THE EXTENSION OF THE KREIN-ŠMULIAN THEOREM FOR ORDER-CONTINUOUS BANACH LATTICES

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Abstract. If $X$ is a Banach space and $C \subset X$ a convex subset, for $x^{**} \in X^{**}$ and $A \subset X^{**}$ let $d(x^{**}, C) = \inf\{\|x^{**} - x\| : x \in C\}$ be the distance from $x^{**}$ to $C$ and $\hat{d}(A, C) = \sup\{d(a, C) : a \in A\}$. Among other things, we prove that if $X$ is an order-continuous Banach lattice and $K$ is a $w^*$-compact subset of $X^{**}$ we have: (i) $\hat{d}(\overline{co}^{w^*}(K), X) \leq 2\hat{d}(K, X)$ and, if $K \cap X$ is $w^*$-dense in $K$, then $\hat{d}(\overline{co}^{w^*}(K), X) = \hat{d}(K, X)$; (ii) if $X$ fails to have a copy of $\ell_1(\mathbb{N})$, then $\hat{d}(\overline{co}^{w^*}(K), X) = \hat{d}(K, X)$; (iii) if $X$ has a 1-symmetric basis, then $\hat{d}(\overline{co}^{w^*}(K), X) = \hat{d}(K, X)$.

1. Introduction. If $X$ is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of $X$, respectively, and $X^*$ its topological dual. The weak-topology of $X$ is denoted by $w$ and the weak*-topology of $X^*$ by $w^*$. If $C$ is a convex subset of $X^{**}$, for $x^{**} \in X^{**}$ and $A \subset X^{**}$, let $d(x^{**}, A) = \inf\{\|x^{**} - x\| : x \in A\}$ be the distance from $x^{**}$ to $C$ and $\hat{d}(A, C) = \sup\{d(a, C) : a \in A\}$. Observe that: (i) $\hat{d}(\overline{co}(A), C) = \hat{d}(co(A), C) = \hat{d}(A, C)$ where $co(A)$ is the convex hull of $A$; (ii) if $X = \{z \in X^{**} : z(x) = 0, \forall x \in X\}$ and $Q : X^{**} \to X^{**}$ is the canonical quotient mapping, then:

$$d(x^{**}, X) = \sup\{z(x^{**}) : z \in S(X^\perp)\} = \|Qx^{**}\|.$$  

With this terminology, the Krein-Šmulian Theorem (see [3, p. 51]) states the following: if $X$ is a Banach space and $K \subset X^{**}$ a $w^*$-compact subset such that $\hat{d}(K, X) = 0$ (thus, $K$ is a weakly compact subset of $X$), then $\hat{d}((\overline{co})^{w^*}(K), X) = 0$, that is, $\overline{co}^{w^*}(K) \subset X$ and $\overline{co}^{w^*}(K) = \overline{co}(K)$ is also a weakly compact subset of $X$ ($\overline{co}(K) = \|\cdot\|$-closure of $co(K)$ and $\overline{co}^{w^*}(K) = w^*$-closure of $\overline{co}(K)$). So, in view of this situation, we can pose two natural questions:

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(A) If $K \subset X^{**}$ is a $w^*$-compact subset, does the equality $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X)$ always hold?

The answer to this question is negative. In fact, we constructed (see [5], [7]) a Banach space $X$ such that: (i) there exists a $w^*$-compact subset $H \subset B(X^{**})$ such that $H \cap X$ is $w^*$-dense in $H$, $\hat{d}(H, X) = \frac{1}{2}$ and $\hat{d}(\overline{w}^*(H), X) = 1$; (ii) there exists a $w^*$-compact subset $K \subset B(X^{**})$ such that $\hat{d}(K, X) = \frac{1}{3}$ and $\hat{d}(\overline{w}^*(K), X) = 1$.

(B) Does there exist a universal constant $1 \leq M < \infty$ such that always $X$ has $M$-control inside its bidual $X^{**}$, that is, $\hat{d}(\overline{w}^*(K), X) \leq M \hat{d}(K, X)$ for every $w^*$-compact subset of $X^{**}$?

The answer to this question is affirmative. In [5] we proved the following result, which extends the Krein–Šmulian Theorem: if $K \subset X^{**}$ is a $w^*$-compact subset and $Z \subset X$ a subspace of $X$, then $\hat{d}(\overline{w}^*(K), Z) \leq 5\hat{d}(K, Z)$ and, if $Z \cap K$ is $w^*$-dense in $K$, then $\hat{d}(\overline{w}^*(K), Z) \leq 2\hat{d}(K, Z)$. So, in view of these results we have: (i) the universal constant $M$ of our extension of the Krein–Šmulian Theorem satisfies $3 \leq M \leq 5$; (ii) for the category of $w^*$-compact subsets $K \subset X^{**}$ such that $X \cap K$ is $w^*$-dense in $K$, the constant $M$ is exactly 2.

Although the answer to question (A) is, in general, negative there are many Banach spaces $X$ for which $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X)$. This is the case (see [5]), for instance, if $\ell_1 \not\subset X^*$, if the unit ball $B(X^*)$ of the dual $X^*$ is $w^*$-angelic (for example, if $X$ is weakly compactly generated (WCG) or weakly Lindelöf determined (WLD)), if $X = \ell_1(I)$, if $K$ is fragmented by the norm of $X^{**}$, etc.

In order to find classes of Banach space with 1-control in its bidual or, at least, a better control than in the above general case, we examine in this paper the class of Banach lattices. First, we have the following remarks:

(a) In [7, Prop. 10] we have constructed a Banach lattice $X$ and a $w^*$-compact subset $K \subset B(X^{**})$ such that $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X) = \frac{1}{3}$ and $\hat{d}(\overline{w}^*(K), X) = 1$. So, concerning the control inside the bidual, the class of Banach lattices behaves as in the general case.

(b) In [6] we proved that if $\varphi$ is an Orlicz function, $I$ an infinite set and $X = \ell_\varphi(I)$ the corresponding Orlicz space, equipped either with the Luxemburg or the Orlicz norm, then for every $w^*$-compact subset $K \subset X^{**}$ we have $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X)$ if and only if $\varphi$ satisfies the $\Delta_2$ condition at 0, that is, if and only if $\ell_\varphi(I)$ has a 1-symmetric basis.

In view of these results it is natural to ask the following questions. If $X$ is a Banach space with a 1-symmetric basis, does $X$ have 1-control inside $X^{**}$? What happens if $X$ is an order-continuous Banach lattice? The purpose of this note is to consider these problems. Among other things, we prove that if $X$ is an order-continuous Banach lattice and $K$ is a $w^*$-compact subset of $X^{**}$ then: (i) $\hat{d}(\overline{w}^*(K), X) \leq 2\hat{d}(K, X)$ and, if $K \cap X$ is $w^*$-dense in $K$, then $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X)$; (ii) if $X$ fails to have a copy of $\ell_1(\mathbb{N}_1)$, then $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X)$; (iii) if $X$ has a 1-symmetric basis, then $\hat{d}(\overline{w}^*(K), X) = \hat{d}(K, X)$.

Our notation is standard as in the books [9],[10]. $|A|$ denotes the cardinality of a set $A$, $\omega_1$ the first uncountable ordinal and $\mathbb{N}_1 = |\omega_1|$. Let $X$ be a Banach space. If $A$ is a subset
of \( X \), then \([A]\) is the subspace generated by \( A \) and \([A]\) the closure of \([A]\). The Banach space \( X \) is said to be weakly compactly generated (for short, WCG) if there exists a w-compact subset \( W \) of \( X \) such that \( X = [W] \). The Banach space \( X \) is said to have \( M \)-control inside its bidual \( X^{**} \) if \( \hat{d}(\co^w(K), X) \leq M \hat{d}(K, X) \) for every w*-compact subset \( K \) of \( X^{**} \). The notions (countable or uncountable) of 1-unconditional decomposition, 1-unconditional basis, 1-symmetric basis and transfinite basis can be found in [9] and [12]. Concerning the notions of Banach lattices, order-continuity (for short, o-continuity), etc., let us refer the reader to the books [10] and [11].

2. The structure of \( X, X^* \) and \( X^{**} \) when \( X = \sum_{\alpha \in A} \oplus X_\alpha \). Let us consider the structure of \( X, X^* \) and \( X^{**} \) when \( X \) is a 1-unconditional direct sum \( X = \sum_{\alpha \in A} \oplus X_\alpha \) of a family of Banach subspaces \( \{X_\alpha : \alpha \in A\} \) of \( X \).

**Definition 1.** A Banach space \( X \) is said to be a 1-unconditional direct sum of a family of Banach subspaces \( \{X_\alpha : \alpha \in A\} \) of \( X \), for short, \( X = \sum_{\alpha \in A} X_\alpha \) 1-unconditional, when \( X = [\bigcup_{\alpha \in A} X_\alpha] \) and, if \( x_\alpha \in X_\alpha, \epsilon_\alpha = \pm 1, \alpha \in A \), and \( A \) is a finite subset of \( A \), then \( \| \sum_{\alpha \in A} \epsilon_\alpha x_\alpha \| \leq \| \sum_{\alpha \in A} x_\alpha \| \).

