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SINGULARITIES IN MUCKENHOUPT WEIGHTED FUNCTION SPACES

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Abstract. We study weighted function spaces of Lebesgue, Besov and Triebel-Lizorkin type where the weight function belongs to some Muckenhoupt $\mathcal{A}_p$ class. The singularities of functions in these spaces are characterised by means of envelope functions.

Introduction. The purpose of this paper is to use the recently introduced concept of growth envelopes and continuity envelopes in function spaces, respectively, in order to characterise weighted spaces of type $L_p(\mathbb{R}^n, w)$, $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ where w belongs to some Muckenhoupt class $\mathcal{A}_p$. The idea to consider growth envelopes and continuity envelopes in (unweighted) function spaces originates from such classical results as the famous Sobolev embedding theorem [36], or, secondly, from the Brézis-Wainger result [5] on the almost Lipschitz continuity of functions from a Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, $1 < p < \infty$. Basically, the unboundedness of functions that belong to (classical) Sobolev spaces $W_p^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, $1 \le p < \infty$, (and more general scales of spaces) is characterised. By Sobolev's embedding theorem it is known that for $k \le \frac{n}{p}$, $1 \le p < \infty$, there are (essentially) unbounded functions in $W_p^k(\mathbb{R}^n)$. More precisely, in case of $k < \frac{n}{p}$, one has $W_p^k(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n)$ for $p \le r \le p^* = \frac{np}{n-kp}$, whereas in the limiting case, $k = \frac{n}{p}$,

$$W_p^{n/p}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n) \quad \text{whenever} \quad p \le r < \infty, \tag{0.1}$$

but $W_p^{n/p}(\mathbb{R}^n) \not\hookrightarrow L_\infty(\mathbb{R}^n)$ unless $p = 1$. Beyond the 'critical line' $k = \frac{n}{p}$, i.e., for $k > \frac{n}{p}$ or $k = n$ and $p = 1$, we have $W_p^k(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$.

In the past a lot of work has been devoted to Sobolev type embeddings, in particular to refinements of the limiting case (0.1) in terms of wider classes of function spaces. We do not want to report on this elaborate history here; apart from the original papers

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assertions of this type are indispensable parts in books dealing with Sobolev spaces and related questions, cf. [1], [45], [26], [10].

In order to study the growth or unboundedness of such functions (distributions) the *growth envelope* $\mathfrak{E}_{\mathsf{G}}(X) = (\mathcal{E}_{\mathsf{G}}^X(t), u_{\mathsf{G}}^X)$ of a function space $X \subset L_1^{\mathrm{loc}}$ is introduced, where

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup\{f^*(t) : \|f|X\| \le 1\}, \quad t > 0,$$

is the *growth envelope function* of X and $u_{\mathsf{G}}^X \in (0,\infty]$ is some additional index providing a finer description. Here f^* denotes the non-increasing rearrangement of f, as usual. When $X \hookrightarrow C$ it makes sense to replace $f^*(t)$ with $\frac{\omega(f,t)}{t}$ and to study questions of the smoothness of functions, where $\omega(f,t)$ is the modulus of continuity. This leads to the *continuity envelope function*,

$$\mathcal{E}_{\mathsf{C}}^X(t) \sim \sup\left\{\frac{\omega(f,t)}{t} : \|f|X\| \le 1\right\}, \quad 0 < t < 1,$$

and the *continuity envelope* $\mathfrak{E}_{\mathsf{C}}$. These concepts were introduced in [43], [18], where the latter book also contains a recent survey of the present state-of-the-art (concerning extensions and more general approaches) as well as applications and further references.

A first motivation to study weighted spaces resulted from questions of local versus global behaviour of $\mathcal{E}_{\mathsf{G}}^X(t)$, that is, for $t \to 0$ and $t \to \infty$, respectively. The idea was to check that so-called 'admissible' weights, say, $w_\alpha(x) = (1+|x|^2)^{\alpha/2}$, have no influence on the local singularity behaviour (measured in envelopes), unlike (particular examples of) Muckenhoupt weights, e.g., $w^\alpha(x) = |x|^\alpha$, $\alpha > 0$. Secondly, we expected that globally their influence is the same. These assumptions could be verified in [19], see also [18]. However, we only studied the above model cases essentially, sticking to growth envelopes.

The present paper contains new and more general results, not only covering continuity envelope functions, but modifying the model weight function as well as proving a first result in the general case. We restrict ourselves to Muckenhoupt weights and consider first

$$w_{\alpha,\beta}(x) = \left\{\begin{array}{ll} |x|^\alpha & \text{if} \quad |x| \le 1\,, \\ |x|^\beta & \text{if} \quad |x| > 1\,, \end{array}\right\} \quad \text{with} \quad \beta \ge 0, \quad -n < \alpha \le \beta.$$

We can prove in this situation that

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p} - \frac{\max(\alpha,0)}{np} + \frac{s}{n}}, \quad 0 < t < 1,$$

and

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p} - \frac{\beta}{np}}, \quad t \to \infty,$$

where we have to assume $-n + \frac{\max(\alpha,0)}{p} < s - \frac{n}{p} < \frac{\max(\alpha,0)}{p}$. In case of Lebesgue spaces the counterpart reads as

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \sim t^{-\frac{\max(\alpha,0)}{np} - \frac{1}{p}}, \quad 0 < t < 1,$$

and

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \sim t^{-\frac{\beta}{np} - \frac{1}{p}}, \quad t \to \infty.$$

In the general case $w \in \mathcal{A}_p$ we obtain that

$$c_1 \sup_{|E|=t} \left(\int_E w(x)\,\mathrm{d}x\right)^{-1/p} \le \mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \le c_2 \sup_{|E|=t} \frac{1}{|E|}\left(\int_E w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'},$$

where the supremum is taken over all sets $E \subset \mathbb{R}^n$ with measure $|E| = t$. Finally, coming to continuity envelopes, we prove that for $\frac{\max(\alpha,0)}{p} < s - \frac{n}{p} < \frac{\max(\alpha,0)}{p} + 1$,

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{C}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-1+s-\frac{n}{p}-\frac{\max(\alpha,0)}{p}}, \quad 0 < t < 1.$$

The main tools to prove such results are unweighted counterparts, sharp embeddings and atomic decompositions of corresponding spaces. It is obvious but surprising at first glance that parameters $\alpha < 0$ do not change the corresponding singularity behaviour. The first observation of this kind, though in a different context, is contained in [21].

Note that envelope results have some interesting applications to limiting embeddings, Hardy-type inequalities, and the estimate of approximation numbers of related compact embeddings, we refer to [18] for further details.

The paper is organised as follows. Section 1 collects standard notation and fundamentals about Muckenhoupt weights and weighted function spaces, in Section 2 we briefly recall the concepts of envelope functions including basic examples. The main results are contained in Sections 3 and 4 concerning growth envelope functions and continuity envelope functions, respectively.

