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GENERALIZED GRADIENTS FOR LOCALLY LIPSCHITZ INTEGRAL FUNCTIONALS ON NON-L^p-TYPE SPACES OF MEASURABLE FUNCTIONS

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Abstract. Let (Ω, μ) be a measure space, E be an arbitrary separable Banach space, $E^*_{\omega^*}$ be the dual equipped with the weak* topology, and $g : \Omega \times E \to \mathbb{R}$ be a Carathéodory function which is Lipschitz continuous on each ball of E for almost all $s \in \Omega$. Put $G(x) := \int_\Omega g(s, x(s))d\mu(s)$. Consider the integral functional G defined on some non-L^p-type Banach space X of measurable functions $x\colon \Omega \to E$. We present several general theorems on sufficient conditions under which any element $\gamma \in X^*$ of Clarke's generalized gradient (multivalued C-subgradient) $\partial_C G(x)$ has the representation $\gamma(v) = \int_\Omega \langle \zeta(s), v(s)\rangle d\mu(s)\, (v \in X)$ via some measurable function $\zeta\colon \Omega \to E^*_{w^*}$ of the associate space X' such that $\zeta(s) \in \partial_C g(s, x(s))$ for almost all $s \in \Omega$. Here, given a fixed $s \in \Omega$, $\partial_C g(s, u_0)$ denotes Clarke's generalized gradient for the function $g(s, \cdot)$ at $u_0 \in E$. Concerning X, we suppose that it is either a so-called non-solid Banach M-space (in particular, non-solid generalized Orlicz space) or Köthe–Bochner space (solid space).

Introduction. The purpose of the present paper is to formulate and prove several general results (see Theorems 3.2-3.3, Remark 3.4 in Section 3, Theorems 4.2-4.5 in Section 4) for the calculation of Clarke's generalized gradients (also called multivalued C-

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subgradients in nonsmooth analysis) of locally Lipschitz integral functionals. We succeed to obtain the results for these functionals defined on a non-L^p-type space X of measurable functions which is either a so-called non-solid Banach M-space (in particular, non-solid generalized Orlicz space) or Köthe–Bochner space (solid space).

Theorem 4.2 in the case of the solid space X is a generalization of the L^p-result of F. H. Clarke, J. P. Aubin/F. H. Clarke in 1976-1983 [5, 11, 12]. Note that this L^p-result is intensively used in the theory of partial differential inclusions and non-smooth mechanics (see, e.g., [9, 16, 23, 24]).

In Section 6 we show that Theorem 4.2 with its proof in a very special case for the solid regular Orlicz–Bochner space $X = L^\Phi(E)$ implies an alternative proof for the L^Φ-result of R. Płuciennik/S. Tian/Y. Wang in 1990 [36, Theorem 2]. We observe that the L^p-result of F. Clarke under his condition (A) (see [12, Theorems 2.7.5, 2.7.3]) can be standardly generalized to the case of Lipschitz integral functionals defined on a Köthe–Bochner space X (solid space), and so we shall not present this generalization herein. It turns out that the L^p-result of F. Clarke [12, Theorem 2.7.5] under his condition (B) together with its proof given in [12, Proofs of Theorems 2.7.5, 2.7.2] can be generalized in several directions via introducing the so-called U-property such as in Lemmas 3.1, 4.1 and via using verifiable "majorant" conditions such as in Theorems 3.2–3.3, 4.2–4.3. Analyzing the above conditions (A)-(B), we propose the "majorant" conditions $(K6)$–$(K7)$ and $(K8)$–$(K9)$ in Theorems 4.4–4.5 and we need not apply Lebourg's mean value theorem [12, Theorem 2.3.7] for proving Theorems 4.4–4.5.

By the non-smooth variational methods [9, 24], the calculation formula for Clarke's generalized gradient established in Corollary 6.2 of Theorems 4.3, 4.5, can be applied to the solvability problem for the Dirichlet elliptic inclusion in $\mathbb{R}^2$ involving exponential-growth-type multivalued right-hand side in the Orlicz space L^{Φ_0}, where $\Phi_0(\alpha) = \exp(\alpha^2) - 1$ (see details in our paper [32]). Observe that the above mentioned L^Φ-result of [36, Theorem 2] cannot be applied to this problem, since the "exponential-type" function Φ_0 does not satisfy the condition (Δ) of [36]. The above inclusion in L^{Φ_0} was studied by topological methods in [1, 2, 30, 31].

