_0 = P^{**}_0(\eta_0)$. Obviously $w_0 \in W$.

Claim 1. $d(w_0, X_0) < a$.

Indeed, let $x \in X$ be arbitrary. Then

$$d(w_0, X_0) \leq \|w_0 - P^{**}_0 x\| = \|P^{**}_0(\eta_0) - P^{**}_0 x\| \leq \|\eta_0 - x\|.$$

That is, $d(w_0, X_0) \leq d(\eta_0, X) \leq \hat{d}(K, X) < a$.

Claim 2. $d(w_0, X_0) \geq b - a$.

Indeed, as $\varphi_0 \in B(X^{***}) \cap X^\perp_0$ and

$$\langle \varphi_0, w_o\rangle = \langle \varphi_0, P^{**}_0\eta_0\rangle = \langle P^{***}_0\varphi_0, \eta_0\rangle = \langle \varphi_0, \eta_0\rangle \geq b - a,$$

we conclude that $d(w_0, X_0) \geq b - a$.

As $a < b - a$ we get a contradiction which proves the statement in the case (A).

Case B. In this case we have:

(i) The finite subsets $\mathcal{A}_k$, $k \geq 1$, of $\mathcal{A}$ are disjoint.

(ii) $y_k \in \sum_{\alpha\in\mathcal{A}_k} \oplus_0 X^*_\alpha \subset Y_0$, $\|y_k\| \leq 1 + \epsilon_{k-1}$, $k \geq 2$, and $\langle \eta_j, y_k\rangle > b - a$ for $j \geq k$ with $j, k \in \mathbb{N}$.

(iii) For every $n \in \mathbb{N}$ we have $\| \sum_{i=1}^{n} y_i \| \leq 2$.

Let η_0 be a w*-cluster point of the sequence $\{\eta_k : k \geq 1\}$ in X^{**}. Obviously $\eta_0 \in K$. Moreover, since $\langle \eta_j, y_k \rangle > b - a,\ \forall j \geq k$, we get $\langle \eta_0, y_k \rangle \geq b - a > 0,\ \forall k \geq 1$. Thus $\langle \eta_0, \sum_{i=1}^{n} y_i \rangle \geq n(b - a),\ \forall n \geq 1$. Since $\| \sum_{i=1}^{n} y_i \| \leq 2,\ \forall n \geq 1$, we get a contradiction which proves the statement (B). ■

PROPOSITION 4. *Let X be a Banach space, which is the 1-unconditional direct sum $X = \sum_{\alpha \in \mathcal{A}} \oplus X_\alpha$ of the family $\{X_\alpha : \alpha \in \mathcal{A}\}$ of WCG Banach spaces. If $K \subset X^{**}$ is a w^*-compact subset such that $K \cap X$ is w^*-dense in K, then $\hat{d}(\overline{\mathrm{co}}^{w^*}(K), X) = \hat{d}(K, X)$.*

Proof. The proof is analogous to the one of case (A) of Proposition 3. For reader's convenience we give the details of the proof. Suppose that there exists a w*-compact subset $K \subset B(X^{**})$ such that

$$\hat{d}(\overline{\mathrm{co}}^{w^*}(K), X) > b > a > \hat{d}(K, X) > 0.$$

By Lemma 12 of [5] we have:

FACT. *There exist $\psi \in S(X^{***}) \cap X^{\perp}$ and a w^*-compact subset $\emptyset \neq H \subset K$ such that for every w^*-open subset V of X^{**} with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ with $\langle \psi, \xi \rangle > b$.*

STEP 1. By the Fact there exists a vector $\xi_1 \in \overline{\mathrm{co}}^{w^*}(H)$ such that $\langle \psi, \xi_1 \rangle > b$. Since $B(X^*)$ is w*-dense in $B(X^{***})$, we can find a vector $x_1^* \in B(X^*)$ such that $\langle \xi_1, x_1^* \rangle > b$. Let $V_1 = \{u \in X^{**} : \langle u, x_1^* \rangle > b\}$, which is a w*-open subset of X^{**} such that $V_1 \cap H \neq \emptyset$ and so, for the sake of density, also $V_1 \cap K \cap X \neq \emptyset$. Thus there exists a vector $\eta_1 \in K \cap X$ such that $\langle \eta_1, x_1^* \rangle > b$. Since $\eta_1 \in X$, the support $\mathcal{A}_1 := \mathrm{supp}(\eta_1) = \{\alpha \in \mathcal{A} : \eta_{1\alpha} \neq 0\}$ of η_1 is countable, we say, $\mathcal{A}_1 = \{\alpha_{1n} : n \geq 1\}$.