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1. Weighted function spaces. We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. If $a \in \mathbb{R}$, then $a_+ = \max(a, 0)$. For $0 < u \le \infty$, the number u' is given by $1/u' = (1 - 1/u)_+$. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. As we shall always deal with function spaces on $\mathbb{R}^n$, we may often omit the '$\mathbb{R}^n$' from their notation for convenience.

1.1. *Muckenhoupt weights.* We briefly recall some fundamentals on Muckenhoupt classes $\mathcal{A}_p$. By a weight w we shall always mean a locally integrable function $w \in L_1^{\mathrm{loc}}$, positive a.e. in the sequel. Let M stand for the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where $\mathcal{B}$ is the collection of all open balls $B(x,r) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$, $r > 0$.

Definition 1.1. Let w be a positive, locally integrable function on $\mathbb{R}^n$.

(i) Let $1 < p < \infty$. Then w belongs to the *Muckenhoupt class* $\mathcal{A}_p$ if there exists a constant $0 < A < \infty$ such that for all balls B,

$$(1.1)\qquad \left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)^{1/p}\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \le A,$$

where p' is given by $1/p' + 1/p = 1$, as usual.

(ii) A weight w belongs to the *Muckenhoupt class* $\mathcal{A}_1$ if there exists a constant $A > 0$ such that

$$Mw(x) \le Aw(x) \quad \text{for almost all} \quad x \in \mathbb{R}^n.$$

(iii) The *Muckenhoupt class* $\mathcal{A}_\infty$ is given by $\mathcal{A}_\infty = \bigcup_{p>1}\mathcal{A}_p$.

Since the pioneering work of Muckenhoupt [27], [28], [29], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [13], [38], [39, Ch. IX], and [37, Ch. V] for a complete account of the theory of Muckenhoupt weights.

Remark 1.2. For convenience, we recall a few basic properties only. The class $\mathcal{A}_p$ is stable with respect to translation, dilation and multiplication by a positive scalar. A weight $w \in \mathcal{A}_p$ possesses the doubling property, and $w \in \mathcal{A}_p$ implies $w^{-p'/p} \in \mathcal{A}_{p'}$, $1 < p < \infty$. In addition to the (more or less obvious) monotonicity $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$ for $1 \le p_1 < p_2 \le \infty$, the so-called 'reverse Hölder inequality', a fundamental feature of $\mathcal{A}_p$ weights (see [37, Ch. V, Prop. 3, Cor.]) leads to the somehow surprising property that for any $w \in \mathcal{A}_p$ there exists some number $r < p$ such that $w \in \mathcal{A}_r$.

Example 1.3. Obviously, one of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_p$, $1 < p < \infty$, is given by

$$(1.2)\qquad w_\alpha(x) = |x|^\alpha, \quad x \in \mathbb{R}^n, \quad -n < \alpha < \infty.$$

It is well known that $w_\alpha \in \mathcal{A}_1$ if, and only if, $\alpha \le 0$ and $w_\alpha \in \mathcal{A}_p$ if, and only if, $\alpha < n(p-1)$. In this paper we shall study a slightly more general weight with possibly different polynomial growth both near zero and infinity,

$$(1.3)\qquad w_{\alpha,\beta}(x) = \begin{Bmatrix} |x|^\alpha & \text{if} & |x| \le 1\,, \\ |x|^\beta & \text{if} & |x| > 1\,, \end{Bmatrix} \quad \text{with} \quad \alpha > -n, \quad \beta > -n.$$

Obviously this refines the approach (1.2), i.e., $w_{\alpha,\alpha} = w_\alpha$. One verifies that $w_{\alpha,\beta} \in \mathcal{A}_p$ if

$$-n < \alpha < n(p-1) \qquad \text{and} \qquad -n < \beta < n(p-1).$$

1.2. *Function spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ with $w \in \mathcal{A}_\infty$.* Let $w \in \mathcal{A}_\infty$ be a Muckenhoupt weight and $0 < p < \infty$. All spaces are defined on $\mathbb{R}^n$ in this section. The weighted Lebesgue space $L_p(w)$ contains all measurable functions such that

$$(1.4)\qquad \|f\,|L_p(w)\| = \|w^{1/p} f|L_p\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}$$

is finite. Note that for $p = \infty$ one obtains the classical (unweighted) Lebesgue space,

$$(1.5)\qquad L_\infty(w) = L_\infty, \quad w \in \mathcal{A}_\infty.$$

We thus restrict ourselves to $p < \infty$ in what follows.

The Schwartz space $\mathcal{S}$ and its dual $\mathcal{S}'$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1, \tag{1.6}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^{\infty}$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'$, then the Paley-Wiener-Schwartz theorem implies that $\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.4. Let $0 < q \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$, and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. Assume $w \in \mathcal{A}_\infty$.

(i) The *weighted Besov space* $B^s_{p,q}(w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\,|B^s_{p,q}(w)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)|L_p(w)\|^q\Big)^{1/q} \tag{1.7}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

(ii) The *weighted Triebel-Lizorkin space* $F^s_{p,q}(w)$ is the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\,|F^s_{p,q}(w)\| = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)|^q\Big)^{1/q}|L_p(w)\Big\| \tag{1.8}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Remark 1.5. The spaces $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \ge 1$), and $\mathcal{S} \hookrightarrow B^s_{p,q}(w) \hookrightarrow \mathcal{S}'$, similarly for the F-case, where the first embedding is dense if $q < \infty$; cf. [6]. Moreover, for $w_0 \equiv 1 \in \mathcal{A}_\infty$ we re-obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces; we refer, in particular, to the series of monographs by Triebel, [40], [41], [42] and [44] for a comprehensive treatment of the unweighted spaces.

The above spaces with weights of type $w \in \mathcal{A}_\infty$ have been studied by Bui first in [6], [7], with subsequent papers [8], [9]. It turned out that many of the results from the unweighted situation have weighted counterparts: e.g., we have $F^0_{p,2}(w) = h_p(w)$, $0 < p < \infty$, where the latter are Hardy spaces, see [6, Thm. 1.4], and, in particular, $h_p(w) = L_p(w) = F^0_{p,2}(w)$, $1 < p < \infty$, $w \in \mathcal{A}_p$, see [38, Ch. VI, Thm. 1]. Concerning (classical) Sobolev spaces $W^k_p(w)$ (built upon $L_p(w)$ in the usual way) we have

$$W^k_p(w) = F^k_{p,2}(w), \quad k \in \mathbb{N}_0, \quad 1 < p < \infty, \quad w \in \mathcal{A}_p, \tag{1.9}$$

cf. [6, Thm. 2.8]. Further results, concerning, for instance, embeddings, (real) interpolation, extrapolation, lift operators, duality assertions can be found in [6], [7], [13].

More recently, Rychkov extended in [32] the above class of weights in order to incorporate locally regular weights, too, creating in that way the class $\mathcal{A}_p^{\text{loc}}$. Recent works are due to Roudenko [12, 30, 31], and Bownik [3, 4]. We partly rely on our approach [20].

