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SOME REMARKS PROVIDING DISCONTINUOUS MAPS ON SOME $C_p(X)$ SPACES

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Abstract. Let X be a completely regular Hausdorff topological space and $C_p(X)$ the space of continuous real-valued maps on X endowed with the pointwise topology. A simple and natural argument is presented to show how to construct on the space $C_p(X)$, if X contains a homeomorphic copy of the closed interval $[0,1]$, real-valued maps which are everywhere discontinuous but continuous on all compact subsets of $C_p(X)$.

A topological space X is called a k_R*-space* if each real-valued map on X which is continuous on each compact subset of X is continuous. In [6] it has been proved that if X is a Lindelof Čech complete space, then $C_p(X)$ is a k_R-space iff X is scattered. Very recently this result has been extended by Cascales and Namioka in [3] to K-analytic spaces X. Hence, if X is a compact non-scattered space, one gets on $C_p(X)$ noncontinuous real-valued maps which are continuous on all compact subsets of $C_p(X)$. We refer the reader to [1], [2] for some other known results of this type. It is well-known that a linear map between topological vector spaces E and F is either continuous everywhere or discontinuous at each point; for the background we refer the reader to [5]. Any construction of everywhere discontinuous nonlinear maps which are everywhere sequentially continuous seems to be non-trivial. Even less evident is a construction of a real-valued

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map which is everywhere discontinuous but continuous on each compact subset. The aim of this self-contained note is to present an elementary argument which provides such maps on $C_p(X)$. Although the reader could also deduce the Proposition below reading carefully the proof of Corollary 4.2 ((vi) $\Rightarrow$ (b)) of [3], we decided to present this direct argument.

PROPOSITION. *Let X be a compact space with a nonzero nonnegative regular Borel probability λ which vanishes on singletons of X. Then there exists a map $T : C_p(X) \to \mathbb{R}$ discontinuous at each point of $C_p(X)$ but continuous on every compact subset Y of E.*

We will use the following fact, see also Proposition 3.29 and Corollary 12.2 of [4].

LEMMA. *Let X be a countably compact topological space and H a relatively compact countable subset of $C_p(X)$. Then $\overline{H}$ is metrizable.*

Cascales and Namioka [3, Theorem 4.1 and Corollary 4.2] have proved that a compact space X is scattered iff the closure (in $\mathbb{R}^X$) of any countable subset of $C_p(X)$ is metrizable.

Proof. Let $\mathcal{T}$ be the weakest (pseudometrizable) topology on X that makes continuous the elements of H and let D be a countable $\mathcal{T}$ dense subset of X.

If $f \in \overline{H}$ is not $\mathcal{T}$-continuous at $s \in X$, then there exists some open neighbourhood U of $f(s)$ such that $f(W) \nsubseteq U$ for each $\mathcal{T}$-neighbourhood W of s.

Then, if $f_1 \in H$ such that $|f_1(s) - f(s)| < 2^{-1}$, there exists $s_1 \in X$ such that $|f_1(s) - f_1(s_1)| < 2^{-1}$. But $f(s_1) \notin U$. An easy induction provides sequences (f_n) in H and (s_n) in X such that $|f_n(s) - f(s)| < 2^{-n}$, $|f_n(s_m) - f(s_m)| < 2^{-n}$, for $m < n$, $|f_p(s) - f_p(s_n)| < 2^{-n}$, for $p \leq n$, and $f(s_n) \notin U$ for each $n \in \mathbb{N}$. Let t and g be accumulation points of (s_n) and (f_n), respectively. Then $g(s) = f(s)$, $g(t) = f(t)$, $g(s) = g(t)$ and $f(t) \notin U$. The contradiction $f(s) \notin U$ implies that f is $\mathcal{T}$-continuous.

The topology $\mathcal{T}_D$ of pointwise convergence in D induces a metrizable topology on $\overline{H}$ because if $f, g \in \overline{H}$ and $f(x) = g(x)$ when $x \in D$, then the $\mathcal{T}$-continuity implies that $f(x) = g(x)$ for each $x \in X$. It is clear, by compactness, that on $\overline{H}$ the topology induced by $C_p(X)$ is equal to the metrizable topology induced by $\mathcal{T}_D$. ■

Now we are ready to prove the Proposition.