STEP 2. As $V_1 \cap H \neq \emptyset$, by the Fact there exists a vector $\xi_2 \in \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$ such that $\langle \psi, \xi_2 \rangle > b$. Let $L_1 := \bigcup_{i,j=1}^{1} K_{\alpha_{ij}}$ and $\epsilon_1 := 2^{-1} \wedge (\psi(\xi_2) - b)$. As $L_1 \cup \{\xi_2\}$ is a w-compact subset of X^{**}, by Lemma 2 there exists a vector $x_2^* \in X^*$ such that $\|x_2^*\| \leq 1 + \epsilon_1$ and

$$\forall k \in L_1 \cup \{\xi_2\},\ \langle \psi, k \rangle - \epsilon_1 < \langle k, x_2^* \rangle < \langle \psi, k \rangle + \epsilon_1.$$

In particular $\langle \xi_2, x_2^* \rangle > b$ and $|\langle x_2^*, k \rangle| \leq 2^{-1},\ \forall k \in L_1$. As $\langle \xi_2, x_2^* \rangle > b$ and $\xi_2 \in \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$, if we put $W_2 := \{u \in X^{**} | \langle u, x_2^* \rangle > b\}$, then $W_2 \cap V_1 \cap H \neq \emptyset$ and, for the sake of density, also $W_2 \cap V_1 \cap K \cap X \neq \emptyset$. Denote $V_2 := W_2 \cap V_1$ and choose $\eta_2 \in V_2 \cap K \cap X$, which satisfies $x_i^*(\eta_2) > b,\ i = 1, 2$. Since $\eta_2 \in X$, the support $\mathcal{A}_2 := \mathrm{supp}(\eta_2) = \{\alpha \in \mathcal{A} : \eta_{2\alpha} \neq 0\}$ of η_2 is countable, we say, $\mathcal{A}_2 = \{\alpha_{2n} : n \geq 1\}$.

Further we proceed by iteration. We get the sequences $\{x_k^* : k \geq 1\} \subset X^*$, $\{\eta_k : k \geq 1\} \subset K \cap X$, $\{L_k : k \geq 1\}$ and $\{\mathcal{A}_k : k \geq 1\}$, such that

(a) $\mathcal{A}_k := \mathrm{supp}(\eta_k) = \{\alpha \in \mathcal{A} : \eta_{k\alpha} \neq 0\}$ is the support of η_k and is countable, say, $\mathcal{A}_k = \{\alpha_{kn} : n \geq 1\},\ \forall k \geq 1$.

(b) The subsets $L_k = \bigcup_{i,j=1}^{k} K_{\alpha_{ij}}$ are w-compact subsets of $B(X),\ \forall k \geq 1$.

(c) $\|x_{k+1}^*\| \leq 1 + \epsilon_k,\ \langle \eta_j, x_k^* \rangle > b,\ j \geq k \geq 1$, and for every $h \in L_k$ we have $|\langle x_{k+1}^*, h \rangle| \leq 2^{-k},\ \forall k \geq 1$.

Let $\mathcal{A}_0 := \bigcup_{n\geq 1}\mathcal{A}_n$, $X_0 := \sum_{\alpha\in\mathcal{A}_0}\oplus X_\alpha$ and let $P_0 : X \to X_0$ be the canonical projection, with norm $\|P_0\| = 1$. Observe that X_0 is WCG because it is a countable sum of WCG Banach spaces. The space X admits the monotone decomposition

$$X = X_0 \overset{m}{\oplus} X_1 \text{ where } X_1 := \sum_{\alpha\in\mathcal{A}\setminus\mathcal{A}_0} X_\alpha.$$

Thus we get the monotone decompositions

$$X^* = X_0^* \overset{m}{\oplus} X_1^*,\ X^{**} = X_0^{**} \overset{m}{\oplus} X_1^{**},\ X^{***} = X_0^{***} \overset{m}{\oplus} X_1^{***},\ \text{etc.},$$

with projections $P_0 : X \to X_0$, $P_0^* : X^* \to X_0^*$, $P_0^{**} : X^{**} \to X_0^{**}$, $P_0^{***} : X^{***} \to X_0^{***}$, etc. Observe that $P_0(\eta_k) = \eta_k \in X_0$, $\forall k \geq 1$. Let η_0 be a w*-cluster point of the sequence $\{\eta_k : k \geq 1\}$ in X^{**}. Obviously $\eta_0 \in X_0^{**} \cap K$ because X_0^{**} is w*-closed in X^{**} (actually $X_0^{**} = \overline{X_0}^{w^*}$) and $\{\eta_k : k \geq 1\} \subset X_0 \cap K$. Since $\langle x_i^*, \eta_k\rangle > b$, $\forall i \leq k$, we get $\langle \eta_0, x_i^*\rangle \geq b$, $\forall i \geq 1$, and also $\langle \eta_0, P_0^* x_i^*\rangle \geq b$, $\forall i \geq 1$, because $\langle \eta_0, P_0^* x_i^*\rangle = \langle P_0^{**}\eta_0, x_i^*\rangle = \langle \eta_0, x_i^*\rangle \geq b$. Let φ_0 be a w*-cluster point of $\{P_0^* x_k^* : k \geq 1\}$ in X^{***}. Then

(i) $\varphi_0 \in B(X^{***})$. Actually φ_0 is in $P_0^{***}(X^{***}) = X_0^{***}$, (that is, $P_0^{***}(\varphi_0) = \varphi_0$) because X_0^{***} is w*-closed in X^{***} and $\{P_0^* x_k^* : k \geq 1\} \subset X_0^* \subset X_0^{***}$.