Remark 1.6. In the past, Besov and Triebel-Lizorkin spaces with so-called 'admissible' weights (of at most polynomial growth) were studied in detail, i.e., different characterisations, the continuity and compactness of corresponding embeddings and further topics. As a proto-type one can think of

$$v(x) = (1+|x|^2)^{\alpha/2}, \quad \alpha \in \mathbb{R}, \quad x \in \mathbb{R}^n. \tag{1.10}$$

Then the definition of $B^s_{p,q}(v)$ and $F^s_{p,q}(v)$ is literally the same as in Definition 1.4. As for literature we refer to [14], [15], [16], see also [11, Ch. 4] and [33] for a more general approach. Quite recently, there was a refreshed interest in this topic leading to the papers [22], [23], [24], [25] and [35]. Applications are described in [16].

One remarkable point is that for such weights $f \in B^s_{p,q}(v)$ if, and only if, $vf \in B^s_{p,q}$, with equivalent norms, $\|f|B^s_{p,q}(v)\| \sim \|vf|B^s_{p,q}\|$. In view of (1.4) one has thus to modify v by $v^{1/p}$ in order to compare results in corresponding spaces.

We formulate two special embedding results for weighted spaces of type $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$ that will be used in the sequel. The first one will enable us to reduce all (non-limiting) cases to the study of B-spaces only. Let $0<p<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, and $w\in\mathcal{A}_\infty$. Then

$$B^s_{p,\min(p,q)}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max(p,q)}(w), \tag{1.11}$$

see [6, Thm. 2.6]. In other words, whenever the q-parameter has no influence on the result we can immediately transfer the B-results to the F-spaces by (1.11). The next result establishes a link between weighted and unweighted Besov spaces; it is proved in [21].

Proposition 1.7. *Let*

$$-\infty < s_2 \le s_1 < \infty, \quad 0<p_1\le p_2\le\infty, \quad 0<q_1\le q_2\le\infty, \tag{1.12}$$

and $w_{\alpha,\beta}$ *be given by* (1.3). *Then*

$$B^{s_1}_{p_1,q_1}(w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_2,q_2} \tag{1.13}$$

if, and only if,

$$\beta \ge 0 \quad \text{and} \quad \delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \ge \frac{\max(\alpha,0)}{p_1}. \tag{1.14}$$

Remark 1.8. We briefly compare this result with the *admissible* weights v discussed in Remark 1.6. Restricted to the parameters given by (1.12) we proved in [15, Thm. 2.3] that

$$B^{s_1}_{p_1,q_1}(v) \hookrightarrow B^{s_2}_{p_2,q_2} \quad \text{if, and only if,} \quad \delta\ge 0 \quad \text{and} \quad v(x)\ge c \tag{1.15}$$

for some $c>0$ and all $x\in\mathbb{R}^n$. In particular, with v given by (1.10) we have to assume, in addition, that $\alpha\ge 0$. For extensions and more general results in this context we refer to [15].

2. Envelope functions

2.1. *Growth envelope functions.* Let for some measurable function $f : \mathbb{R}^n \to \mathbb{C}$, finite a.e., its decreasing rearrangement f^* be defined as usual,

$$f^*(t) = \inf\{s \geq 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \leq t\} \quad , \quad t \geq 0. \tag{2.1}$$

Definition 2.1. Let X be some quasi-normed function space on $\mathbb{R}^n$. The *growth envelope function* $\mathcal{E}_{\mathsf{G}}^X : (0,\infty) \to [0,\infty]$ of X is defined by

$$\mathcal{E}_{\mathsf{G}}^X(t) = \sup_{\|f|X\| \leq 1} f^*(t), \quad t > 0. \tag{2.2}$$

It is well-known that $\mathcal{E}_{\mathsf{G}}^X$ is monotonically decreasing and right-continuous, that $X \hookrightarrow L_\infty$ if, and only if, $\mathcal{E}_{\mathsf{G}}^X$ is bounded, and that there are function spaces X which do not possess a growth envelope function in the sense that $\mathcal{E}_{\mathsf{G}}^X$ is not finite for any $t > 0$; we refer to [17] and [18] for a more general approach as well as a detailed account of basic properties of $\mathcal{E}_{\mathsf{G}}^X$. We only mention the convenient 'monotonicity' feature for further use :

$$X_1 \hookrightarrow X_2 \quad \text{implies} \quad \mathcal{E}_{\mathsf{G}}^{X_1}(t) \leq c\,\mathcal{E}_{\mathsf{G}}^{X_2}(t) \qquad \text{for some } c > 0 \text{ and all } t > 0. \tag{2.3}$$

Remark 2.2. Let X be a rearrangement-invariant Banach function space, $t > 0$, and $A_t \subset \mathbb{R}^n$ with $|A_t| = t$, then the *fundamental function* φ_X of X is defined by $\varphi_X(t) = \|\chi_{A_t}|X\|$. In [18, Sect. 2.3] we proved that in this case

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \frac{1}{\varphi_X(t)} \sim \|\chi_{A_t}|X\|^{-1}, \quad t > 0. \tag{2.4}$$

Example 2.3. Basic examples for spaces X are the well-known Lorentz spaces; for definitions and further details we refer to [2, Ch. 4, Def. 4.1], for instance. This scale represents a natural refinement of the scale of Lebesgue spaces. In [18, Sect. 3.2] we proved that for $0 < p < \infty$, $0 < q \leq \infty$,

$$\mathcal{E}_{\mathsf{G}}^{L_{p,q}}(t) \sim t^{-\frac{1}{p}}, \qquad t > 0. \tag{2.5}$$

Remark 2.4. We dealt in [18] with the so-called *growth envelope* $\mathfrak{E}_{\mathsf{G}}(X)$ of some function space X : the envelope function $\mathcal{E}_{\mathsf{G}}^X$ is equipped with some additional fine index u_{G}^X that contains further information. In view of (2.5) this reads as $\mathfrak{E}_{\mathsf{G}}(L_{p,q}) = (t^{-\frac{1}{p}}, q)$. However, we do not study this index in the present context (though it might be interesting).

In general, the concept of growth envelopes make sense only for spaces $X \subset L_1^{\mathrm{loc}}$, i.e., when we deal with locally integrable *functions*. For $X = B^s_{p,q}$ we shall thus assume $s > n(\frac{1}{p} - 1)_+$ since by [34, Thm. 3.3.2] this implies $B^s_{p,q} \subset L_1^{\mathrm{loc}}$ for all $0 < p, q \leq \infty$, whereas in the borderline case $s = n(\frac{1}{p} - 1)_+$ one needs additional assumptions on p and q, respectively. This setting will not be treated here. We recall some unweighted results.