If \( X = \sum_{\alpha \in A} \oplus X_\alpha \) is a 1-unconditional direct sum, then

(i) For each subset \( A \subset A \) there exists a projection \( P_A : X \to X \) such that \( \| P_A \| = 1 \) and \( P_A(X) = \sum_{\alpha \in A} \oplus X_\alpha \).

(ii) Every \( x \in X \) has a unique representation of the form \( x = \sum_{\alpha \in A} x_\alpha \) with \( x_\alpha \in X_\alpha \) such that the subset \( \{\alpha \in A : x_\alpha \neq 0\} \) is countable, the above series converges unconditionally and \( \| \sum_{\alpha \in A} \epsilon_\alpha x_\alpha \| = \| x \| \), where \( \epsilon_\alpha = \pm 1, \forall \alpha \in A \).

(iii) If \( u \in X^* \), the \( \alpha \)-th coordinate \( u_\alpha \) of \( u \) will be the restriction \( u_\alpha := u \mid X_\alpha \in X_\alpha^* \) of \( u \) to \( X_\alpha \). We will identify \( u \) with the family \( (u_\alpha)_{\alpha \in A} \) of its coordinates.

We consider each dual \( X_\alpha^* \) canonically and isometrically embedded into \( X^* \) as follows. If \( P_\alpha : X \to X_\alpha \) is the projection associated to \( X_\alpha \), then \( P_\alpha^*(X_\alpha^*) \) is a subspace of \( X^* \) isometric to \( X_\alpha^* \). We identify \( X_\alpha^* \) with \( P_\alpha^*(X_\alpha^*) \). Consider in \( X^* \) the closed subspace \( Y_0 := [\bigcup_{\alpha \in A} X_\alpha^*] \), which is actually the 1-unconditional direct sum of the closed subspaces \( \{X_\alpha^* : \alpha \in A\} \), that is, \( Y_0 = \sum_{\alpha \in A} \oplus X_\alpha^* \) 1-unconditional. Let \( Y_0^* \) be the dual of \( Y_0 \).

**Fact 1.** \( Y_0^* \) can be embedded canonically and isometrically into \( X^{**} \).

Indeed, if \( z \in Y_0^* \), for each \( \alpha \in A \) let \( z_\alpha := z \mid X_\alpha^* \) be the \( \alpha \)-th coordinate of \( z \) and identify \( z \) with the family \( (z_\alpha)_{\alpha \in A} \) of its coordinates. In order to embed \( Y_0^* \) into \( X^{**} \), define the mapping \( h : Y_0^* \to X^{**} \) as follows:

\[ \forall z \in Y_0^*, \forall u \in X^*, \ h(z)(u) = \sum_{\alpha \in A} z_\alpha(u_\alpha). \]

Concerning the definition of \( h \) we have to check several things, namely:

(\(\text{A}\)) First, we need to be sure that the series \( \sum_{\alpha \in A} z_\alpha(u_\alpha) \) converges. If \( A_0 \subset A \) is a finite subset, then \( \sum_{\alpha \in A_0} u_\alpha \in Y_0 \) and, if \( \epsilon_\alpha = \pm 1, \forall \alpha \in A \), then

\[ \sum_{\alpha \in A_0} z_\alpha(\epsilon_\alpha u_\alpha) = z \left( \sum_{\alpha \in A_0} \epsilon_\alpha u_\alpha \right) \leq \| z \| \cdot \left\| \sum_{\alpha \in A_0} \epsilon_\alpha u_\alpha \right\| \leq \| z \| \cdot \| u \|. \]
Thus we get: (i) \(|\{\alpha \in \mathcal{A} : z_\alpha(u_\alpha) \neq 0\}| \leq N_0\); (ii) the series \(\sum_{\alpha \in \mathcal{A}} z_\alpha(u_\alpha)\) converges absolutely and, moreover, \(\sum_{\alpha \in \mathcal{A}} z_\alpha(u_\alpha) \leq \|z\| \cdot \|u\|\). Therefore \(h(z) \in X^{**}\) with \(\|h(z)\| \leq \|z\|, \forall z \in Y_0^*,\) and \(h\) is a continuous linear mapping such that \(\|h\| \leq 1\).

(B) Let us see that \(h\) is an isometry. As \(h(z) \mid Y_0 = z, \forall z \in Y_0^*,\) we have

\[
\|h(z)\| = \sup\{\langle h(z), u \rangle : u \in B(X^*)\} \geq \sup\{\langle z, u \rangle : u \in B(Y_0)\} = \|z\|.
\]

On the other hand, \(\|h(z)\| \leq \|z\|\). So, \(h\) is an isometry.

We know that the subspace \(Y_0^\perp := \{z \in X^{**} : \langle z, y \rangle = 0, \forall y \in Y_0\}\) of \(X^{**}\) is isometrically isomorphic to the dual \((X^*)^*\).

**Fact 2.** \(X^{**} = h(Y_0^*) \oplus Y_0^\perp,\) that is, \(X^{**}\) is the monotone direct sum of \(h(Y_0^*)\) and \(Y_0^\perp\).

Observe that this means that every \(z \in X^{**}\) has a unique decomposition \(z = z_1 + z_2\) with \(z_1 \in h(Y_0^*)\) and \(z_2 \in Y_0^\perp\) so that \(\|z\| \geq \|z_1\| \vee \|z_2\|\).

Indeed, it is clear that \((Y_0^*)^0 \cap Y_0^\perp = \{0\}\). Let \(z \in X^{**}\) and put \(w_1 := z \mid Y_0\). Let us see that \(z - h(w_1) \in Y_0^\perp\). For every \(\alpha \in \mathcal{A}\) and every \(v \in X_\alpha^*\) we have

\[
\langle z - h(w_1), v \rangle = \langle z, v \rangle - \langle h(w_1), v \rangle
\]

\[= \langle z \mid X_\alpha^*, v \rangle - \langle w_1, v \rangle = \langle z \mid X_\alpha^*, v \rangle - \langle z \mid X_\alpha^*, v \rangle = 0.
\]

Thus, \(X^{**} = h(Y_0^*) \oplus Y_0^\perp\). Moreover, the above direct sum is monotone because, if \(z = z_1 + z_2 \in X^{**}\) with \(z_1 \in h(Y_0^*)\) and \(z_2 \in Y_0^\perp\), we have on the one hand

\[
\|z\| \geq \sup\{\langle z_1 + z_2, u \rangle : u \in B(Y_0)\} = \sup\{\langle z_1, u \rangle : u \in B(Y_0)\} = \|z_1\|.
\]

On the other hand, given \(\epsilon > 0\), choose \(v \in B(X^*)\) such that \(\|z_2\| - \frac{\epsilon}{2} \leq \langle z_2, v \rangle\). We know that \(\langle z_1, v \rangle = \sum_{\alpha \in \mathcal{A}} z_1\alpha(v_\alpha)\) (where \(z_1\alpha := z \mid X_\alpha^*\)) so that there exists a finite subset \(\mathcal{A}_0 \subset \mathcal{A}\) such that \(\sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}_0} z_1\alpha(v_\alpha) \leq \frac{\epsilon}{2}\). Thus, if \(u = v - \sum_{\alpha \in \mathcal{A}_0} v_\alpha\), then \(u \in B(X^*)\), \(\langle z_2, u \rangle = \langle z_2, v \rangle\) and

\[
\|\langle z_1, u \rangle\| = \left| \sum_{\alpha \in \mathcal{A}} z_1\alpha(u_\alpha) \right| = \left| \sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}_0} z_1\alpha(v_\alpha) \right| \leq \frac{\epsilon}{2}.
\]

Hence

\[
\|z_2\| - \frac{\epsilon}{2} \leq \langle z_2, v \rangle = \langle z_2, u \rangle \leq \langle z_1 + z_2, u \rangle + \frac{\epsilon}{2} = \langle z, u \rangle + \frac{\epsilon}{2} \leq \|z\| + \frac{\epsilon}{2}.
\]

As \(\epsilon > 0\) is arbitrary, we get \(\|z_2\| \leq \|z\|\) and so the direct sum \(X^{**} = h(Y_0^*) \oplus Y_0^\perp\) is monotone.

Finally observe that the canonical copy \(J(X)\) of \(X\) in \(X^{**}\) is inside \(h(Y_0^*)\) although \(J(X) \neq h(Y_0^*)\) in general.

### 3. 1-unconditional direct sums of WCG subspaces

Let us investigate the control inside its bidual of a Banach space which is a 1-unconditional direct sum of WCG subspaces. First, we need the following lemma.