(ii) Since for every $k \geq 1$ and every $h \in L_k$ we have

$$|\langle P_0^*(x_{k+1}^*), h\rangle| = |\langle x_{k+1}^*, P_0(h)\rangle| = |\langle x_{k+1}^*, h\rangle| \leq 2^{-k},$$

we get $\varphi_0 \in X_0^\perp$.

(iii) $\langle \varphi_0, \eta_0\rangle \geq b$ because $\langle \eta_0, P_0^*(x_k^*)\rangle \geq b$, $\forall k \geq 1$.

Let $W := P_0^{**}(K) \subset B(X_0^{**})$. W is clearly a w*-compact subset of X_0^{**}. Let $w_0 = P_0^{**}(\eta_0)$. Obviously $w_0 \in W$.

Claim 1. $d(w_0, X_0) < a$.

Indeed, first as $W = P_0^{**}(K)$, $X_0 = P_0^{**}(X)$ and $\|P_0^{**}\| = 1$, we have $\hat{d}(W, X_0) \leq \|P_0^{**}\|\hat{d}(K, X) < a$. Now it is enough to observe that $w_0 \in W$.

Claim 2. $d(w_0, X_0) \geq b$.

Indeed, since $\varphi_0 \in B(X^{***}) \cap X_0^\perp$ and

$$\langle \varphi_0, w_0\rangle = \langle \varphi_0, P_0^{**}\eta_0\rangle = \langle P_0^{***}\varphi_0, \eta_0\rangle = \langle \varphi_0, \eta_0\rangle \geq b,$$

we conclude that $d(w_0, X_0) \geq b$.

As $a < b$ we get a contradiction which proves the statement. ∎

Let X be a Banach space which admits the decomposition $X = \sum_{\alpha\in\mathcal{A}}\oplus X_\alpha$ as a 1-unconditional direct sum of closed subspaces X_α. We say that the decomposition $X = \sum_{\alpha\in\mathcal{A}}\oplus X_\alpha$ is of *countable type* if for every $u \in X^*$ the support $\operatorname{supp}(u) := \{\alpha \in \mathcal{A} : u_\alpha \neq 0\}$ of u is countable, $(u_\alpha)_{\alpha\in\mathcal{A}}$ being the set of coordinates of u, that is, $u_\alpha := u_{\restriction X_\alpha} = u \circ P_\alpha$, where $P_\alpha : X \to X_\alpha$ is the canonical projection. For instance, if I is an infinite set, $M : \mathbb{R} \to [0, +\infty]$ an Orlicz function such that its complementary Orlicz function M^* (see [2], [9, Chapter 4]) satisfies $M^*(t) > 0$ for $t > 0$, then the Orlicz space $h_M(I) := \{f \in \mathbb{R}^I : \sum_{i\in I} M(f_i/\lambda) < \infty, \forall\lambda > 0\}$ has countable decomposition (with respect to the canonical basis of $h_M(I)$), because every element of its dual $h_M(I)^* := \ell_{M^*}(I)$ has countable support.

LEMMA 5. *Let X be a Banach space which admits a decomposition $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ as a 1-unconditional direct sum of closed subspaces. The following statement are equivalent:*

(1) *The decomposition $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ is not of countable type.*

(2) *X has an isomorphic copy of $\ell_1(\aleph_1)$ disjointly disposed with respect to the decomposition $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$, that is, there exists a subset $\mathcal{A}_1 \subset \mathcal{A}$ with cardinality $|\mathcal{A}_1| = \aleph_1$ and for each $\alpha \in \mathcal{A}_1$ an element $v_\alpha \in X_\alpha$ so that the family $\{v_\alpha : \alpha \in \mathcal{A}_1\}$ is equivalent to the canonical basis of $\ell_1(\aleph_1)$.*

Proof. (1) $\Rightarrow$ (2). If the decomposition $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ is not of countable type, there exists some $u \in X^*$ such that the subset $\mathcal{A}_0 := \{\alpha \in \mathcal{A} : u_\alpha \neq 0\}$ satisfies $|\mathcal{A}_0| \geq \aleph_1$, where $u_\alpha := u_{\restriction X_\alpha} = u \circ P_\alpha$ and $P_\alpha : X \to X_\alpha$ is the canonical projection. By passing to a subset if necessary, we can find a real number $\epsilon > 0$, a subset $\mathcal{A}_1 \subset \mathcal{A}_0$ with $|\mathcal{A}_1| = \aleph_1$ and a family $\{v_\alpha : \alpha \in \mathcal{A}_1\}$ with $v_\alpha \in B(X_\alpha)$ so that $\langle u, v_\alpha\rangle = \langle u_\alpha, v_\alpha\rangle > \epsilon$. From this we can prove that the family $\{v_\alpha : \alpha \in \mathcal{A}_1\}$ is equivalent to the canonical basis of $\ell_1(\aleph_1)$ and generates a copy of $\ell_1(\aleph_1)$, which is disjointly disposed with respect to the decomposition $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$.