Proposition 2.5. *Let $0 < q \leq \infty$, $0 < p < \infty$, and $s > n(\frac{1}{p} - 1)_+$.*

(i) *Assume $s < \frac{n}{p}$. Then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}, \quad 0 < t < 1. \tag{2.6}$$

(ii) *Assume $s = \frac{n}{p}$ and let $1 < q \leq \infty$ in case of B-spaces, and $1 < p < \infty$ in case of F-spaces, respectively. Then*

$$\mathcal{E}_{\mathsf{G}}^{B^{n/p}_{p,q}}(t) \sim |\log t|^{\frac{1}{q'}}, \qquad \mathcal{E}_{\mathsf{G}}^{F^{n/p}_{p,q}}(t) \sim |\log t|^{\frac{1}{p'}}, \quad 0 < t < \tfrac{1}{2}. \tag{2.7}$$

(iii) *We have*

$$\mathcal{E}_{\mathsf{G}}^{B^{s}_{p,q}}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^{s}_{p,q}}(t) \sim t^{-\frac{1}{p}}, \quad t \to \infty. \tag{2.8}$$

For a proof of this result we refer to [43, Thms. 13.2, 15.2] concerning (i)–(ii), and to [18, Sects. 8.1, 8.3, 10.3]. In the latter book one can also find a treatment of borderline cases and a recent account of further related results.

2.2. *Continuity envelope functions.* Let C be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. Recall that the classical Lipschitz space Lip^1 is defined as the space of all functions $f \in C$ such that

$$\|f|\mathrm{Lip}^1\| = \|f|C\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t} \tag{2.9}$$

is finite, the expression (2.9) defining its norm, where $\omega(f,t)$ stands for the modulus of continuity,

$$\omega(f,t) = \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |f(x+h) - f(x)|, \quad t > 0.$$

DEFINITION 2.6. Let $X \hookrightarrow C$ be some quasi-normed function space on $\mathbb{R}^n$. The *continuity envelope function* $\mathcal{E}_{\mathsf{C}}^X : (0,\infty) \to [0,\infty]$ of X is defined by

$$\mathcal{E}_{\mathsf{C}}^X(t) = \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}, \qquad t > 0. \tag{2.10}$$

It is well-known that $\mathcal{E}_{\mathsf{C}}^X$ is equivalent to some monotonically decreasing function, that $X \hookrightarrow \mathrm{Lip}^1$ if, and only if, $\mathcal{E}_{\mathsf{C}}^X$ is bounded, and that

$$X_1 \hookrightarrow X_2 \quad \text{implies} \quad \mathcal{E}_{\mathsf{C}}^{X_1}(t) \le c\,\mathcal{E}_{\mathsf{C}}^{X_2}(t) \qquad \text{for some } c > 0 \text{ and all } t > 0; \tag{2.11}$$

we refer to [18, Ch. 5] for a more general approach and further details.

REMARK 2.7. Again, we dealt in the papers and books mentioned above usually with the so-called *continuity envelope* $\mathfrak{E}_{\mathsf{C}}(X)$ of some function space X where $\mathcal{E}_{\mathsf{C}}^X$ is equipped with some additional fine index u_{C}^X.

EXAMPLE 2.8. Let for $0 < a < 1$, $b \ge 0$, the Lipschitz spaces Lip^a, $\mathrm{Lip}^{(1,-b)}$ represent the natural extensions of (2.9): we collect all $f \in C$ such that

$$\|f|\mathrm{Lip}^a\| = \|f|C\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t^a},$$

and

$$\|f|\mathrm{Lip}^{(1,-b)}\| = \|f|C\| + \sup_{t\in(0,\frac{1}{2})} \frac{\omega(f,t)}{t|\log t|^b},$$

respectively, are finite. In [18, Sect. 5.3] we proved that

$$\mathcal{E}_{\mathsf{C}}^{\mathrm{Lip}^a}(t) \sim t^{-(1-a)}, \qquad 0 < t < 1, \tag{2.12}$$

and

$$\mathcal{E}_{\mathsf{C}}^{\mathrm{Lip}^{(1,-b)}}(t) \sim |\log t|^b, \qquad 0 < t < \tfrac{1}{2}. \tag{2.13}$$

The counterpart of Proposition 2.5 reads as follows.

PROPOSITION 2.9. *Let $0 < q \le \infty$, $0 < p < \infty$, and $\frac{n}{p} \le s \le \frac{n}{p} + 1$.*

(i) *Assume $\frac{n}{p} < s < 1 + \frac{n}{p}$. Then*

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}(t) \sim \mathcal{E}_{\mathsf{C}}^{F^s_{p,q}}(t) \sim t^{-1-\frac{n}{p}+s}, \quad 0 < t < 1. \tag{2.14}$$

(ii) *Assume $s = \frac{n}{p} + 1$ and let $1 < q \le \infty$ in case of B-spaces, and $1 < p < \infty$ in case of F-spaces, respectively. Then*

$$\mathcal{E}_{\mathsf{C}}^{B^{1+n/p}_{p,q}}(t) \sim |\log t|^{\frac{1}{q'}}, \qquad \mathcal{E}_{\mathsf{C}}^{F^{1+n/p}_{p,q}}(t) \sim |\log t|^{\frac{1}{p'}}, \qquad 0 < t < \tfrac{1}{2}. \tag{2.15}$$

(iii) *Assume $s = \frac{n}{p}$ and let $0 < q \le 1$ in case of B-spaces, and $0 < p \le 1$ in case of F-spaces, respectively. Then*

$$\mathcal{E}_{\mathsf{C}}^{B^{n/p}_{p,q}}(t) \sim \mathcal{E}_{\mathsf{C}}^{F^{n/p}_{p,q}}(t) \sim t^{-1}, \quad 0 < t < 1. \tag{2.16}$$

For a proof of this result we refer to [43, Thms. 13.2, 15.2] concerning (ii), and to [18, Ch. 9] for all cases, as well as further details and related results. Note that for B-spaces the case $p = \infty$ can be included (with the corresponding result).

3. Growth envelope functions in weighted function spaces.

Let $w \sim w_{\alpha,\beta}$ be given by (1.3) with $\alpha > -n$, $\beta > -n$ for $1 < p < \infty$. Moreover, we shall only deal with spaces on $\mathbb{R}^n$ in this section and will thus omit it from their notation.

3.1. *Weighted Lebesgue spaces.* First we recall what is already known in case of $\alpha = \beta \ge 0$. In [19] (see also [18]) we proved the following result.

PROPOSITION 3.1. *Let $1 < p < \infty$, $0 \le \alpha < n(p-1)$, and $w \sim w_{\alpha,\alpha}$ given by (1.3). Then*

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \sim t^{-\frac{\alpha}{np}-\frac{1}{p}}, \quad 0 < t < 1, \tag{3.1}$$

and

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \sim t^{-\frac{\alpha}{np}-\frac{1}{p}}, \quad t \to \infty. \tag{3.2}$$

REMARK 3.2. Note that for admissible weights v of type (1.10) the parallel result reads as

$$\mathcal{E}_{\mathsf{G}}^{L_p(v)}(t) \sim t^{-\frac{1}{p}}, \quad 0 < t < 1, \quad 0 < p < \infty, \quad \alpha \ge 0,$$

whereas the global behaviour $\mathcal{E}_{\mathsf{G}}^{L_p(v)}(t)$ for $t \to \infty$ is the same as (3.2), cf. [19].

We extend this result now to weights $w \sim w_{\alpha,\beta}$ with parameters $-n < \alpha \le \beta$, $\beta \ge 0$.