**Lemma 2.** Let \(X\) be a Banach space and \(K\) a w-compact subset of \(X^*\). Given \(z \in B(X^{**})\) and \(\epsilon > 0\), there exists \(x \in X\) such that \(\|x\| \leq 1 + \epsilon\) and

\[
\forall k \in K, \ z(k) - \epsilon \leq x(k) \leq z(k) + \epsilon.
\]
The Extension of the Krein-Šmulian Theorem

Proof. Without loss of generality, we suppose that $K$ is convex and symmetric with respect to 0 (otherwise, pick $\overline{\co(K \cup -K)}$ instead of $K$). Consider the Banach space $Z = X \oplus_1 \mathbb{R}$. Then $Z^* = X^* \oplus_\infty \mathbb{R}$ and $Z^{**} = X^{**} \oplus_1 \mathbb{R}$. Let $H_1 := \{(k, z(k) - \frac{\epsilon}{2}) : k \in K \}$ and $H_2 := \{(k, z(k) + \frac{\epsilon}{2}) : k \in K \}$ be two w-compact convex disjoint subsets of $Z^*$ such that, if $H = H_2 - H_1$, then $H \subset Z^*$ is a w-compact convex subset (and so a $w^*$- compact subset) of $Z^*$ fulfilling that $H \cap \overline{B(0; \frac{\epsilon}{2})} = \emptyset$. Thus, if we pick $\rho > 0$ with $\frac{\rho}{2} \leq \rho < 1$, then $H \cap B(0; \frac{\rho}{2}) = \emptyset$. By the Hahn-Banach Theorem there exists a vector $\varphi \in B(Z)$ such that $\langle h, \varphi \rangle \geq \frac{\rho \epsilon}{2}$, $\forall h \in H$. If $\varphi = x_0 + t_0$, with $x_0 \in X$, $t_0 \in \mathbb{R}$ and $\|\varphi\| = \|x_0\| + |t_0| \leq 1$, then for every $(k_1, z(k_1) - \frac{\epsilon}{2}) \in H_1$ and every $(k_2, z(k_2) + \frac{\epsilon}{2}) \in H_2$ we have

$$\varphi((k_2, z(k_2) + \frac{\epsilon}{2})) - \varphi((k_1, z(k_1) - \frac{\epsilon}{2})) \geq \frac{\rho \epsilon}{2}.$$ 

Thus

$$x_0(k_2) + t_0 z(k_2) + t_0 \frac{\epsilon}{2} \geq x_0(k_1) + t_0 z(k_1) - t_0 \frac{\epsilon}{2} + \frac{\rho \epsilon}{2},$$

whence choosing $k_1 = k_2$ in (3.1), we get $t_0 \epsilon \geq \frac{\rho \epsilon}{2}$, that is, $\frac{\rho \epsilon}{2} \leq t_0 \leq 1$. So, $\|x_0\| \leq 1 - \frac{\rho \epsilon}{2}$.

Putting $k_1 = 0$ in (3.1) we get

$$\forall k \in K, \ x_0(k) + t_0 z(k) + t_0 \frac{\epsilon}{2} \geq -t_0 \frac{\epsilon}{2} + \frac{\rho \epsilon}{2}.$$ 

Thus

$$\forall k \in K, \ -\frac{1}{t_0} x_0(k) \leq z(k) + \frac{\epsilon}{2} \frac{2t_0 - \rho}{t_0} \leq z(k) + \epsilon.$$ 

On the other hand, putting $k_2 = 0$ in (3.1) we obtain

$$\forall k \in K, \ t_0 \frac{\epsilon}{2} \geq x_0(k) + t_0 z(k) - t_0 \frac{\epsilon}{2} + \frac{\rho \epsilon}{2}.$$ 

Thus

$$\forall k \in K, \ z(k) - \epsilon \leq z(k) - \frac{\epsilon}{2} \frac{2t_0 - \rho}{t_0} \leq -\frac{1}{t_0} x_0(k).$$ 

Therefore, if $x = -\frac{1}{t_0} x_0$, then $x$ satisfies the statement of the Lemma. $\blacksquare$

Proposition 3. Let $X$ be a Banach space, which is a 1-unconditional direct sum of a family $\{X_\alpha : \alpha \in A\}$ of WCG Banach spaces, we say, $X = \sum_{\alpha \in A} X_\alpha$. Then

(A) $X$ has 2-control inside the bidual $X^{**}$.

(B) If the spaces $X_\alpha$ are reflexive and $X := \sum_{\alpha \in A} \oplus_{\ell_1} X_\alpha$ (that is, $X$ is the direct $\ell_1$-sum of the family $\{X_\alpha : \alpha \in A\}$), then $X$ has 1-control in the bidual $X^{**}$.

Proof. We adopt the notation of the above paragraphs. So, let $Y_0 = \sum_{\alpha \in A} \oplus_{c_0} X_\alpha$, $X^{**} = h(Y^{**}_0) \oplus Y_0^{**}$, etc. Observe that in the case (B) we have $Y_0 = \sum_{\alpha \in A} \oplus_{c_0} X_\alpha$, that is, $Y_0$ is the direct $c_0$-sum of the subspaces $\{X_\alpha^* : \alpha \in A\}$. Let $K_\alpha$ be a w-compact subset of $X_\alpha$ such that $0 \in K_\alpha$ and $X_\alpha = [K_\alpha]$, $\alpha \in A$. In the case (B) we pick $K_\alpha := B(X_\alpha)$. Suppose that there exist a $\omega^*$-compact subset $K \subset B(X^{**})$ and some real numbers $a, b > 0$ such that

1. $\hat{d}(\overline{C} \omega^*(K), X) > b > 2a > \hat{d}(K, X) > 0$ in the case (A).
2. $\hat{d}(\overline{C} \omega^*(K), X) > b > a > \hat{d}(K, X) > 0$ in the case (B).
By Lemma 12 of [5] we have

**FACT.** There exist \( \psi \in S^{(X^{**})} \cap X^\perp \) and a \( w^* \)-compact subset \( \emptyset \neq H \subset K \) such that for every \( w^* \)-open subset \( V \) of \( X^{**} \) with \( V \cap H \neq \emptyset \) there exists \( \xi \in \overline{co}^{w^*}(V \cap H) \) such that \( \langle \psi, \xi \rangle > b \).

Now we proceed step by step:

**STEP 1.** By the Fact there exists a vector \( \xi_1 \in \overline{co}^{w^*}(H) \) such that \( \langle \psi, \xi_1 \rangle > b \). Since \( B(X^*) \) is \( w^* \)-dense in \( B(X^{**}) \), we can find a vector \( x_1^* \in B(X^*) \) such that \( \langle \xi_1, x_1^* \rangle > b \) and another vector \( \eta_1 \in H \) so that \( \langle \eta_1, x_1^* \rangle > b \). Let \( \eta_1 = v_1 + w_1 \) with \( v_1 \in h(Y_0^*) \) and \( w_1 \in Y_0^\perp \). Then \( a > d(\eta_1, X) \geq d(\eta_1, h(Y_0^*)) = \|w_1\|, \) whence
\[
\langle v_1, x_1^* \rangle = \langle \eta_1, x_1^* \rangle - \langle w_1, x_1^* \rangle > b - a.
\]

As \( \langle v_1, x_1^* \rangle = \sum_{\alpha \in A} v_{1\alpha}(x_{1\alpha}^*) > b - a \), we can find a finite subset \( A_1 \subset A \) such that, if \( y_1 \) is the restriction of \( x_1^* \) to \( \sum_{\alpha \in A_1} X_\alpha \) (so \( y_1 = \sum_{\alpha \in A_1} x_{1\alpha}^* \in B(\sum_{\alpha \in A_1} \oplus X_\alpha) \subset B(Y_0^*) \)), then \( \langle \eta_1, y_1 \rangle = \langle v_1, y_1 \rangle > b - a \).

**STEP 2.** Let \( V_1 = \{ u \in X^{**} : \langle u, y_1 \rangle > b - a \} \), which is a \( w^* \)-open subset of \( X^{**} \) with \( V_1 \cap H \neq \emptyset \), because \( \eta_1 \in V_1 \cap H \). By the Fact there exists \( \xi_2 \in \overline{co}^{w^*}(V_1 \cap H) \) with \( \langle \psi, \xi_2 \rangle > b \). Let \( 0 < 2\epsilon_1 < 2^{-1} \wedge (\langle \psi, \xi_2 \rangle - b) \wedge (a(d(K, X))^{-1} - 1) \). Consider in \( X^{**} \) the subset \( L_1 := \{ \xi_2 \} \cup (\sum_{\alpha \in A_1} K_\alpha) \). Clearly \( L_1 \) is a \( w^* \)-compact subset of \( X^{**} \). Moreover, in the case (B), we have \( B(\sum_{\alpha \in A_1} \oplus X_\alpha) \subset L_1 \). By Lemma 2 there exists a vector \( x_2^* \in X^* \) such that \( \|x_2^*\| \leq 1 + \epsilon_1 \) and
\[
\forall k \in L_1, \langle \psi, k \rangle - \epsilon_1 < \langle k, x^*_2 \rangle < \langle \psi, k \rangle + \epsilon_1.
\]

In particular, \( \langle \xi_2, x^*_2 \rangle > b + \epsilon_1 \) and \( \|x^*_2, k\| \leq \epsilon_1 \leq 2^{-2}, \forall k \in \sum_{\alpha \in A_1} K_\alpha \). Since \( \langle \xi_2, x^*_2 \rangle > b + \epsilon_1 \), we can choose \( \eta_2 \in V_1 \cap H \) such that \( \langle \eta_2, x^*_2 \rangle > b + \epsilon_1 \) and also \( \langle \eta_2, y_1 \rangle > b - a \) because \( \eta_2 \in V_1 \). Let \( \eta_2 = v_2 + w_2 \) with \( v_2 \in h(Y_0^*) \) and \( w_2 \in Y_0^\perp \). Observe that
\[
\|w_2\| = d(\eta_2, h(Y_0^*)) \leq d(\eta_2, X) \leq d(K, X) < a \text{ and } \|\langle w_2, x_2^* \rangle \| \leq (1 + \epsilon_1)d(K, X) \leq a.
\]

Now we choose \( y_2 \) and \( A_2 \) in the cases (A) and (B):