(2) $\Rightarrow$ (1). Let $\mathcal{A}_1 \subset \mathcal{A}$ be a subset with cardinality $|\mathcal{A}_1| = \aleph_1$ and for each $\alpha \in \mathcal{A}_1$ let v_α be an element of X_α so that the family $\{v_\alpha : \alpha \in \mathcal{A}_1\}$ is equivalent to the canonical basis $\{e_\alpha : \alpha \in \mathcal{A}_1\}$ of $\ell_1(\mathcal{A}_1)$. Let $T : \ell_1(\mathcal{A}_1) \to X$ be the isomorphism between $\ell_1(\mathcal{A}_1)$ and the closed subspace generated by $\{v_\alpha : \alpha \in \mathcal{A}_1\}$ so that $T(e_\alpha) = v_\alpha$. Since $T^* : X^* \to \ell_\infty(\mathcal{A}_1)$ is a quotient mapping and so $T^*(X^*) = \ell_\infty(\mathcal{A}_1)$, if $w_0 \in \ell_\infty(\mathcal{A}_1)$ is such that $w_0(\alpha) = 1$, $\forall \alpha \in \mathcal{A}_1$, there exists a vector $u \in X^*$ such that $T^*(u) = w_0$. Then for every $\alpha \in \mathcal{A}_1$ we have

$$\langle u, v_\alpha\rangle = \langle u, Te_\alpha\rangle = \langle T^*u, e_\alpha\rangle = \langle w_0, e_\alpha\rangle = 1,$$

and this proves that u is an element of X^* that does not have countable support with respect to the decomposition $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$. ■

PROPOSITION 6. *Let X be a Banach space that admits a decomposition of countable type $X = \sum_{i\in I} \oplus X_i$ as a 1-unconditional direct sum of WCG closed subspaces $\{X_i : i \in I\}$. Then X has 1-control in its bidual X^{**}.*

Proof. If I is a countable set, then X is a countable direct sum of WCG Banach subspaces and, so, it is WCG. Thus the statement follows in this case from [5, Cor. 4]. So, suppose in the sequel that I is uncountable. We assume that the statement is not true and we are going to get a contradiction. Let K be a w*-compact subset of X^{**} and $z_0 \in \overline{\text{co}}^{w^*}(K)$ a vector such that $d(z_0, X) > b > a > \hat{d}(K, X)$. Choose $\psi \in B(X^{***}) \cap X^\perp$ such that $\langle \psi, z_0\rangle > b$. We adopt the following notation. For each $i \in I$ let $K_i \subset X_i$ be a w-compact subset of X_i which generates X_i. If J is a subset of I, let $X(J)$ denote the subspace $X(J) := \sum_{i\in J} \oplus X_i$ (so $X = X(I)$) and let P_J denote the canonical projection $P_J : X \to X(J)$ with norm $\|P_J\| = 1$. We identify the subspace $P_J^*(X^*)$ of the dual X^* with $X(J)^*$.

STEP 1. Since $\langle \psi, z_0\rangle > b$, there exists $x_1^* \in B(X^*)$ such that $\langle z_0, x_1^*\rangle > b$ (because $B(X^*)$ is w*-dense in $B(X^{***})$). By hypothesis the support $\mathrm{supp}(x_1^*) = \{\alpha \in I : 0 \neq x_{1\alpha}^*\} := J_1$ of x_1^* is countable. Let $J_1 := \{\alpha_{1j} : j \geq 1\}$ and $I_1 := J_1$. Thus, if $P_{I_1} : X \to \sum_{i\in I_1} \oplus X_i$ is the corresponding canonical projection, then $P_{I_1}^*(x_1^*) = x_1^*$ (that is, $x_1^* \in X(I_1)^*$) and moreover

$$\langle P_{I_1}^{**}(z_0), x_1^*\rangle = \langle z_0, P_{I_1}^*(x_1^*)\rangle = \langle z_0, x_1^*\rangle > b.$$

STEP 2. Let $K_{\alpha_{11}}$ be the w-compact subset that generates $X_{\alpha_{11}}$, and put $L_2 := \{z_0\} \cup K_{\alpha_{11}}$, which is a w-compact subset of X^{**}. Let $\epsilon_2 = 2^{-2} \wedge (\langle \psi, z_0\rangle - b)$. By Lemma 2 there exists a vector $x_2^* \in X^*$ such that $\|x_2^*\| \leq 1 + \epsilon_2$ and

$$\forall k \in L_2,\ \langle \psi, k\rangle - \epsilon_2 < \langle k, x_2^*\rangle < \langle \psi, k\rangle + \epsilon_2.$$