In Section 1 we give some terminology and auxiliary facts for Banach lattices, Köthe–Bochner spaces, Banach M-spaces, and Clarke's generalized gradients. In Section 2 we give the auxiliary Lemmas 2.1, 2.3, and then the known Lemma 2.4 for convex integral functionals defined on these spaces. In Section 5 we give Corollaries 5.1–5.2 of Theorems 3.2–3.3 for the case of the non-solid generalized Orlicz space $X = L^M(\Omega, \mathbb{R}^m)$ with $m \geq 2$. In Section 6 we give Corollary 6.2 of Theorems 4.3, 4.5 for the solid Orlicz–Bochner space $X = L^\Phi(E)$.

1. Some terminology and auxiliary facts. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space with a complete σ-finite σ-additive measure μ on some σ-algebra $\mathfrak{A}$ of subsets of Ω. Throughout this paper E denotes a separable Banach space. Denote [3] by $\mathrm{Cl}(E)$ (resp., $\mathrm{Cp}(E)$, $\mathrm{Cv}(E)$, $\mathrm{CvCp}(E)$, etc.) the family of all nonempty closed (resp., compact, convex, convex compact, etc.) subsets of E. By $\overline{\mathrm{bco}}\, A$ we denote the *balanced closed convex hull of* $A \subset E$. Put $B_E(u, r) := \{\omega \in E : \|\omega - u\|_E \leq r\}$ for $r \in (0, \infty)$. Given a Suslin locally convex space F, we denote by $\mathcal{B}(F)$ the σ-algebra of Borel subsets of F. Then a multifunction $\Gamma\colon \Omega \to 2^F$ is called (see, e.g., [8, 13, 15]) measurable if $\Gamma^-(C) := \{s : \Gamma(s) \cap C \neq \emptyset\} \in \mathfrak{A}$ for $C \in \mathcal{B}(F)$. $\mathrm{Sel}\,\Gamma$ denotes the set of all measurable selections of Γ. The theory of measurable selections is given in [8, 13, 15] (in particular, for $\Gamma\colon \Omega \to 2^{E^*_{w^*}}$, since (by [8, p. 198], [7, p. 4,11]) the dual space $E^*_{w^*}$ with the topology $w^* = \sigma(E^*, E)$ is a Lusin space for the case of separability of E).

Given a function $x\colon \Omega \to E$ and a multifunction $H\colon \Omega \times E \to \mathrm{Cp}(E^*_{\omega^*})$, define $N_H(x) := \mathrm{Sel}\, H(\cdot, x(\cdot))$. General theorems on the boundedness of the (Nemytskij) multivalued superposition operator N_H can be found in [3, 4, 26, 34, 39]. We say that *the Filippov implicit function property is valid for the multifunction* H if given any measurable function $\xi\colon \Omega \to E^*_{w^*}$ and any measurable multifunction $\mathbf{b}\colon \Omega \to \mathrm{Cp}(E)$ with $\xi(s) \in H(s, \mathbf{b}(s))$ almost everywhere (a.e.), there exists a measurable function $z\colon \Omega \to E$ with $z(s) \in \mathbf{b}(s)$ and $\xi(s) \in H(s, z(s))$ a.e. The multifunction H is called $(\mathfrak{A} \times \mathcal{B}(E), \mathcal{B}(E^*_{w^*}))\,(\mathrm{mod}\ 0)$-measurable if exists Ω_0 with $\mu(\Omega \backslash \Omega_0) = 0$ such that H is $(\mathfrak{A} \times \mathcal{B}(E), \mathcal{B}(E^*_{w^*}))$-measurable on $\Omega_0 \times E$. The multifunction H is called *multi-superpositionally measurable* if the multifunction $s \mapsto N_H(\mathbf{c})(s) := H(s, \mathbf{c}(s))$ is measurable for any measurable multifunction $s \mapsto \mathbf{c}(s) \in \mathrm{Cp}(E)$. Many sufficient conditions for the above properties for H can be found e.g. in [2, 8, 16, 42]. By Filippov's theorem [2, Theorem 6.1] *all the above properties are valid for any Carathéodory multifunction* $H\colon \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^m)$.

Further, $L^0(\Omega, F)$ denotes the space of all (equivalent classes of) measurable functions $x\colon \Omega \to F$. A Banach space $K \subset L^0(\Omega, \mathbb{R})$ with norm $\|\cdot\|_K$ is called [4, 22, 41] a Banach lattice with monotone norm (also under the name, Köthe space, Banach ideal space), if $x \in K$ and $y \in L^0(\Omega, \mathbb{R})$ and $|y(s)| \leq |x(s)|$ a.e., then $y \in K$ and $\|y\|_K \leq \|x\|_K$. Put $K_+ := \{x \in K : x(s) \geq 0 \text{ a.e.}\}$. We shall use only K with $\mathrm{supp}\, K = \Omega$. Given a Banach lattice $K \subset L^0(\Omega, \mathbb{R})$, define [21, 41] the Köthe–Bochner space $X = K(E) \subset L^0(\Omega, E)$ as the Banach space of all measurable functions $x\colon \Omega \to E$ such that $\|x(\cdot)\|_E \in K$, with norm $\|x\|_X := \|\|x(\cdot)\|_E\|_K$. Let L^∞ be the Banach algebra of all essentially bounded measurable scalar-valued functions defined on Ω. A Banach space $X \subset L^0(\Omega, E)$ with norm $\|\cdot\|_X$ is called [29] a Banach M-space (or Banach L^∞-module [27, 28, 33], or ideal* space [41]) if $x \in X$ and $\alpha \in L^\infty$ imply that $\alpha x \in X$ and $\|\alpha x\|_X \leq \|\alpha\|_{L^\infty} \|x\|_X$. If the above condition is valid only for $\alpha \in (L^\infty)_+$, then X is called a *Banach* M_+*-space.*