**CASE A.** We have
\[
\langle v_2, x^*_2 \rangle = \langle \eta_2, x_2^* \rangle - \langle w_2, x_2^* \rangle \leq \langle \eta_2, x_2^* \rangle - \|w_2, x_2^*\| > b - a.
\]

Thus, as \( \langle v_2, x^*_2 \rangle = \sum_{\alpha \in A} \langle v_{2\alpha}, x_{2\alpha}^* \rangle > b - a \), we can find a finite subset \( A_2 \subset A \) satisfying \( A_1 \subset A_2 \subset A \) such that, if \( y_2 \) is the restriction of \( x_2^* \) to \( \sum_{\alpha \in A_2} \oplus X_\alpha \) (so \( y_2 = \sum_{\alpha \in A_2} x_{2\alpha}^* \in \sum_{\alpha \in A_2} \oplus X_\alpha \subset Y_0 \) with \( \|y_2\| \leq 1 + \epsilon_1 \)), then \( \langle \eta_2, y_2 \rangle = \langle v_2, y_2 \rangle > b - a \). Observe that for every \( k \in \bigcup_{\alpha \in A_1} K_\alpha \), we have \( \psi(k) = 0 \), whence
\[
\|\langle y_2, k \rangle \| = \|\langle x_2^*, k \rangle \| \leq \epsilon_1 \leq 2^{-2}.
\]

**CASE B.** Let \( \gamma_{21} := x_2^* \upharpoonright \sum_{\alpha \in A_1} \oplus X_\alpha \) (that is, \( \gamma_{21} = \sum_{\alpha \in A_1} x_{2\alpha}^* \)) and \( \gamma_{22} = x_2^* - \gamma_{21} \). Since \( \|x_2^*, k\| \leq \epsilon_1, \forall k \in \sum_{\alpha \in A_1} K_\alpha \), and \( B(\sum_{\alpha \in A_1} \oplus X_\alpha) \subset \sum_{\alpha \in A_1} K_\alpha \), then
\[
\|\gamma_{21}\| \leq \epsilon_1.
\]

So
\[
\langle v_2, \gamma_{22} \rangle = \langle \eta_2, x_2^* - \langle w_2, x_2^* \rangle - \langle v_2, \gamma_{21} \rangle \geq \eta_2, x_2^* \rangle - \epsilon_1 - a > b - a.
\]

Since \( \langle v_2, \gamma_{22} \rangle = \sum_{\alpha \in A \setminus A_1} \langle v_{2\alpha}, x_{2\alpha}^* \rangle > b - a \), we can find a finite subset \( A_2 \subset A \setminus A_1 \) such that, if \( y_2 \) is the restriction of \( x_2^* \) to \( \sum_{\alpha \in A_2} \oplus X_\alpha \) (so \( y_2 = \sum_{\alpha \in A_2} x_{2\alpha}^* \in \sum_{\alpha \in A_2} \oplus X_\alpha \subset Y_0 \)) with \( \|y_2\| \leq 1 + \epsilon_1 \), then \( \langle \eta_2, y_2 \rangle = \langle v_2, y_2 \rangle > b - a \).
Further we proceed by iteration. We obtain the sequences \( \{y_k : k \geq 1\} \subset Y_0, \{\eta_k : k \geq 1\} \subset K \) and \( \{A_k : k \geq 1\}, A_k \subset A \), fulfilling the following conditions:

**Case A.** In this case we have:

(i) The finite subsets \( A_k \) of \( A \) satisfy \( A_k \subset A_{k+1} \) for \( k \geq 1 \).

(ii) \( y_k \in \sum_{\alpha \in A_k} X^*_\alpha \subset Y_0, \|y_k\| \leq 1 + \epsilon_{k-1}, k \geq 2, \) and \( \langle \eta_j, y_k \rangle > b - a \) for \( j \geq k \) with \( j, k \in \mathbb{N} \).

(iii) For every \( h \in \bigcup_{\alpha \in A_k} K_\alpha \) we have \( \|\langle y_{k+1}, h \rangle\| \leq 2^{-k-1}, \forall k \geq 1 \).

Let \( A_0 := \bigcup_{n \geq 1} A_n, X_0 := \sum_{\alpha \in A_0} X_\alpha \) and let \( P_0 : X \to X_0 \) be the canonical projection on \( X_0 \), with norm \( \|P_0\| = 1 \). The space \( X \) admits the monotone decomposition

\[
X = X_0 \oplus X_1 \text{ where } X_1 := \sum_{\alpha \in A \setminus A_0} X_\alpha.
\]

Therefore we get the following monotone decompositions

\[
X^* = X_0^* \oplus X^*_1, \quad X^{**} = X_0^{**} \oplus X^{**}_1, \quad X^{***} = X_0^{***} \oplus X^{***}_1, \quad \text{etc.,}
\]

with projections \( P_0 : X \to X_0, P_* : X^* \to X^*_0, P^{**} : X^{**} \to X^{**}_0, P^{***} : X^{***} \to X^{***}_0, \) etc. Observe that \( P_0^*(y_k) = y_k, \forall k \geq 1 \), that is, \( y_k \in X_0^*, \forall k \geq 1 \). Let \( \eta_0 \) be a \( w^* \)-cluster point of the sequence \( \{\eta_k : k \geq 1\} \) in \( X^{**} \). Obviously \( \eta_0 \in K \). Moreover, since \( \langle \eta_j, y_k \rangle > b - a, \forall j \geq k \), we get \( \langle \eta_0, y_k \rangle \geq b - a, \forall k \geq 1 \). Let \( \varphi_0 \) be a \( w^* \)-cluster point of \( \{y_k : k \geq 1\} \) in \( X^{***} \). Then

(i) \( \varphi_0 \in B(X^{***}) \). Actually \( \varphi_0 \) is in \( P_0^{***}(X^{***}) = X^{***}_0 \), that is, \( P_0^{***}(\varphi_0) = \varphi_0 \).

(ii) By construction \( \varphi_0 \mid_{K_\alpha} = 0, \forall \alpha \in A_0 \). Thus \( \varphi_0 \in X_0^\perp \), because \( \bigcup_{\alpha \in A_0} K_\alpha \) generates \( X_0 \).

(iii) \( \langle \varphi_0, \eta_0 \rangle \geq b - a \) because \( \langle \eta_0, y_k \rangle \geq b - a, \forall k \geq 1 \).

Let \( W := P_0^{**}(K) \subset B(X_0^{**}) \), which is a \( w^* \)-compact subset of \( X_0^{**} \), and \( w_0 = P_0^{**}(\eta_0) \). Obviously \( w_0 \in W \).

**Claim 1.** \( d(w_0, X_0) < a \).

Indeed, let \( x \in X \) be arbitrary. Then

\[
d(w_0, X_0) \leq \|w_0 - P_0^{**}x\| = \|P_0^{**}(\eta_0) - P_0^{**}x\| \leq \|\eta_0 - x\|.
\]

That is, \( d(w_0, X_0) \leq d(\eta_0, X) \leq \tilde{d}(K, X) < a \).

**Claim 2.** \( d(w_0, X_0) \geq b - a \).

Indeed, as \( \varphi_0 \in B(X^{***}) \cap X_0^\perp \) and

\[
\langle \varphi_0, w_0 \rangle = \langle \varphi_0, P_0^{**}\eta_0 \rangle = \langle P_0^{***}\varphi_0, \eta_0 \rangle = \langle \varphi_0, \eta_0 \rangle \geq b - a,
\]

we conclude that \( d(w_0, X_0) \geq b - a \).

As \( a < b - a \) we get a contradiction which proves the statement in the case (A).

**Case B.** In this case we have:

(i) The finite subsets \( A_k, k \geq 1 \), of \( A \) are disjoint.

(ii) \( y_k \in \sum_{\alpha \in A_k} X^*_\alpha \subset Y_0, \|y_k\| \leq 1 + \epsilon_{k-1}, k \geq 2, \) and \( \langle \eta_j, y_k \rangle > b - a \) for \( j \geq k \) with \( j, k \in \mathbb{N} \).
(iii) For every \( n \in \mathbb{N} \) we have \( \left\| \sum_{i=1}^{n} y_i \right\| \leq 2 \).

Let \( \eta_0 \) be a \( w^* \)-cluster point of the sequence \( \{ \eta_k : k \geq 1 \} \) in \( X^{**} \). Obviously \( \eta_0 \in K \). Moreover, since \( \langle \eta_j, y_k \rangle > b - a \), \( \forall j \geq k \), we get \( \langle \eta_0, y_k \rangle \geq b - a > 0 \), \( \forall k \geq 1 \). Thus \( \langle \eta_0, \sum_{i=1}^{n} y_i \rangle \geq n(b - a), \forall n \geq 1 \). Since \( \| \sum_{i=1}^{n} y_i \| \leq 2 \), \( \forall n \geq 1 \), we get a contradiction which proves the statement (B). ■

**Proposition 4.** Let \( X \) be a Banach space, which is the 1-unconditional direct sum \( X = \sum_{\alpha \in A} \oplus X_\alpha \) of the family \( \{ X_\alpha : \alpha \in A \} \) of WCG Banach spaces. If \( K \subset X^{**} \) is a \( w^* \)-compact subset such that \( K \cap X \) is \( w^* \)-dense in \( K \), then \( \hat{d}(\overline{\text{co}}^w(K), X) = \hat{d}(K, X) \).