In particular, $\langle z_0, x_2^*\rangle > b$ and $|\langle k, x_2^*\rangle| \leq 2^{-2},\ \forall k \in K_{\alpha_{11}}$. Let $J_2 := \mathrm{supp}(x_2^*)$ be the support of x_2^*, which is countable by hypothesis. Put $J_2 := \{\alpha_{2j} : j \geq 1\}$ and $I_2 := J_1 \cup J_2$. Then $P_{I_2}^*(x_i^*) = x_i^* \in X(I_2)^*,\ i = 1, 2$, and moreover

$$\langle P_{I_2}^{**}(z_0), x_i^*\rangle = \langle z_0, P_{I_2}^*(x_i^*)\rangle = \langle z_0, x_i^*\rangle > b,\ i = 1, 2.$$

STEP 3. Let $L_3 := \{z_0\} \cup (\bigcup\{K_{\alpha_{ij}} : 1 \leq i, j \leq 2\})$, which is a w-compact subset of X^{**}. Let $\epsilon_3 = 2^{-3} \wedge (\langle \psi, z_0\rangle - b)$. By Lemma 2 there exists a vector $x_3^* \in X^*$ such that $\|x_3^*\| \leq 1 + \epsilon_3$ and

$$\forall k \in L_3,\ \langle \psi, k\rangle - \epsilon_3 < \langle k, x_3^*\rangle < \langle \psi, k\rangle + \epsilon_3.$$

In particular, $\langle z_0, x_3^*\rangle > b$ and $|\langle k, x_3^*\rangle| \leq 2^{-3},\ \forall k \in K_{\alpha_{ij}},\ 1 \leq i, j \leq 2$. Let $J_3 := \mathrm{supp}(x_3^*)$ be the support of x_3^*, that is countable by hypothesis, and put $J_3 := \{\alpha_{3j} : j \geq 1\}$ and $I_3 := J_1 \cup J_2 \cup J_3$. Then $P_{I_3}^*(x_i^*) = x_i^* \in X(I_3)^*,\ i = 1, 2, 3$, and moreover

$$\langle P_{I_3}^{**}(z_0), x_i^*\rangle = \langle z_0, P_{I_3}^*(x_i^*)\rangle = \langle z_0, x_i^*\rangle > b,\ i = 1, 2, 3.$$

Further we proceed by iteration. Let $I_0 := \bigcup_{n\geq 1} I_n$. Observe that I_0 is a countable subset of I such that $P_{I_0}^*(x_i^*) = x_i^* \in X(I_0)^*,\ i \geq 1$. Let $\psi_0 \in B(X^{***})$ be a w*-cluster point of the sequence $\{x_n^* : n \geq 1\}$ in X^{***}. Then

(α) Since $X(I_0)^{***}$ is a w*-closed subset of X^{***} and $x_n^* \in B(X(I_0)^*) \subset B(X(I_0)^{***})$, $n \geq 1$, we get $\psi_0 \in B(X(I_0)^{***})$ and so $P_{I_0}^{***}(\psi_0) = \psi_0$.

(β) Since $|\langle u, x_{n+1}^*\rangle| \leq 2^{-n-1},\ \forall u \in K_{\alpha_{ij}},\ 1 \leq i, j \leq n$, and the subset $\bigcup_{i,j\geq 1} K_{\alpha_{ij}}$ generates the space $X(I_0)$, we conclude that $\psi_{0\restriction X(I_0)} \equiv 0$, that is, $\psi_0 \in B(X(I_0)^{\perp})$.

(γ) Since $\langle z_0, x_k^*\rangle > b,\ \forall k \geq 1$, then $\langle \psi_0, z_0\rangle \geq b$.

Let $H := P_{I_0}^{**}(K)$ and $w_0 := P_{I_0}^{**}(z_0)$. Clearly, H is a w*-compact subset of $X(I_0)^{**}$ (for the topology $\sigma(X(I_0)^{**}, X(I_0)^*)$ and $w_0 \in \overline{\mathrm{co}}^{w^*}(H)$.

CLAIM 1. $\hat{d}(H, X(I_0)) < a$ and $d(w_0, X(I_0)) < a$.

Indeed

$$\hat{d}(H, X(I_0)) = \hat{d}(P_{I_0}^{**}(K), P_{I_0}^{**}(X)) \leq \|P_{I_0}^{**}\|\hat{d}(K, X) < a.$$

By [5, Cor. 4] we have $d(w_0, X(I_0)) < a$, because $X(I_0)$ is a countable direct sum of WCG spaces and so it is WCG.

CLAIM 2. $d(w_0, X(I_0)) \geq b$.