PROPOSITION 3.3. *Let $1 < p < \infty$, $\beta \ge 0$, $-n < \alpha \le \beta$, and $w \sim w_{\alpha,\beta}$ given by (1.3). Then*

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \sim t^{-\frac{\max(\alpha,0)}{np}-\frac{1}{p}}, \quad 0 < t < 1, \tag{3.3}$$

and

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \sim t^{-\frac{\beta}{np}-\frac{1}{p}}, \quad t \to \infty. \tag{3.4}$$

Proof. Step 1. In view of the construction (1.3) we obviously have

$$L_p(w_{\alpha,\beta}) \hookrightarrow L_p(w_{\gamma,\gamma}) \quad \text{if, and only if,} \quad \alpha \le \gamma \le \beta. \tag{3.5}$$

Thus (2.3) and (3.2) imply (with $\gamma = \beta \ge 0$)

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \le c t^{-\frac{\beta}{np}-\frac{1}{p}} \quad \text{for} \quad t \to \infty. \tag{3.6}$$

Moreover, if we apply (3.5) with $\gamma = \alpha_+$, then by (2.3) and (3.1),

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \le c t^{-\frac{\alpha_+}{np}-\frac{1}{p}} \quad \text{for} \quad 0 < t < 1. \tag{3.7}$$

For the converse estimates, assume first $0 \le \alpha \le \beta$. Let

$$f_s = s^{-\frac{1}{p}-\frac{\alpha}{np}} \chi_{A_s}, \quad A_s \subset \mathbb{R}^n, \quad |A_s| = s. \tag{3.8}$$

Assume, for convenience, $A_s = K_{cs^{1/n}}(0)$, the ball centered at the origin with radius $cs^{1/n}$, and c appropriately chosen such that $|A_s| = s$. Then

$$f_s^*(t) = s^{-\frac{1}{p}-\frac{\alpha}{np}} \chi_{[0,s)}(t), \quad t > 0, \tag{3.9}$$

and for $0 < s < 1$,

$$\begin{aligned} \|f_s|L_p(w_{\alpha,\beta})\| &= s^{-\frac{1}{p}-\frac{\alpha}{np}} \left(\int_{A_s} w_{\alpha,\beta}(x)\,\mathrm{d}x \right)^{1/p} \sim s^{-\frac{1}{p}-\frac{\alpha}{np}} \left(\int_0^{cs^{1/n}} |x|^\alpha \,\mathrm{d}x \right)^{1/p} \\ &\sim s^{-\frac{1}{p}-\frac{\alpha}{np}} \left(s^{\frac{1}{n}(\alpha+n)} \right)^{1/p} \sim c', \end{aligned}$$

i.e. (up to possible normalising factors) we have $\|f_s|L_p(w_{\alpha,\beta})\| \le 1$. Hence,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \ge \sup_{s>0} f_s^*(t) \ge c \sup_{s>t} s^{-\frac{1}{p}-\frac{\alpha}{np}} \sim t^{-\frac{\alpha}{np}-\frac{1}{p}}, \quad 0 < t < 1, \tag{3.10}$$

where we used (3.9) and $\alpha > -n$. If $s \to \infty$, we consider

$$g_s = s^{-\frac{1}{p}-\frac{\beta}{np}} \chi_{A_s} \quad \text{with} \quad A_s = \left\{ x \in \mathbb{R}^n : c_1 s^{\frac{1}{n}} \le |x| \le c_2 s^{\frac{1}{n}} \right\}, \tag{3.11}$$

where c_1, c_2 are chosen appropriately such that $|A_s| = s$. Consequently,

$$g_s^*(t) = s^{-\frac{1}{p}-\frac{\beta}{np}} \chi_{[0,s)}(t), \quad t > 0, \tag{3.12}$$

and

$$\|g_s|L_p(w_{\alpha,\beta})\| = s^{-\frac{1}{p}-\frac{\beta}{np}} \left(\int_{A_s} w_{\alpha,\beta}(x)\,\mathrm{d}x \right)^{1/p} \sim s^{-\frac{1}{p}-\frac{\beta}{np}} s^{\frac{\beta}{np}} |A_s|^{\frac{1}{p}} \sim c$$

since $w_{\alpha,\beta}(x) \sim |x|^\beta \sim s^{\frac{\beta}{n}}$ for $x \in A_s$ and $s >> 1$. Note that we (only) used $|A_s| = s$ in this argument. Now we can proceed as above, that is,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \ge \sup_{s>0} g_s^*(t) \ge c \sup_{s>t} s^{-\frac{1}{p}-\frac{\beta}{np}} \sim t^{-\frac{\beta}{np}-\frac{1}{p}}, \quad t \to \infty, \tag{3.13}$$

where we additionally used $\beta > -n$. In view of (3.6), (3.7) and (3.10) this concludes the proof for $0 \le \alpha \le \beta$.

Step 2. Let $-n < \alpha \le 0 \le \beta$. As already pointed out, estimate (3.13) remains valid, so that by (3.6), (3.7) it is left to prove in this case

$$\mathcal{E}_{\mathsf{G}}^{L_p(w_{\alpha,\beta})}(t) \ge c t^{-\frac{1}{p}}, \quad 0 < t < 1. \tag{3.14}$$

We modify the extremal functions (3.8) by

$$h_s = s^{-\frac{1}{p}} \chi_{A_s}, \quad A_s = K_{cs^{1/n}}(x_0), \quad |x_0| = 2, \quad 0 < s < 1, \tag{3.15}$$

and c is again chosen such that $|A_s| = s$. Then $w_{\alpha,\beta}(x) \sim 1$ for all $x \in A_s$ such that

$$\|h_s|L_p(w_{\alpha,\beta})\| \sim s^{-\frac{1}{p}} \left(\int_{A_s} w_{\alpha,\beta}(x)\, \mathrm{d}x \right)^{1/p} \sim s^{-\frac{1}{p}} |A_s|^{\frac{1}{p}} \sim c', \tag{3.16}$$

and $h_s^*(t) = s^{-\frac{1}{p}} \chi_{[0,s)}(t)$, $t > 0$. Parallel to (3.10) this leads to (3.14). ∎

REMARK 3.4. Let $-n < \alpha \le \beta < 0$. One can easily check that our proof above covers the lower estimates (3.13) and (3.14) in this case, too. So a reasonable conjecture seems that Proposition 3.3 remains true for $\alpha \le \beta < 0$, but the corresponding upper estimates are not yet proved. Furthermore, one could also consider values $\beta \le \alpha$ and benefit from the lower estimates (3.10), (3.13) and (3.14) correspondingly.

Before we come to weighted spaces of Besov and Triebel-Lizorkin type we try to extract some ideas from the above proof to formulate the result on a more abstract level.

PROPOSITION 3.5. *Let $1 < p < \infty$, $w \in \mathcal{A}_p$ and $L_p(w)$ be given by (1.4). Then there are positive constants c_1, c_2 such that for all $t > 0$,*

$$c_1 \sup_{|E|=t} \left(\int_E w(x)\, \mathrm{d}x \right)^{-1/p} \le \mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \le c_2 \sup_{|E|=t} \frac{1}{|E|} \left(\int_E w(x)^{-p'/p}\, \mathrm{d}x \right)^{1/p'}, \tag{3.17}$$

where the supremum is taken over all sets $E \subset \mathbb{R}^n$ with $|E| = t$.