**Proof.** The proof is analogous to the one of case (A) of Proposition 3. For reader’s convenience we give the details of the proof. Suppose that there exists a \( w^* \)-compact subset \( K \subset B(X^{**}) \) such that

\[ \hat{d}(\overline{\text{co}}^w(K), X) > b > a > \hat{d}(K, X) > 0. \]

By Lemma 12 of [5] we have:

**Fact.** There exist \( \psi \in S(X^{**}) \cap X^\perp \) and a \( w^* \)-compact subset \( \emptyset \neq H \subset K \) such that for every \( w^* \)-open subset \( V \) of \( X^{**} \) with \( V \cap H \neq \emptyset \) there exists \( \xi \in \overline{\text{co}}^w(V \cap H) \) with \( \langle \psi, \xi \rangle > b \).

**Step 1.** By the Fact there exists a vector \( \xi_1 \in \overline{\text{co}}^w \psi \) such that \( \langle \psi, \xi_1 \rangle > b \). Since \( B(X^*) \) is \( w^* \)-dense in \( B(X^{**}) \), we can find a vector \( x_1^* \in B(X^*) \) such that \( \langle \xi_1, x_1^* \rangle > b \). Let \( V_1 = \{ u \in X^* : \langle u, x_1^* \rangle > b \} \), which is a \( w^* \)-open subset of \( X^{**} \) such that \( V_1 \cap H \neq \emptyset \) and so, for the sake of density, also \( V_1 \cap K \cap X \neq \emptyset \). Thus there exists a vector \( \eta_1 \in K \cap X \) such that \( \langle \eta_1, x_1^* \rangle > b \). Since \( \eta_1 \in X \), the support \( A_1 := \text{supp}(\eta_1) = \{ \alpha \in A : \eta_1 \alpha \neq 0 \} \) of \( \eta_1 \) is countable, we say, \( A_1 = \{ \alpha_{1n} : n \geq 1 \} \).

**Step 2.** As \( V_1 \cap H \neq \emptyset \), by the Fact there exists a vector \( \xi_2 \in \overline{\text{co}}^w(V_1 \cap H) \) such that \( \langle \psi, \xi_2 \rangle > b \). Let \( L_1 := \bigcup_{i,j=1}^{1} K_{\alpha_{ij}} \) and \( \epsilon_1 := 2^{-1} - \langle \psi, \xi_2 \rangle - b \). As \( L_1 \cup \{ \xi_2 \} \) is a \( w \)-compact subset of \( X^{**} \), by Lemma 2 there exists a vector \( x_2^* \in X^* \) such that \( \| x_2^* \| \leq 1 + \epsilon_1 \) and

\[ \forall k \in L_1 \cup \{ \xi_2 \}, \langle \psi, k \rangle - \epsilon_1 < \langle k, x_2^* \rangle < \langle \psi, k \rangle + \epsilon_1. \]

In particular \( \langle \xi_2, x_2^* \rangle > b \) and \( \| (x_2^*, k) \| \leq 2^{-1}, \forall k \in L_1 \). As \( \langle \xi_2, x_2^* \rangle > b \) and \( \xi_2 \in \overline{\text{co}}^w(V_1 \cap H) \), if we put \( W_2 := \{ u \in X^{**} : \langle u, x_2^* \rangle > b \} \), then \( W_2 \cap V_1 \cap H \neq \emptyset \) and, for the sake of density, also \( W_2 \cap V_1 \cap K \cap X \neq \emptyset \). Denote \( V_2 := W_2 \cap V_1 \) and choose \( \eta_2 \in V_2 \cap K \cap X \), which satisfies \( x_2^*(\eta_2) > b \), \( i = 1, 2 \). Since \( \eta_2 \in X \), the support \( A_2 := \text{supp}(\eta_2) = \{ \alpha \in A : \eta_2 \alpha \neq 0 \} \) of \( \eta_2 \) is countable, we say, \( A_2 = \{ \alpha_{2n} : n \geq 1 \} \).

Further we proceed by iteration. We get the sequences \( \{ x_k^* : k \geq 1 \} \subset X^* \), \( \{ \eta_k : k \geq 1 \} \subset K \cap X \), \( \{ L_k : k \geq 1 \} \) and \( \{ A_k : k \geq 1 \} \), such that

(a) \( A_k := \text{supp}(\eta_k) = \{ \alpha \in A : \eta_k \alpha \neq 0 \} \) is the support of \( \eta_k \) and is countable, say, \( A_k = \{ \alpha_{kn} : n \geq 1 \} \), \( \forall k \geq 1 \).

(b) The subsets \( L_k = \bigcup_{i,j=1}^{k} K_{\alpha_{ij}} \) are \( w \)-compact subsets of \( B(X) \), \( \forall k \geq 1 \).

(c) \( \| x_{k+1}^* \| \leq 1 + \epsilon_k \), \( \langle \eta_j, x_k^* \rangle > b \), \( j \geq k \geq 1 \), and for every \( h \in L_k \) we have \( \| (x_{k+1}^*, h) \| \leq 2^{-k}, \forall k \geq 1 \).
Let \( A_0 := \bigcup_{n \geq 1} A_n, X_0 := \sum_{\alpha \in A_0} X_\alpha \) and let \( P_0 : X \to X_0 \) be the canonical projection, with norm \( \| P_0 \| = 1 \). Observe that \( X_0 \) is WCG because it is a countable sum of WCG Banach spaces. The space \( X \) admits the monotone decomposition

\[
X = X_0 \oplus X_1 \text{ where } X_1 := \sum_{\alpha \in A \setminus A_0} X_\alpha.
\]

Thus we get the monotone decompositions

\[
X^* = X_0^* \oplus X_1^*, X^{**} = X_0^{**} \oplus X_1^{**}, X^{***} = X_0^{***} \oplus X_1^{***}, \text{ etc.,}
\]

with projections \( P_0 : X \to X_0, P_0^* : X^* \to X_0^*, P_0^{**} : X^{**} \to X_0^{**}, P_0^{***} : X^{***} \to X_0^{***}, \text{ etc.} \) Observe that \( P_0(\eta_k) = \eta_k \in X_0, \forall k \geq 1 \). Let \( \eta_0 \) be a w*-cluster point of the sequence \( \{ \eta_k : k \geq 1 \} \) in \( X^{**} \). Obviously \( \eta_0 \in X_0^{**} \cap K \) because \( X_0^{**} \) is w*-closed in \( X^{**} \) (actually \( X_0^{**} = \overline{X_0^{w*}} \)) and \( \{ \eta_k : k \geq 1 \} \subset X_0 \cap K \). Since \( \langle x_i^*, \eta_k \rangle > b, \forall i \leq k \), we get

\[
\langle \eta_0, x_i^* \rangle \geq b, \forall i \geq 1 \text{ and also } \langle \eta_0, P_0^{**} x_i^* \rangle \geq b, \forall i \geq 1, \text{ because } \langle \eta_0, P_0^{**} x_i^* \rangle = \langle P_0^{**} \eta_0, x_i^* \rangle = \langle \eta_0, x_i^* \rangle \geq b.
\]

Let \( \varphi_0 \) be a w*-cluster point of \( \{ P_0^{**} x_k^* : k \geq 1 \} \) in \( X^{***} \). Then

(i) \( \varphi_0 \in B(X^{***}) \). Actually \( \varphi_0 \) is in \( P_0^{**}(X^{**}) = X_0^{**} \), (that is, \( P_0^{**}(\varphi_0) = \varphi_0 \)) because \( X_0^{**} \) is w*-closed in \( X^{**} \) and \( \{ P_0^{**} x_k^* : k \geq 1 \} \subset X_0^* \subset X_0^{**} \).

(ii) Since for every \( k \geq 1 \) and every \( h \in L_k \) we have

\[
|\langle P_0^{**}(x_{k+1}^*), h \rangle| = |\langle x_{k+1}^*, P_0(h) \rangle| = |\langle x_{k+1}^*, h \rangle| \leq 2^{-k},
\]

we get \( \varphi_0 \in X_0^\perp \).

(iii) \( \langle \varphi_0, \eta_0 \rangle \geq b \) because \( \langle \eta_0, P_0^{**} x_k^* \rangle \geq b, \forall k \geq 1 \).

Let \( W := P_0^{**}(K) \subset B(X_0^{**}) \). \( W \) is clearly a w*-compact subset of \( X_0^{**} \). Let \( w_0 = P_0^{**}(\eta_0) \). Obviously \( w_0 \in W \).

**Claim 1.** \( d(w_0, X_0) < a \).

Indeed, first as \( W = P_0^{**}(K), X_0 = P_0^{**}(X) \) and \( \| P_0^{**} \| = 1 \), we have

\[
d(W, X_0) \leq \| P_0^{**} \| \cdot d(K, X_0) < a.
\]

Now it is enough to observe that \( w_0 \in W \).

**Claim 2.** \( d(w_0, X_0) \geq b \).

Indeed, since \( \varphi_0 \in B(X^{***}) \cap X_0^\perp \) and

\[
\langle \varphi_0, w_0 \rangle = \langle \varphi_0, P_0^{**} \eta_0 \rangle = \langle P_0^{**} \varphi_0, \eta_0 \rangle = \langle \varphi_0, \eta_0 \rangle \geq b,
\]

we conclude that \( d(w_0, X_0) \geq b \).