Indeed, by (γ) we have

$$\langle \psi_0, w_0\rangle = \langle \psi_0, P^{**}_{I_0}(z_0)\rangle = \langle P^{***}_{I_0}(\psi_0), z_0\rangle = \langle \psi_0, z_0\rangle \geq b.$$

As $\psi_0 \in X(I_0)^\perp$ and $\|\psi_0\| \leq 1$, we get $d(w_0, X(I_0)) \geq b$.

Since $b > a$, we obtain a contradiction which proves the statement. ∎

PROPOSITION 7. *Let X be a Banach space that has a transfinite basis $\{e_\alpha : 1 \leq \alpha < \omega_1\}$ with constant 1 (that is, if $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n+m} < \omega_1$ and $(\lambda_i)_{i=1}^{n+m} \in \mathbb{R}^{n+m}$, then $\|\sum_{i=1}^n \lambda_i e_{\alpha_i}\| \leq \|\sum_{i=1}^{n+m} \lambda_i e_{\alpha_i}\|$) so that every $z \in X^*$ has countable support. Then X has 1-control in its bidual X^{**}.*

Proof. This proof is analogous to the one of Proposition 6. We suppose that the statement is not true and we are going to obtain a contradiction. Assume that there exist a w*-compact subset $K \subset X^{**}$ and a vector $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$ such that $d(z_0, X) > b > a > \hat{d}(K, X)$. Choose a vector $\psi \in B(X^{***}) \cap X^\perp$ such that $\langle \psi, z_0\rangle > b$. We adopt the following notation. If $\beta < \omega_1$, let $X(\beta)$ denote the subspace $X(\beta) := \sum_{i<\beta} \oplus [e_i]$ (so $X = X(\omega_1)$) and let P_β be the canonical projection $P_\beta : X \to X(\beta)$ with norm $\|P_\beta\| = 1$. We identify the subspace $P^*_\beta(X^*)$ of X^* with $X(\beta)^*$.

STEP 1. Since $\langle \psi, z_0\rangle > b$, there exists $x_1^* \in B(X^*)$ such that $\langle z_0, x_1^*\rangle > b$ (because $B(X^*)$ is w*-dense in $B(X^{***})$). By hypothesis the support of x_1^* is countable and so there exists $\beta_1 < \omega_1$ such that $\mathrm{supp}(x_1^*) \subset [1, \beta_1)$. Thus, $P^*_{\beta_1}(x_1^*) = x_1^*$ (that is, $x_1^* \in X(\beta_1)^*$) and moreover

$$\langle P^{**}_{\beta_1}(z_0), x_1^*\rangle = \langle z_0, P^*_{\beta_1}(x_1^*)\rangle = \langle z_0, x_1^*\rangle > b.$$

STEP 2. Let $[1, \beta_1) = \{\alpha_{1j} : j \geq 1\}$. Since $\langle \psi, z_0\rangle > b$ and $\langle \psi, e_{\alpha_{11}}\rangle = 0$, there exists $x_2^* \in B(X^*)$ such that $\langle z_0, x_2^*\rangle > b$ and $\langle x_2^*, e_{\alpha_{11}}\rangle = 0$. By hypothesis the support of x_2^* is countable and so there exists $\beta_1 < \beta_2 < \omega_1$ such that $\mathrm{supp}(x_2^*) \subset [1, \beta_2)$. Thus, for $i = 1, 2$, we have $P^*_{\beta_2}(x_i^*) = x_i^*$ (that is, $x_i^* \in X(\beta_2)^*$) and moreover

$$\langle P^{**}_{\beta_2}(z_0), x_i^*\rangle = \langle z_0, P^*_{\beta_2}(x_i^*)\rangle = \langle z_0, x_i^*\rangle > b.$$

STEP 3. Let $[1, \beta_2) = \{\alpha_{2j} : j \geq 1\}$. Since $\langle \psi, z_0\rangle > b$ and $\langle \psi, e_{\alpha_{ij}}\rangle = 0,\ i, j = 1, 2$, there exists $x_3^* \in B(X^*)$ such that $\langle z_0, x_3^*\rangle > b$ and $\langle x_3^*, e_{\alpha_{ij}}\rangle = 0,\ i, j = 1, 2$. By hypothesis the support of x_3^* is countable and so there exists $\beta_2 < \beta_3 < \omega_1$ such that $\mathrm{supp}(x_3^*) \subset [1, \beta_3)$. Thus, for $i = 1, 2, 3$, we have $P^*_{\beta_3}(x_i^*) = x_i^*$ (that is, $x_i^* \in X(\beta_3)^*$) and moreover

$$\langle P^{**}_{\beta_3}(z_0), x_i^*\rangle = \langle z_0, P^*_{\beta_3}(x_i^*)\rangle = \langle z_0, x_i^*\rangle > b.$$

Further we proceed by iteration. Let $\beta_0 := \sup\{\beta_n : n \geq 1\}$, that satisfies $\beta_0 < \omega_1$ and $P^*_{\beta_0}(x_i^*) = x_i^*$, and so $x_i^* \in X(\beta_0)^*,\ i \geq 1$. Let $\psi_0 \in B(X^{***})$ a w*-cluster point of the sequence $\{x_n^* : n \geq 1\}$ en X^{***}. Then

(α) Since $X(\beta_0)^{***}$ is a w*-closed subset of X^{***} and $x_n^* \in B(X(\beta_0)^*) \subset B(X(\beta_0)^{***})$, $n \geq 1$, we get $\psi_0 \in B(X(\beta_0)^{***})$ and so $P^{***}_{\beta_0}(\psi_0) = \psi_0$.