Proof. We first deal with the estimate from below and adapt the 'extremal' functions (3.15) (see also (3.8) and (3.11)) appropriately. Let for $s > 0$,

$$h_s = s^{-\frac{1}{p}} \chi_{E_s}, \quad E_s \subset \mathbb{R}^n, \quad |E_s| = s, \tag{3.18}$$

where the set $E_s \subset \mathbb{R}^n$ generalises A_s from above which was chosen according to our needs, i.e., to reflect the 'typical' behaviour of our weight function. (In our case $w \sim w_{\alpha,\beta}$ this refers to $|x| \to 0$ and $|x| \to \infty$, respectively.) In general, however, we have no further information about 'appropriate' sets E_s, but at least we can conclude that

$$\|h_s|L_p(w)\| \sim s^{-\frac{1}{p}} \left(\int_{E_s} w(x)\, \mathrm{d}x \right)^{1/p}.$$

Since $h_s^*(t) = s^{-\frac{1}{p}} \chi_{[0,s)}(t)$,

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \ge \sup_s \frac{h_s^*(t)}{\|h_s|L_p(w)\|} \sim \sup_{s>t} \left(\int_{E_s} w(x)\, \mathrm{d}x \right)^{-1/p},$$

leading finally to the lower estimate in (3.17). Of course, we use $w \in \mathcal{A}_p$ here.

Conversely, let $f \in L_p(w)$ with $\|f|L_p(w)\| \le 1$. First we apply an abstract result about *resonant* measure spaces according to [2, Ch. 2, Def. 2.3, Thm. 2.7], that is,

$$\int_0^\infty f^*(t) g^*(t)\, \mathrm{d}t = \sup_{k^*=g^*} \int_{\mathbb{R}^n} |f(x)||k(x)|\, \mathrm{d}x,$$

where the supremum is taken over all function k on $\mathbb{R}^n$ which are equimeasurable with g. We choose $g = \chi_B$ with $|B| = t$, hence $g^* = \chi_{[0,t)}$, then $|k|$ is the characteristic function of some set $E_t \subset \mathbb{R}^n$ with $|E_t| = t$. Thus monotonicity leads to

$$tf^*(t) \le \int_0^t f^*(u)\,\mathrm{d}u = \sup_{|E_t|=t} \int_{E_t} |f(x)|\,\mathrm{d}x. \tag{3.19}$$

Moreover, our assumption together with Hölder's inequality implies

$$\int_{E_t} |f(x)|\,\mathrm{d}x \le \|f|L_p(w)\| \left(\int_{E_t} w(x)^{-p'/p}\,\mathrm{d}x \right)^{1/p'} \le c \left(\int_{E_t} w(x)^{-p'/p}\,\mathrm{d}x \right)^{1/p'}, \tag{3.20}$$

such that (3.19), (3.20) and Definition 2.1 lead to

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \le c \sup_{|E|=t} \frac{1}{|E|} \left(\int_E w(x)^{-p'/p}\,\mathrm{d}x \right)^{1/p'}. \ \blacksquare$$

Remark 3.6. Note that (3.17) is closely connected with the Muckenhoupt condition (1.1) for the weight $w \in \mathcal{A}_p$. More precisely, if the supremum over *all* sets $E \subset \mathbb{R}^n$ with $|E| = t$ could be replaced by the supremum over *all balls* $B \subset \mathbb{R}^n$ with $|B| = t$, then we immediately obtain

$$\mathcal{E}_{\mathsf{G}}^{L_p(w)}(t) \sim \sup_{|B|=t} \left(\int_B w(x)\,\mathrm{d}x \right)^{-1/p} = \sup_{|B|=t} \|\chi_B|L_p(w)\|^{-1} \tag{3.21}$$

as a consequence of (1.1). But this is not yet covered by the above proof. However, the examples studied so far in (2.5) ($w \equiv 1$), Propositions 3.1 ($w = w_{\alpha,\alpha}$, $\alpha \ge 0$) and 3.3 ($w = w_{\alpha,\beta}$, $\beta \ge 0$, $-n < \alpha \le \beta$) exemplify (3.17) (and also (3.21)). Moreover, observe the similarity between (2.4) and (3.21), though $L_p(w)$ is not a rearrangement-invariant space in general. Obviously the $\sup_{|B|=t}$ disappears in (2.4) (just because of the rearrangement-invariance).

3.2. *Weighted Besov and Triebel-Lizorkin spaces.* We come to weighted spaces of type $B^s_{p,q}(w)$ and $F^s_{p,q}(w)$. First we collect what is already known (in addition to the unweighted result recalled in Proposition 2.5).

Proposition 3.7. *Let* $0 < q \le \infty$, $0 < p < \infty$, $n(\frac{1}{p} - 1)_+ < s < \frac{n}{p}$, $\alpha \ge 0$, *and* $w \sim w_{\alpha,\alpha}$ *given by* (1.3).

(i) *Let* $0 \le \frac{\alpha}{p} < n - \frac{n}{p} + s$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w)}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}-\frac{\alpha}{np}}, \quad 0 < t < 1. \tag{3.22}$$

(ii) *Let* $1 < p < \infty$, *and* $0 \le \alpha < n(p-1)$. *Then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w)}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w)}(t) \sim t^{-\frac{\alpha}{np}-\frac{1}{p}}, \quad t \to \infty. \tag{3.23}$$

Remark 3.8. Again, as in the L_p case we have for admissible weights v of type (1.10) that

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(v)}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}}(t) \sim t^{-\frac{1}{p}+\frac{s}{n}}, \quad 0 < t < 1,$$

whereas the global behaviour coincides with (3.23). Moreover, we determined the corresponding growth envelopes for $B^s_{p,q}(w)$, $F^s_{p,q}(w)$ (and $B^s_{p,q}(v)$, $F^s_{p,q}(v)$) in [18].