As \( a < b \) we get a contradiction which proves the statement. \( \blacksquare \)

Let \( X \) be a Banach space which admits the decomposition \( X = \sum_{\alpha \in A} X_\alpha \) as a 1-unconditional direct sum of closed subspaces \( X_\alpha \). We say that the decomposition \( X = \sum_{\alpha \in A} X_\alpha \) is of **countable type** if for every \( u \in X^* \) the support \( \text{supp}(u) := \{ \alpha \in A : u_\alpha \neq 0 \} \) of \( u \) is countable, \( \langle u_\alpha \rangle_{\alpha \in A} \) being the set of coordinates of \( u \), that is, \( u_\alpha := u|_{X_\alpha} = u \circ P_\alpha \), where \( P_\alpha : X \to X_\alpha \) is the canonical projection. For instance, if \( I \) is an infinite set, \( M : \mathbb{R} \to [0, \infty] \) an Orlicz function such that its complementary Orlicz function \( M^* \) (see [2], [9, Chapter 4]) satisfies \( M^*(t) > 0 \) for \( t > 0 \), then the Orlicz space \( h_M(I) := \{ f \in \mathbb{R}^I : \sum_{i \in I} M(f_i/\lambda) < \infty, \forall \lambda > 0 \} \) has countable decomposition (with respect to
the canonical basis of $h_M(I)$), because every element of its dual $h_M(I)^* := \ell_{M^*}(I)$ has countable support.

Lemma 5. Let $X$ be a Banach space which admits a decomposition $X = \sum_{\alpha \in A} \oplus X_\alpha$ as a 1-unconditional direct sum of closed subspaces. The following statement are equivalent:

1. The decomposition $X = \sum_{\alpha \in A} \oplus X_\alpha$ is not of countable type.
2. $X$ has an isomorphic copy of $\ell_1(\mathbb{N}_1)$ disjointly disposed with respect to the decomposition $X = \sum_{\alpha \in A} \oplus X_\alpha$, that is, there exists a subset $A_1 \subset A$ with cardinality $|A_1| = \mathbb{N}_1$ and for each $\alpha \in A_1$ an element $v_\alpha \in X_\alpha$ so that the family $\{v_\alpha : \alpha \in A_1\}$ is equivalent to the canonical basis of $\ell_1(\mathbb{N}_1)$.

Proof. (1) $\Rightarrow$ (2). If the decomposition $X = \sum_{\alpha \in A} \oplus X_\alpha$ is not of countable type, there exists some $u \in X^*$ such that the subset $A_0 := \{\alpha \in A : u_\alpha \neq 0\}$ satisfies $|A_0| \geq \mathbb{N}_1$, where $u_\alpha := u|_{X_\alpha} = u \circ P_\alpha$ and $P_\alpha : X \to X_\alpha$ is the canonical projection. By passing to a subset if necessary, we can find a real number $\epsilon > 0$, a subset $A_1 \subset A_0$ with $|A_1| = \mathbb{N}_1$ and a family $\{v_\alpha : \alpha \in A_1\}$ with $v_\alpha \in B(X_\alpha)$ so that $\langle u, v_\alpha \rangle = \langle u_\alpha, v_\alpha \rangle > \epsilon$. From this we can prove that the family $\{v_\alpha : \alpha \in A_1\}$ is equivalent to the canonical basis of $\ell_1(\mathbb{N}_1)$ and generates a copy of $\ell_1(\mathbb{N}_1)$, which is disjointly disposed with respect to the decomposition $X = \sum_{\alpha \in A} \oplus X_\alpha$.

(2) $\Rightarrow$ (1). Let $A_1 \subset A$ be a subset with cardinality $|A_1| = \mathbb{N}_1$ and for each $\alpha \in A_1$ let $v_\alpha$ be an element of $X_\alpha$ so that the family $\{v_\alpha : \alpha \in A_1\}$ is equivalent to the canonical basis $\{e_\alpha : \alpha \in A_1\}$ of $\ell_1(A_1)$. Let $T : \ell_1(A_1) \to X$ be the isomorphism between $\ell_1(A_1)$ and the closed subspace generated by $\{v_\alpha : \alpha \in A_1\}$ so that $T(e_\alpha) = v_\alpha$. Since $T^* : X^* \to \ell_\infty(A_1)$ is a quotient mapping and so $T^*(X^*) = \ell_\infty(A_1)$, if $w_0 \in \ell_\infty(A_1)$ is such that $w_0(\alpha) = 1, \forall \alpha \in A_1, \thereexists a vector u \in X^*$ such that $T^*(u) = w_0$. Then for every $\alpha \in A_1$ we have

$$\langle u, v_\alpha \rangle = \langle u, Te_\alpha \rangle = \langle T^*u, e_\alpha \rangle = \langle w_0, e_\alpha \rangle = 1,$$

and this proves that $u$ is an element of $X^*$ that does not have countable support with respect to the decomposition $X = \sum_{\alpha \in A} \oplus X_\alpha$.

Proposition 6. Let $X$ be a Banach space that admits a decomposition of countable type $X = \sum_{i \in I} \oplus X_i$ as a 1-unconditional direct sum of WCG closed subspaces $\{X_i : i \in I\}$. Then $X$ has 1-control in its bidual $X^{**}$.

Proof. If $I$ is a countable set, then $X$ is a countable direct sum of WCG Banach subspaces and, so, it is WCG. Thus the statement follows in this case from [5, Cor. 4]. So, suppose in the sequel that $I$ is uncountable. We assume that the statement is not true and we are going to get a contradiction. Let $K$ be a $w^*$-compact subset of $X^{**}$ and $z_0 \in \overline{co}^{w^*}(K)$ a vector such that $d(z_0, X) > b > a > \hat{d}(K, X)$. Choose $\psi \in B(X^{**}) \cap X^\perp$ such that $\langle \psi, z_0 \rangle > b$. We adopt the following notation. For each $i \in I$ let $K_i \subset X_i$ be a $w$-compact subset of $X_i$ which generates $X_i$. If $J$ is a subset of $I$, let $X(J)$ denote the subspace $X(J) := \sum_{i \in J} \oplus X_i$ (so $X = X(I)$) and let $P_J$ denote the canonical projection $P_J : X \to X(J)$ with norm $\|P_J\| = 1$. We identify the subspace $P_J^*(X^*)$ of the dual $X^*$ with $X(J)^*$.
STEP 1. Since $\langle \psi, z_0 \rangle > b$, there exists $x^*_1 \in B(X^*)$ such that $\langle z_0, x^*_1 \rangle > b$ (because $B(X^*)$ is $w^*$-dense in $B(X^{***})$). By hypothesis the support $\text{supp}(x^*_1) = \{ \alpha \in I : 0 \neq x^*_1 \alpha \} := J_1$ of $x^*_1$ is countable. Let $J_1 := \{ \alpha_{i,j} : j \geq 1 \}$ and $I_1 := J_1$. Thus, if $P_{I_1} : X \to \sum_{i \in I_1} \oplus X_i$ is the corresponding canonical projection, then $P_{I_1}^*(x^*_1) = x^*_1$ (that is, $x^*_1 \in X(I_1)^*$) and moreover\[
\langle P_{I_1}^*(z_0), x^*_1 \rangle = \langle z_0, P_{I_1}^*(x^*_1) \rangle = \langle z_0, x^*_1 \rangle > b.
\]

STEP 2. Let $K_{\alpha_{i,j}}$ be the w-compact subset that generates $X_{\alpha_{i,j}}$, and put $L_2 := \{ z_0 \} \cup K_{\alpha_{i,j}}$, which is a w-compact subset of $X^**$. Let $\epsilon_2 = 2^{-2} \wedge (\langle \psi, z_0 \rangle - b)$. By Lemma 2 there exists a vector $x^*_2 \in X^*$ such that $\| x^*_2 \| \leq 1 + \epsilon_2$ and
\[
\forall k \in L_2, \ \langle \psi, k \rangle - \epsilon_2 < \langle k, x^*_2 \rangle < \langle \psi, k \rangle + \epsilon_2.
\]
In particular, $\langle z_0, x^*_2 \rangle > b$ and $|\langle k, x^*_2 \rangle| \leq 2^{-2}, \ \forall k \in K_{\alpha_{i,j}}$. Let $J_2 := \text{supp}(x^*_2)$ be the support of $x^*_2$, which is countable by hypothesis. Put $J_2 := \{ \alpha_{i,j} : j \geq 1 \}$ and $I_2 := J_1 \cup J_2$. Then $P_{I_2}^*(x^*_2) = x^*_1 \in X(I_2)^*$, $i = 1, 2$, and moreover\[
\langle P_{I_2}^*(z_0), x^*_2 \rangle = \langle z_0, P_{I_2}^*(x^*_2) \rangle = \langle z_0, x^*_1 \rangle > b, \ i = 1, 2.
\]