(β) Since $\langle x^*_{n+1}, e_{\alpha_{ij}}\rangle = 0,\ \forall 1 \leq i, j \leq n$, and the family $\{e_{\alpha_{ij}} : 1 \leq i, j\}$ generates the space $X(\beta_0)$, we conclude that $\psi_{0\restriction X(\beta_0)} \equiv 0$, that is, $\psi_0 \in B(X(\beta_0)^\perp)$.

(γ) Since $\langle z_0, x_k^*\rangle > b,\ \forall k \geq 1$, then $\langle \psi_0, z_0\rangle \geq b$.

Let $H := P^{**}_{\beta_0}(K)$ and $w_0 := P^{**}_{\beta_0}(z_0)$. Clearly, H is a w*-compact subset of $X(\beta_0)^{**}$ and $w_0 \in \overline{\mathrm{co}}^{w^*}(H)$.

Claim 1. *$\hat{d}(H, X(\beta_0)) < a$ and $d(w_0, X(\beta_0)) < a$.*

Indeed

$$\hat{d}(H, X(\beta_0)) = \hat{d}(P^{**}_{\beta_0}(K), P^{**}_{\beta_0}(X)) \le \|P^{**}_{\beta_0}\| \hat{d}(K, X) < a.$$

By [5, Cor. 4] we get $d(w_0, X(\beta_0)) < a$, because $X(\beta_0)$ is separable and so WCG.

Claim 2. *$d(w_0, X(\beta_0)) \ge b$.*

Indeed, by (γ) we have

$$\langle \psi_0, w_0 \rangle = \langle \psi_0, P^{**}_{\beta_0}(z_0) \rangle = \langle P^{***}_{\beta_0}(\psi_0), z_0 \rangle = \langle \psi_0, z_0 \rangle \ge b.$$

As $\psi_0 \in X(\beta_0)^{\perp}$ and $\|\psi_0\| \le 1$, we get $d(w_0, X(\beta_0)) \ge b$.

Since $b > a$, we obtain a contradiction, which proves the statement. ■

Proposition 8. *Let X be a Banach space which is the transfinite direct sum of the family $\{X_\alpha : 1 \le \alpha < \omega_1\}$ of WCG subspaces, $X = \sum_{\alpha<\omega_1} \oplus X_\alpha$, with constant 1 (that is, if $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{n+m} < \omega_1$, $x_{\alpha_i} \in X_{\alpha_i}$ and $(\lambda_i)_{i=1}^{n+m} \in \mathbb{R}^{n+m}$, then $\|\sum_{i=1}^{n} \lambda_i x_{\alpha_i}\| \le \|\sum_{i=1}^{n+m} \lambda_i x_{\alpha_i}\|$) so that every $z \in X^*$ has countable support. Then X has 1-control in its bidual X^{**}.*

Proof. The proof is analogous to the one of Proposition 7 by using the w-compact subsets $K_\alpha \subset X_\alpha$ that generate X_α, $\alpha < \omega_1$, and Lemma 2, as in Proposition 6. ■

4. Application to the order-continuous Banach lattices. First we show that, if X is an o-continuous Banach lattice, then X is a 1-unconditional direct sum of disjoint closed ideals which are WCG. This result is well known (see [1]) but we give its proof for the reader's convenience.

Lemma 9. *Let X be an o-continuous Banach lattice with a weak unit $e > 0$. Then X is WCG.*

Proof. It is well known (see [10, p. 28]) that the interval $[0, e] := \{x \in X : 0 \le x \le e\}$ is a w-compact subset of X. Let us see that $X = \overline{[[0, e]]}$, that is, X is the closure of the space generated by $[0, e]$. Pick a positive element $x \in X^+$. Then $ne \wedge x \uparrow x$ for $n \to \infty$, whence $\|x - ne \wedge x\| \downarrow 0$ because X is o-continuous. So $\bigcup_{n\ge1}[0, ne] = \bigcup_{n\ge1} n[0, e]$ is dense in the positive cone X^+. As $X = X^+ - X^+$, we conclude that X is the closure of the subspace generated by $[0, e]$. ■

Lemma 10. *If X is an o-continuous Banach lattice, then X is the 1-unconditional direct sum $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ of a family of closed ideals $\{X_\alpha : \alpha \in \mathcal{A}\}$ mutually disjoint, such that each X_α has weak unit and so it is WCG.*