THEOREM 3.9. *Let $0 < q \le \infty$, $0 < p < \infty$, $s > 0$, $\beta \ge 0$, $-n < \alpha \le \beta$, and $w_{\alpha,\beta}$ be given by* (1.3). *Assume that*

$$-n + \frac{\max(\alpha,0)}{p} < s - \frac{n}{p} < \frac{\max(\alpha,0)}{p}. \tag{3.24}$$

(i) *Then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}-\frac{\max(\alpha,0)}{np}+\frac{s}{n}}, \quad 0 < t < 1. \tag{3.25}$$

(ii) *Let $1 < p < \infty$, then*

$$\mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{G}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-\frac{1}{p}-\frac{\beta}{np}}, \quad t \to \infty. \tag{3.26}$$

Proof. Step 1. Note first that it is sufficient to deal with B-spaces only due to (1.11). We discuss condition (3.24) first. As mentioned in Section 2.1 briefly, the concept of growth envelopes requires $X \subset L_1^{\mathrm{loc}}$ and $X \not\hookrightarrow L_\infty$ for the underlying function space X in general. We use Proposition 1.7 in the form

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^\sigma_{\tau,q} \quad \text{and} \quad \frac{\alpha_+}{p} = \delta = s - \frac{n}{p} - \sigma + \frac{n}{\tau}, \tag{3.27}$$

and $s \ge \sigma$, $0 < p \le \tau \le \infty$, and $0 < q \le \infty$. For unweighted B-spaces it is known that $B^\sigma_{\tau,q} \subset L_1^{\mathrm{loc}}$ if $\sigma > \max(\frac{n}{\tau} - n, 0)$ (neglecting limiting cases). Thus the left-hand side of (3.24) implies that we can always suitably choose σ and τ such that (3.27) implies $B^s_{p,q}(w_{\alpha,\beta}) \subset L_1^{\mathrm{loc}}$. Moreover, we may assume that $\tau \ge \max(p,1)$. On the other hand, $B^\sigma_{\tau,q} \hookrightarrow L_\infty$ for $\sigma > \frac{n}{\tau}$ (neglecting limiting cases again) such that $s - \frac{n}{p} > \frac{\alpha_+}{p}$ and (3.27) lead to $B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow L_\infty$ (which is not interesting from our point view). Disregarding limiting cases we are left to consider parameters according to (3.24). (As a by-product of our theorem and general features of growth envelopes we shall obtain that $B^s_{p,q}(w_{\alpha,\beta}) \not\hookrightarrow L_\infty$ when (3.24) is satisfied.)

Step 2. First we deal with the estimates from above and use (3.27) together with the unweighted result Proposition 2.5. By our assumption (3.24) and the embedding (3.27) we conclude $n(\frac{1}{\tau} - 1)_+ < \sigma < \frac{n}{\tau}$ such that (2.6) reads as

$$\mathcal{E}_{\mathsf{G}}^{B^\sigma_{\tau,q}}(t) \sim t^{-\frac{1}{\tau}+\frac{\sigma}{n}}, \quad 0 < t < 1, \tag{3.28}$$

which directly leads to the corresponding upper estimate in (3.25) in view of (2.3) and (3.27). As for the global behaviour we use (3.23) and the embedding $B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^s_{p,q}(w_{\beta,\beta})$ since $\alpha \le \beta$ (recall (3.5) with $\gamma = \beta$). Thus (2.3) and (3.23) (with $\alpha = \beta$) prove the upper estimate in (3.26).

Step 3. It remains to show the converse estimates in (3.25) and (3.26). Let

$$f_j(x) = 2^{-j(s-\frac{n}{p}-\frac{\alpha_+}{p})}\psi(2^j(x-x_0)), \quad x \in \mathbb{R}^n, \tag{3.29}$$

be atoms in $B^s_{p,q}(w_{\alpha,\beta})$, where ψ is the compactly supported C^∞-function in $\mathbb{R}^n$ given by

$$\psi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & , |x| < 1, \\ 0 & , |x| \ge 1; \end{cases} \tag{3.30}$$

we refer to the atomic decomposition result in [20], [3]. Here we can choose $x_0 = 0$ for $\alpha \geq 0$, but $|x_0| \sim 2$ for $\alpha < 0$. Hence, $\|f_j|B^s_{p,q}(w_{\alpha,\beta})\| \leq c$, and

$$f_j^*(t) \sim 2^{-j(s-\frac{n}{p}-\frac{\alpha_+}{p})}\psi^*(2^{jn}t), \quad j \in \mathbb{N}.$$

One easily calculates that

$$\psi^*(t) \sim \begin{cases} e^{-\frac{1}{1-(t/|\omega_n|)^{2/n}}} & , t < |\omega_n|, \\ 0 & , t \geq |\omega_n|, \end{cases} \tag{3.31}$$

where $|\omega_n|$ denotes the (surface) measure of the unit sphere in $\mathbb{R}^n$. Let $0 < t < 1$ and choose $j_0 \in \mathbb{N}$ such that $2^{-j_0 n} \sim t$. Thus

$$\begin{aligned} \mathcal{E}_{\mathsf{G}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) &\geq c \sup_{j\in\mathbb{N}} f_j^*(t) \geq c\, f_{j_0}^*(c' 2^{-j_0 n}) \\ &\geq c''\, 2^{-j_0(s-\frac{n}{p}-\frac{\alpha_+}{p})} \sim t^{\frac{s}{n}-\frac{1}{p}-\frac{\alpha_+}{np}}, \end{aligned}$$

i.e., the inequality converse to (3.25).

Step 4. As for the global behaviour we adapt the corresponding argument appropriately in order to show the lower estimate of (3.4). Let

$$\varrho = \varphi_1 = \varphi(2^{-1}\cdot) - \varphi, \tag{3.32}$$

where φ is given by (1.6). Then, obviously, supp $\varrho \subset \{x \in \mathbb{R}^n : 1 < |x| < 4\}$, and for suitably chosen φ, $\varrho^*(|\omega_n|) \geq \frac{1}{2}$. We replace f_j given by (3.29) with

$$g_j(x) = 2^{-j\frac{n+\beta}{p}}\, \varrho(2^{-j}x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}, \tag{3.33}$$

and obtain

$$g_j^*(t) \sim 2^{-j\frac{n+\beta}{p}}\ \varrho^*(2^{-jn}t) \geq c 2^{-j\frac{n+\beta}{p}} \quad \text{for} \quad t \sim 2^{jn}, \quad j \in \mathbb{N}.$$

This will immediately conclude the proof granted that we can show that $g_j \in B^s_{p,q}(w_{\alpha,\beta})$ with $\|g_j|B^s_{p,q}(w_{\alpha,\beta})\| \leq c$ uniformly in $j \in \mathbb{N}$.

Let k be some compactly supported C^∞ function on $\mathbb{R}^n$ with

$$\sum_{m\in\mathbb{Z}^n} k(x-m) = 1, \quad x \in \mathbb{R}^n.$$

Then we have for all $x \in \mathbb{R}^n$, $j \in \mathbb{N}$,

$$g_j(x) = 2^{-j\frac{n+\beta}{p}} \sum_{m\in\mathbb{Z}^n} k(x-m)\ \varrho(2^{-j}x) \sim 2^{-j\frac{n+\beta}{p}} \sum_{|m|\sim 2^j} k(x-m)\varrho(2^{-j}x). \tag{3.34}$$