STEP 3. Let $L_3 := \{ z_0 \} \cup (\bigcup \{ K_{\alpha_{i,j}} : 1 \leq i, j \leq 2 \})$, which is a w-compact subset of $X^{**}$. Let $\epsilon_3 = 2^{-3} \wedge (\langle \psi, z_0 \rangle - b)$. By Lemma 2 there exists a vector $x^*_3 \in X^*$ such that $\| x^*_3 \| \leq 1 + \epsilon_3$ and
\[
\forall k \in L_3, \ \langle \psi, k \rangle - \epsilon_3 < \langle k, x^*_3 \rangle < \langle \psi, k \rangle + \epsilon_3.
\]
In particular, $\langle z_0, x^*_3 \rangle > b$ and $|\langle k, x^*_3 \rangle| \leq 2^{-3}, \ \forall k \in K_{\alpha_{i,j}}, 1 \leq i, j \leq 2$. Let $J_3 := \text{supp}(x^*_3)$ be the support of $x^*_3$, that is countable by hypothesis, and put $J_3 := \{ \alpha_{i,j} : j \geq 1 \}$ and $I_3 := J_1 \cup J_2 \cup J_3$. Then $P_{I_3}^*(x^*_3) = x^*_1 \in X(I_3)^*$, $i = 1, 2, 3$, and moreover\[
\langle P_{I_3}^*(z_0), x^*_3 \rangle = \langle z_0, P_{I_3}^*(x^*_3) \rangle = \langle z_0, x^*_1 \rangle > b, \ i = 1, 2, 3.
\]

Further we proceed by iteration. Let $I_0 := \bigcup_{n \geq 1} I_n$. Observe that $I_0$ is a countable subset of $I$ such that $P_{I_0}^*(x^*_1) = x^*_1 \in X(I_0)^*$, $i \geq 1$. Let $\psi_0 \in B(X^{***})$ be a $w^*$-cluster point of the sequence $\{ x^*_n : n \geq 1 \}$ in $X^{***}$. Then

(a) Since $X(I_0)^{***}$ is a $w^*$-closed subset of $X^{***}$ and $x^*_n \in B(X(I_0)^*) \subset B(X(I_0)^{***})$, $n \geq 1$, we get $\psi_0 \in B(X(I_0)^{***})$ and so $P_{I_0}^{***}(\psi_0) = \psi_0$.

(b) Since $|\langle u, x^*_n \rangle| \leq 2^{-n} - 1, \ \forall u \in K_{\alpha_{i,j}}, 1 \leq i, j \leq n$, and the subset $\bigcup_{i,j \geq 1} K_{\alpha_{i,j}}$ generates the space $X(I_0)$, we conclude that $\psi_0|_{X(I_0)} \equiv 0$, that is, $\psi_0 \in B(X(I_0)^{**})$.

(c) Since $\langle z_0, x^*_k \rangle > b, \ \forall k \geq 1$, then $\langle \psi_0, z_0 \rangle \geq b$.

Let $H := P_{I_0}^{***}(K)$ and $w_0 := P_{I_0}^{***}(z_0)$. Clearly, $H$ is a $w^*$-compact subset of $X(I_0)^{**}$ (for the topology $\sigma(X(I_0)^**, X(I_0)^*)$) and $w_0 \in \overline{co}^{w^*}(H)$.

CLAIM 1. $\hat{d}(H, X(I_0)) < a$ and $d(w_0, X(I_0)) < a$.

Indeed\[
\hat{d}(H, X(I_0)) = \hat{d}(P_{I_0}^{**}(K), P_{I_0}^{**}(X)) \leq ||P_{I_0}^{**}|| \hat{d}(K, X) < a.
\]
By [5, Cor. 4] we have $d(w_0, X(I_0)) < a$, because $X(I_0)$ is a countable direct sum of WCG spaces and so it is WCG.

CLAIM 2. $d(w_0, X(I_0)) \geq b$. 

Indeed, by (γ) we have
\[ \langle \psi_0, w_0 \rangle = \langle \psi_0, P^{**}_0(z_0) \rangle = \langle P^{**}_0(\psi_0), z_0 \rangle = \langle \psi_0, z_0 \rangle \geq b. \]
As \( \psi_0 \in X(I_0)^\perp \) and \( \|\psi_0\| \leq 1 \), we get \( d(w_0, X(I_0)) \geq b \).

Since \( b > a \), we obtain a contradiction which proves the statement.

**Proposition 7.** Let \( X \) be a Banach space that has a transfinite basis \( \{ e_\alpha : 1 \leq \alpha < \omega_1 \} \) with constant 1 (that is, if \( 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n+m} < \omega_1 \) and \( (\lambda_i)_{i=1}^{n+m} \in \mathbb{R}^{n+m} \), then \( \| \sum_{i=1}^n \lambda_i e_\alpha_i \| \leq \| \sum_{i=1}^{n+m} \lambda_i e_\alpha_i \| \) so that every \( z \in X^* \) has countable support. Then \( X \) has 1-control in its bidual \( X^{**} \).

**Proof.** This proof is analogous to the one of Proposition 6. We suppose that the statement is not true and we are going to obtain a contradiction. Assume that there exist a \( w^* \)-compact subset \( K \subset X^{**} \) and a vector \( z_0 \in \text{co} \{ \omega_i \}^*(K) \) such that \( d(z_0, X) > b > a > \hat{d}(K, X) \). Choose a vector \( \psi \in B(X^{**} \cap \text{X}^\perp \) such that \( \langle \psi, z_0 \rangle \geq b \). We adopt the following notation. If \( \beta < \omega_1 \), let \( X(\beta) \) denote the subspace \( X(\beta) := V_{\beta} \oplus e_{\beta} \) (so \( X = X(\omega_1) \)) and let \( P_{\beta} \) be the canonical projection \( P_{\beta} : X \to X(\beta) \) with norm \( \| P_{\beta} \| = 1 \). We identify the subspace \( P_{\beta}^*(X^*) \) of \( X^* \) with \( X(\beta)^* \).

**Step 1.** Since \( \langle \psi, z_0 \rangle > b \), there exists \( x_1^* \in B(X^*) \) such that \( \langle \psi, x_1^* \rangle > b \) (because \( B(X^*) \) is \( w^* \)-dense in \( B(X^{**}) \)). By hypothesis the support of \( x_1^* \) is countable and so there exists \( \beta_1 < \omega_1 \) such that \( \text{supp}(x_1^*) \subset [1, \beta_1] \). Thus, \( P_{\beta_1}^*(x_1^*) = x_1^* \) (that is, \( x_1^* \in X(\beta_1)^* \) and moreover
\[ \langle P_{\beta_1}^*(z_0), x_1^* \rangle = \langle z_0, P_{\beta_1}^*(x_1^*) \rangle = \langle z_0, x_1^* \rangle > b. \]

**Step 2.** Let \( [1, \beta_1) = \{ \alpha_{1j} : j \geq 1 \} \). Since \( \langle \psi, z_0 \rangle > b \) and \( \langle \psi, e_{\alpha_{11}} \rangle = 0 \), there exists \( x_2^* \in B(X^*) \) such that \( \langle z_0, x_2^* \rangle > b \) and \( \langle x_2^*, e_{\alpha_{11}} \rangle = 0 \). By hypothesis the support of \( x_2^* \) is countable and so there exists \( \beta_1 < \beta_2 < \omega_1 \) such that \( \text{supp}(x_2^*) \subset [1, \beta_2] \). Thus, for \( i = 1, 2 \), we have \( P_{\beta_2}^*(x_i^*) = x_i^* \) (that is, \( x_i^* \in X(\beta_2)^* \) and moreover
\[ \langle P_{\beta_2}^*(z_0), x_i^* \rangle = \langle z_0, P_{\beta_2}^*(x_i^*) \rangle = \langle z_0, x_i^* \rangle > b. \]

**Step 3.** Let \( [1, \beta_2) = \{ \alpha_{2j} : j \geq 1 \} \). Since \( \langle \psi, z_0 \rangle > b \) and \( \langle \psi, e_{\alpha_{2j}} \rangle = 0 \), \( i, j = 1, 2 \), there exists \( x_3^* \in B(X^*) \) such that \( \langle z_0, x_3^* \rangle > b \) and \( \langle x_3^*, e_{\alpha_{2j}} \rangle = 0 \). By hypothesis the support of \( x_3^* \) is countable and so there exists \( \beta_2 < \beta_3 < \omega_1 \) such that \( \text{supp}(x_3^*) \subset [1, \beta_3] \). Thus, for \( i = 1, 2, 3 \), we have \( P_{\beta_3}^*(x_i^*) = x_i^* \) (that is, \( x_i^* \in X(\beta_3)^* \) and moreover
\[ \langle P_{\beta_3}^*(z_0), x_i^* \rangle = \langle z_0, P_{\beta_3}^*(x_i^*) \rangle = \langle z_0, x_i^* \rangle > b. \]

Further we proceed by iteration. Let \( \beta_0 := \text{sup} \{ \beta_n : n \geq 1 \} \), that satisfies \( \beta_0 < \omega_1 \) and \( P_{\beta_0}^*(x_i^*) = x_i^* \), and so \( x_i^* \in X(\beta_0)^* \), \( i \geq 1 \). Let \( \psi_0 \in B(X^{**}) \) a \( w^* \)-cluster point of the sequence \( \{ x_n^* : n \geq 1 \} \) en \( X^{**} \). Then

1. \( \alpha \) Since \( X(\beta_0)^{***} \) is a \( w^* \)-closed subset of \( X^{***} \) and \( x_n^* \in B(X(\beta_0)^*) \subset B(X(\beta_0)^{***}) \), \( n \geq 1 \), we get \( \psi_0 \in B(X(\beta_0)^{***}) \) and so \( P_{\beta_0}^*(\psi_0) = \psi_0 \).
2. \( \beta \) Since \( (x_{n+1}^*, e_{\alpha_{ij}}) = 0, \forall 1 \leq i, j \leq n \), and the family \( \{ e_{\alpha_{ij}} : 1 \leq i, j \} \) generates the space \( X(\beta_0) \), we conclude that \( \psi_0(\chi_{\beta_0}) = 0 \), that is, \( \psi_0 \in B(X(\beta_0)^+) \).
3. \( \gamma \) Since \( \langle z_0, x_k^* \rangle > b, \forall k \geq 1 \), then \( \psi_0(z_0) > b \).
Let $H := P^{**}_{\beta_0}(K)$ and $w_0 := P^{**}_{\beta_0}(z_0)$. Clearly, $H$ is a $w^*$-compact subset of $X(\beta_0)^{**}$ and $w_0 \in \overline{co}^{w^*}(H)$.