Proof. By [10, 1.a.9] X admits the expression $X = \sum_{\alpha\in\mathcal{A}} \oplus X_\alpha$ as a direct sum of a family of closed ideals mutually disjoint $\{X_\alpha : \alpha \in \mathcal{A}\}$ (so as a 1-unconditional direct sum), such that each X_α has a weak unit. By Lemma 9 we get the statement. ■

PROPOSITION 11. *Let X be an o-continuous Banach lattice. If K is a w^*-compact subset of X^{**}, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) \le 2\hat{d}(K, X)$ and, if $K \cap X$ is w^*-dense in K, then $\hat{d}(\overline{\text{co}}^{w^*}(K), X) = \hat{d}(K, X)$.*

Proof. This result follows from Lemma 10, Proposition 3 and Proposition 4. ■

PROPOSITION 12. *Let X be an o-continuous Banach lattice that does not have a copy of $\ell_1(\aleph_1)$. Then X has 1-control in its bidual X^{**}.*

Proof. Clearly, if X is an o-continuous Banach lattice that does not have a copy of $\ell_1(\aleph_1)$, then X admits by Lemma 5 and Lemma 10 a decomposition of countable type $X = \sum_{\alpha \in \mathcal{A}} \oplus X_\alpha$ as a 1-unconditional direct sum of closed ideals X_α each having weak unit. So, this result follows from Proposition 6. ■

PROPOSITION 13. *Let X be a Banach space with a 1-unconditional basis $\{e_i : i \in I\}$ equivalent to the canonical basis of $\ell_1(I)$. Then X has 1-control in its bidual X^{**}.*

Proof. The proof is analogous to the one of part (B) of Proposition 3, putting $X_i = [e_i]$, $i \in I$, and taking into account that X^* and the subspace Y_0 of X^* (see the notation of Proposition 3) are canonically isomorphic to $\ell_\infty(I)$ and $c_0(I)$, respectively. ■

A Banach space X has a 1-symmetric basis $\{e_i : i \in I\}$ (see [12, p. 811]) whenever $\{e_i : i \in I\}$ is a 1-unconditional basis of X and, moreover, $\{e_i : i \in I\}$ is symmetric, that is, for any two sequences $\{i_n : n \ge 1\}$ and $\{j_n : n \ge 1\}$ of I, the basic sequences $\{e_{i_n} : n \ge 1\}$ and $\{e_{j_n} : n \ge 1\}$ are equivalent. Of course, the canonical bases of non-separable Orlicz spaces, Lorentz spaces, Orlicz-Lorentz spaces, etc., are 1-symmetric bases.

PROPOSITION 14. *Let X be a Banach space with a 1-symmetric basis. Then X has* 1-*control in its bidual X^{**}.*

Proof. CASE 1. Suppose that every element of the dual X^* has countable support. In this case the result follows from Proposition 6.

CASE 2. Suppose that there exists a vector $u \in B(X^*)$ whit uncountable support. By Proposition 13 it is enough to prove the following claim.

CLAIM. *If there exists a vector $u \in B(X^*)$ with uncountable support, then the* 1-*symmetric basis $\{e_i : i \in I\}$ of X is equivalent to the canonical basis of $\ell_1(I)$.*

Indeed, since $\text{supp}(u) := \{i \in I : u(e_i) \ne 0\}$ is uncountable, we can find a real number $\epsilon > 0$ and an uncountable subset $J \subset \text{supp}(u)$ such that $|u(e_i)| > \epsilon$, $\forall i \in J$. Let us prove that the family $\{e_i : i \in J\}$ is equivalent to the basis of $\ell_1(J)$. Suppose that the basis $\{e_i : i \in J\}$ is normalized and choose a vector of the form $\sum_{1 \le k \le n} \lambda_k e_{i_k}$, $i_k \in J$. Let $\epsilon_k = \pm 1$ so that $u(\lambda_k \epsilon_k e_{i_k}) = |\lambda_k u(e_{i_k})| \ge \epsilon |\lambda_k|$, $1 \le k \le n$. Then

$$\sum_{1 \le k \le n} |\lambda_k| \ge \Big\| \sum_{1 \le k \le n} \lambda_k e_{i_k} \Big\| = \Big\| \sum_{1 \le k \le n} \lambda_k \epsilon_k e_{i_k} \Big\|$$
$$\ge \Big| u\Big(\sum_{1 \le k \le n} \lambda_k \epsilon_k e_{i_k} \Big) \Big| \ge \epsilon \sum_{1 \le k \le n} |\lambda_k|,$$

and this implies that the family $\{e_i : i \in J\}$ is equivalent to the basis of $\ell_1(J)$. As the basis $\{e_i : i \in I\}$ of X is symmetric, finally we conclude that $\{e_i : i \in I\}$ is equivalent to the canonical basis of $\ell_1(I)$, and this proves the Claim and completes the proof of the Proposition. ■

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