Proof of the Proposition. As $C_p(X)$ is homeomorphic to $C_p(X,(0,1))$ it is enough to prove that the map $T : C_p(X,(0,1)) \to \mathbb{R}$ defined by $T(f) := \int_X f d\lambda$ is discontinuous at each point of $E := C_p(X,(0,1))$ but it is continuous on every compact subset Y of E. For $f \in E$ one has $T(f) = \int_X f d\lambda = \alpha > 0$. If T is continuous at f and $0 < \varepsilon < 1$, we find a finite subset $\{x_j : 1 \leq j \leq n\} \subset X$ such that $2^{-1}\alpha < T(h)$ for each $h \in E$ verifying that $|f(x_j) - h(x_j)| < \epsilon$, when $1 \leq j \leq n$. As $\mu(x_j) = 0$ there exist open disjoint neighbourhoods V_j of x_j, $1 \leq j \leq n$, such that $\lambda(V_j) < (4n)^{-1}\alpha$. Let $k = \max\{f(x_j), 1 \leq j \leq n\}$ and let $g \in C(X,[0,k])$ be such that $g(x_j) = f(x_j)$, $1 \leq j \leq n$, and $g(X - \bigcup_{j=1}^{n} V_j) = \{0\}$. Fix $0 < \beta < \min\{2^{-1}(1-k), 4^{-1}\alpha, \varepsilon\}$. Then $h = g + \beta \in E$ satisfies $|f(x_j) - h(x_j)| < \epsilon$, $1 \leq j \leq n$, and $T(h) < T(g) + \beta < 4^{-1}\alpha + 4^{-1}\alpha = 2^{-1}\alpha$, which provides a contradiction.

Sequential continuity of T follows from the Lebesgue dominated convergence theorem and the rest of the Proposition easily follows from the claim below.

Claim. *If $F \subset Y$ and $f \in \overline{F}$, then f is the limit of some sequence in F.*

Indeed, if $p \in \mathbb{N}$ and $x = (x_i) \in X^p$ there exists $g_x \in F$ such that $|f(x_i) - g_x(x_i)| < p^{-1}$, $1 \leq i \leq p$, and therefore there exists an open neighbourhood V_x of x such that $|f(y_i) - g_x(y_i)| < p^{-1}$, $1 \leq i \leq p$, when $(y_i) \in V_x$. By compactness there exists a finite subset S_p in the product X^p such that $X^p = \bigcup_{x \in S_p} V_x$. Let $F_p := \{g_x : x \in S_p\}$. Then $f \in \overline{H}$, where $H = \bigcup_{p \in \mathbb{N}} F_p$. Now the claim is a simple consequence of the preceding lemma. ∎

We complete with the following corollary, which can also be deduced from the proof of [3], Corollary 4.2 ((vi) ⇒ (b)).

Corollary. *If a completely regular Hausdorff space X contains a homeomorphic copy K of $[0, 1]$, then $C_p(X)$ admits a discontinuous everywhere real-valued map which is continuous on every compact subset of $C_p(X)$.*

Proof. Proposition applies for K endowed with the induced Lebesgue measure; let $T : C_p(X) \to \mathbb{R}$ be the map considered in the Proposition. The restriction map $T_1 : f \to f|K$, $f \in C(X)$, is a continuous and open map of $C_p(X)$ onto $C_p(K)$. Then the composition $Q := T \circ T_1$ is everywhere discontinuous on $C_p(X)$ but continuous on every compact subset of $C_p(X)$. This provides a map as required. ∎

Let X be a completely regular Hausdorff space, $E := C_p(X)$ and $DK(E)$ the set of everywhere discontinuous real-valued maps on E which are continuous on every compact subset of E. We have seen that $DK(E) \neq \emptyset$ if X contains a copy of $[0, 1]$. Although algebraically the set $DK(E)$ might be large, we note that if $DK(E) \neq \emptyset$ and X contains an infinite compact subset, then it is straightforward to show that $DK(E)$ is a dense subset of $\mathbb{R}^E$ of the first Baire category (so topologically it is small). We refer the reader, for example, to [1], the proof of Theorem 1.3.4 with possible obvious changes. The same technique can be used to show that if X contains a non-trivial convergent sequence then the set of everywhere discontinuous real-valued maps on E which are sequentially continuous is also of first Baire category.

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