**Claim 1.** $\hat{d}(H, X(\beta_0)) < a$ and $d(w_0, X(\beta_0)) < a$.

Indeed

$$\hat{d}(H, X(\beta_0)) = \hat{d}(P^{**}(K), P^{**}(X)) \leq \|P^{**}\| \hat{d}(K, X) < a.$$ 

By [5, Cor. 4] we get $d(w_0, X(\beta_0)) < a$, because $X(\beta_0)$ is separable and so WCG.

**Claim 2.** $d(w_0, X(\beta_0)) \geq b$.

Indeed, by (γ) we have

$$\langle \psi_0, w_0 \rangle = \langle \psi_0, P^{**}_{\beta_0}(z_0) \rangle = \langle P^{**}_{\beta_0}(\psi_0), z_0 \rangle = \langle \psi_0, z_0 \rangle \geq b.$$ 

As $\psi_0 \in X(\beta_0)^\perp$ and $\|\psi_0\| \leq 1$, we get $d(w_0, X(\beta_0)) \geq b$.

Since $b > a$, we obtain a contradiction, which proves the statement.

**Proposition 8.** Let $X$ be a Banach space which is the transfinite direct sum of the family $\{X_\alpha : 1 \leq \alpha < \omega_1\}$ of WCG subspaces, $X = \sum_{\alpha < \omega_1} X_\alpha$, with constant 1 (that is, if $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n+m} < \omega_1$, $x_{\alpha_i} \in X_{\alpha_i}$ and $(\lambda_i)_{i=1}^{n+m} \in \mathbb{R}^{n+m}$, then $\|\sum_{i=1}^{n} \lambda_i x_{\alpha_i}\| \leq \|\sum_{i=1}^{n+m} \lambda_i x_{\alpha_i}\|$) so that every $z \in X^*$ has countable support. Then $X$ has 1-control in its bidual $X^{**}$.

**Proof.** The proof is analogous to the one of Proposition 7 by using the w-compact subsets $K_\alpha \subset X_\alpha$ that generate $X_\alpha$, $\alpha < \omega_1$, and Lemma 2, as in Proposition 6.

4. **Application to the order-continuous Banach lattices.** First we show that, if $X$ is an o-continuous Banach lattice, then $X$ is a 1-unconditional direct sum of disjoint closed ideals which are WCG. This result is well known (see [1]) but we give its proof for the reader’s convenience.

**Lemma 9.** Let $X$ be an o-continuous Banach lattice with a weak unit $e > 0$. Then $X$ is WCG.

**Proof.** It is well known (see [10, p. 28]) that the interval $[0, e] := \{x \in X : 0 \leq x \leq e\}$ is a w-compact subset of $X$. Let us see that $X = [0, e]$, that is, $X$ is the closure of the space generated by $[0, e]$. Pick a positive element $x \in X^+$. Then $ne \wedge x \uparrow x$ for $n \to \infty$, whence $\|x - ne \wedge x\| \downarrow 0$ because $X$ is o-continuous. So $\bigcup_{n \geq 1} [0, ne] = \bigcup_{n \geq 1} n[0, e]$ is dense in the positive cone $X^+$. As $X = X^+ - X^+$, we conclude that $X$ is the closure of the subspace generated by $[0, e]$.

**Lemma 10.** If $X$ is an o-continuous Banach lattice, then $X$ is the 1-unconditional direct sum $X = \sum_{\alpha \in A} X_\alpha$ of a family of closed ideals $\{X_\alpha : \alpha \in A\}$ mutually disjoint, such that each $X_\alpha$ has weak unit and so it is WCG.

**Proof.** By [10, 1.9] $X$ admits the expression $X = \sum_{\alpha \in A} X_\alpha$ as a direct sum of a family of closed ideals mutually disjoint $\{X_\alpha : \alpha \in A\}$ (so as a 1-unconditional direct sum), such that each $X_\alpha$ has a weak unit. By Lemma 9 we get the statement.
PROPOSITION 11. Let $X$ be an o-continuous Banach lattice. If $K$ is a $w^*$-compact subset of $X^{**}$, then $\hat{d}(\overline{w^*}(K), X) \leq 2\hat{d}(K, X)$ and, if $K \cap X$ is $w^*$-dense in $K$, then $\hat{d}(\overline{w^*}(K), X) = \hat{d}(K, X)$.

Proof. This result follows from Lemma 10, Proposition 3 and Proposition 4. ■

PROPOSITION 12. Let $X$ be an o-continuous Banach lattice that does not have a copy of $\ell_1(\mathbb{N})$. Then $X$ has 1-control in its bidual $X^{**}$.

Proof. Clearly, if $X$ is an o-continuous Banach lattice that does not have a copy of $\ell_1(\mathbb{N})$, then $X$ admits by Lemma 5 and Lemma 10 a decomposition of countable type $X = \sum_{\alpha \in A} X_\alpha$ as a 1-unconditional direct sum of closed ideals $X_\alpha$ each having weak unit. So, this result follows from Proposition 6. ■

PROPOSITION 13. Let $X$ be a Banach space with a 1-unconditional basis $\{e_i : i \in I\}$ equivalent to the canonical basis of $\ell_1(I)$. Then $X$ has 1-control in its bidual $X^{**}$.

Proof. The proof is analogous to the one of part (B) of Proposition 3, putting $X_i = [e_i], i \in I,$ and taking into account that $X^*$ and the subspace $Y_0$ of $X^*$ (see the notation of Proposition 3) are canonically isomorphic to $\ell_\infty(I)$ and $c_0(I)$, respectively. ■

A Banach space $X$ has a 1-symmetric basis $\{e_i : i \in I\}$ (see [12, p. 811]) whenever $\{e_i : i \in I\}$ is a 1-unconditional basis of $X$ and, moreover, $\{e_i : i \in I\}$ is symmetric, that is, for any two sequences $\{i_n : n \geq 1\}$ and $\{j_n : n \geq 1\}$ of $I$, the basic sequences $\{e_{i_n} : n \geq 1\}$ and $\{e_{j_n} : n \geq 1\}$ are equivalent. Of course, the canonical bases of non-separable Orlicz spaces, Lorentz spaces, Orlicz-Lorentz spaces, etc., are 1-symmetric bases.

PROPOSITION 14. Let $X$ be a Banach space with a 1-symmetric basis. Then $X$ has 1-control in its bidual $X^{**}$.

Proof. Case 1. Suppose that every element of the dual $X^*$ has countable support. In this case the result follows from Proposition 6.

Case 2. Suppose that there exists a vector $u \in B(X^*)$ with uncountable support. By Proposition 13 it is enough to prove the following claim.

Claim. If there exists a vector $u \in B(X^*)$ with uncountable support, then the 1-symmetric basis $\{e_i : i \in I\}$ of $X$ is equivalent to the canonical basis of $\ell_1(I)$.

Indeed, since $\text{supp}(u) := \{i \in I : u(e_i) \neq 0\}$ is uncountable, we can find a real number $\epsilon > 0$ and an uncountable subset $J \subset \text{supp}(u)$ such that $|u(e_i)| > \epsilon, \forall i \in J$. Let us prove that the family $\{e_i : i \in J\}$ is equivalent to the basis of $\ell_1(J)$. Suppose that the basis $\{e_i : i \in J\}$ is normalized and choose a vector of the form $\sum_{1 \leq k \leq n} \lambda_k e_{i_k}, \ i_k \in J$. Let $\epsilon_k = \pm 1$ so that $u(\lambda_k \epsilon_k e_{i_k}) = |\lambda_k u(e_{i_k})| \geq \epsilon |\lambda_k|, \ 1 \leq k \leq n$. Then

$$\sum_{1 \leq k \leq n} |\lambda_k| \geq \left\| \sum_{1 \leq k \leq n} \lambda_k e_{i_k} \right\| = \left\| \sum_{1 \leq k \leq n} \lambda_k \epsilon_k e_{i_k} \right\| \geq \epsilon \sum_{1 \leq k \leq n} |\lambda_k|.$$
and this implies that the family \( \{ e_i : i \in J \} \) is equivalent to the basis of \( \ell_1(J) \). As the basis \( \{ e_i : i \in I \} \) of \( X \) is symmetric, finally we conclude that \( \{ e_i : i \in I \} \) is equivalent to the canonical basis of \( \ell_1(I) \), and this proves the Claim and completes the proof of the Proposition. ■

References