On the other hand, $a_{0m}(x) = k(x-m)\varrho(2^{-j}x)$ can be regarded as 1_K-atoms located near Q_{0m}, $m \in \mathbb{Z}^n$, and thus (3.34) represents a special atomic representation of g_j. By the already mentioned atomic decomposition with Muckenhoupt weights, [20], [3], this yields

$$\begin{aligned} \|g_j|B^s_{p,q}(w_{\alpha,\beta})\|^p &\leq c_1\ 2^{-j(n+\beta)} \Big\| \sum_{|m|\sim 2^j} \chi_{0m}|L_p(w_{\alpha,\beta})\Big\|^p \\ &\leq c_2 2^{-j(n+\beta)} \int_{\mathbb{R}^n} \Big(\sum_{|m|\sim 2^j} \chi_{0m}(x)\Big)^p |x|^\beta\ \mathrm{d}x \end{aligned}$$

$$\le c_3 2^{-j(n+\beta)} \sum_{|m|\sim 2^j} \int_{Q_{0m}} |x|^\beta \, dx$$
$$\le c_4 2^{-j(n+\beta)} \sum_{|m|\sim 2^j} |m|^\beta \le c_5,$$

where χ_{0m} stands for the characteristic function of Q_{0m}, $m \in \mathbb{Z}^n$. This completes the argument. ∎

Remark 3.10. The result naturally extends Proposition 3.7 from $\alpha = \beta \ge 0$ to $\beta \ge 0$, $-n < \alpha \le \beta$. Apart from the already mentioned limiting cases in (3.24), i.e., $s = \frac{n+\alpha_+}{p}$ or $s = \frac{n+\alpha_+}{p} - n$, respectively, further parameter settings, e.g., $\alpha \le \beta < 0$ or $\beta \le \alpha$ deserve attention. Even more desirable, however, would be a general result similar to Proposition 3.17. But this will be studied elsewhere. Let us finally remark, that the additional restriction $p > 1$ in (ii) is a consequence of the (method of the) proof and probably not necessary, see also the unweighted result (2.8). Moreover, one can prove for the corresponding indices that $u_{\mathsf{G}}^X = q$ if $X = B^s_{p,q}(w_{\alpha,\beta})$ and $u_{\mathsf{G}}^X = p$ if $X = F^s_{p,q}(w_{\alpha,\beta})$. Note, finally, that the unboundedness of $\mathcal{E}_{\mathsf{G}}^X(t)$ for $t \to 0$ in (3.25) implies $B^s_{p,q}(w_{\alpha,\beta}) \not\hookrightarrow L_\infty$, $F^s_{p,q}(w_{\alpha,\beta}) \not\hookrightarrow L_\infty$ in case of (3.24).

4. Continuity envelope functions in weighted function spaces. We finally deal with continuity envelopes for spaces $B^s_{p,q}(w)$, $F^s_{p,q}(w)$ with $w \in \mathcal{A}_\infty$. Nothing is known so far in such weighted situations; we rely on the unweighted results Proposition 2.9 and the techniques developed above.

Theorem 4.1. *Let $0 < q \le \infty$, $0 < p < \infty$, $\beta \ge 0$, $-n < \alpha \le \beta$, and $w_{\alpha,\beta}$ be given by (1.3). Assume that*

$$\frac{\max(\alpha,0)}{p} < s - \frac{n}{p} < \frac{\max(\alpha,0)}{p} + 1. \tag{4.1}$$

Then

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{C}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-1+s-\frac{n}{p}-\frac{\max(\alpha,0)}{p}}, \quad 0 < t < 1. \tag{4.2}$$

Proof. Step 1. Again it is sufficient to deal with B-spaces only due to (1.11). We discuss condition (4.1) first. As mentioned in Section 2.2, the concept of continuity envelopes requires $X \hookrightarrow C$ and $X \not\hookrightarrow \mathrm{Lip}^1$ for the underlying function space X in general. We use (3.27) and the corresponding unweighted result. Since $B^\sigma_{\tau,q} \hookrightarrow C$ if $\sigma > \frac{n}{\tau}$ and $B^\sigma_{\tau,q} \hookrightarrow \mathrm{Lip}^1$ for $\sigma - \frac{n}{\tau} > 1$ (neglecting limiting cases) we can argue as in Step 1 of the proof of Theorem 3.9 and substantiate (4.1).

Step 2. First we deal with the estimates from above and use (3.27) together with the unweighted result Proposition 2.9(i) for $B^\sigma_{\tau,q}$,

$$\mathcal{E}_{\mathsf{C}}^{B^\sigma_{\tau,q}}(t) \sim t^{-1+\sigma-\frac{n}{\tau}}, \quad 0 < t < 1, \tag{4.3}$$

which directly leads to the corresponding upper estimate in (4.2) in view of (2.11) and (3.27). It remains to show the converse estimate in (4.2). Let

$$f_j(x) = 2^{-j(s-\frac{n}{p}-\frac{\alpha_+}{p})} \varphi(2^j(x-x_0)), \quad j \in \mathbb{N}, \tag{4.4}$$

where φ is the mollified version of

$$\widetilde{\varphi}(x) = \begin{cases} 0, & |x| \geq 1, \\ 1-|x|, & |x| \leq 1, \end{cases} \qquad x \in \mathbb{R}^n,$$

such that $\operatorname{supp} \varphi(2^j\cdot) \subset \{y \in \mathbb{R}^n : |y| \leq c2^{-j}\}$, $j \in \mathbb{N}$, and

$$\frac{\omega(\varphi(2^j\cdot),t)}{t} \sim 2^j, \qquad t \sim 2^{-j}, \quad j \in \mathbb{N}.$$

Then f_j given by (4.4) is a $B^s_{p,q}(w_{\alpha,\beta})$-atom (as we do not need moment conditions). Again we can choose $x_0 = 0$ for $\alpha \geq 0$, but $|x_0| \sim 2$ for $\alpha < 0$. In particular, this means $\|f_j|B^s_{p,q}(w_{\alpha,\beta})\| \sim 1$. Since

$$\frac{\omega(f_j,t)}{t} \sim 2^{-j(s-\frac{n}{p}-\frac{\alpha_+}{p}-1)}, \qquad t \sim 2^{-j}, \quad j \in \mathbb{N},$$

hence

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}(2^{-j}) \geq c\,\frac{\omega(f_j,2^{-j})}{2^{-j}} \geq c' 2^{-j(s-\frac{n}{p}-\frac{\alpha_+}{p}-1)}, \quad j \in \mathbb{N},$$

and by standard arguments the proof of (4.2) is complete. ■

Remark 4.2. In case of Besov spaces one can extend the result to $p = \infty$ which coincides with the unweighted result [18, Prop. 9.1], recall (1.5). There are counterparts for *admissible* weights in the sense of Remark 1.6: One can prove, for instance, that

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(v)}(t) \sim \mathcal{E}_{\mathsf{C}}^{B^s_{p,q}}(t), \quad 0 < t < 1,$$

and the counterpart for F-spaces where v is given by (1.10). Moreover, the indices in those cases are given by $u_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})} = u_{\mathsf{C}}^{B^s_{p,q}(v)} = q$ and $u_{\mathsf{C}}^{F^s_{p,q}(w_{\alpha,\beta})} = u_{\mathsf{C}}^{F^s_{p,q}(v)} = p$.

Remark 4.3. Let us finally remark that envelope result can be applied to obtain Hardy-type inequalities, to establish criteria for limiting embeddings, and, last but not least, to prove surprisingly sharp upper estimates for the asymptotic behaviour of approximation numbers of related compact embeddings. This will not be presented here; we refer to [18, Ch. 11] for further details and references.

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