The spaces K and $K(E)$ are called *solid spaces* (see, e.g., [35] and references cited therein). Spaces of measurable functions which are not from the classes of the above spaces K and $K(E)$ are usually called *non-solid spaces.* Historical comments on different classes of non-solid spaces can be found in [6, 20, 25, 35].

A prominent example of a Banach M-space is the generalized (non-solid in general for the case $m \geq 2$) Orlicz space $L^M = L^M(\Omega, \mathbb{R}^m) \subset L^0(\Omega, \mathbb{R}^m)$ with the Luxemburg norm $\|x\|_{L^M} = \inf\{\lambda > 0 : \int_\Omega M(x(s)/\lambda) d\mu(s) \leq 1\} < \infty$, where $M\colon \mathbb{R}^m \to [0, \infty)$ is a given Young (even, convex) function (see more general cases of M defined on $\Omega \times E$ with $\dim E \geq 2$, e.g. in [10, 14, 17, 18, 19, 20, 25, 26, 35, 40]).

From now on, we suppose that $X = X(\Omega, E)$ satisfies one of the following conditions (cf. [6, 20, 35]):

(M_m) $E = \mathbb{R}^m, m \in \mathbb{N}, X \subset L^0(\Omega, E)$ is a Banach M-space with $\operatorname{supp} X = \Omega$;

(M_∞) $\dim E = +\infty$, $X \subset L^0(\Omega, E)$ is a Banach M_+-space and exist $\alpha, \beta \in L^0(\Omega, (0,\infty))$ such that $L^\infty(\Omega, E; \alpha) \subset X \subset L^1(\Omega, E; \beta)$ continuously, where $L^\infty(\Omega, E; \alpha)$ is equipped with norm $\|x\|_{L^\infty(\alpha)} := \|\frac{x}{\alpha}\|_{L^\infty(E)}$ and $L^1(\Omega, E; \beta)$ is equipped with norm $\|x\|_{L^1(\beta)} := \|\frac{x}{\beta}\|_{L^1(E)}$.

Under the condition (M_m) with $m \geq 2$, the associate space $X' = X'(\Omega, \mathbb{R}^m)$ is defined in [27, 28, 33] by

$$X' := \{x' \in L^0(\Omega, \mathbb{R}^m) : x'(s) \in \operatorname{vsupp} X(s) \text{ a.e.}, \langle x, x'\rangle \in \mathbb{R}\, (\forall x \in X)\}, \tag{1.1}$$

where $\langle x, x'\rangle := \int_\Omega \langle x(s), x'(s)\rangle d\mu(s)$ and the *vector support* $\operatorname{vsupp} X$ (always exists) can be defined by $\operatorname{vsupp} X(s) :=$ the closure of $\{x_1(s), x_2(s), \ldots\}$ a.e. for some sequence $x_n \in X$ such that $x \in X \Rightarrow x(s) \in$ the closure of $\{x_1(s), x_2(s), \ldots\}$ a.e. Under the condition (M_∞), the associate space $X' = X'(\Omega, E^*_{w^*})$ is defined as

$$X' := \{x' \in L^0(\Omega, E^*_{w^*}) : \langle x, x'\rangle \in \mathbb{R}\, (\forall x \in X)\}. \tag{1.2}$$

It is known that X' can be interpreted as a Banach subspace of the dual space X^* by the injection $x' \mapsto \langle \cdot, x'\rangle$ with norm $\|x'\|_{X'} := \sup\{\langle x, x'\rangle : \|x\|_X \leq 1\} < \infty$. If $X = K(E)$ is the Köthe–Bochner space, then $X' = (K(E))' = K'(E^*_{w^*})$, where $K'(E^*_{w^*})$ is the Köthe–Bochner space (modelled by means of K') of measurable functions $y\colon \Omega \to E^*_{w^*}$ (if in addition E^* is separable, then $X' = K'(E^*)$). If $X = L^M$, then $X' = L^{M^*}$ with equivalent norms (in the case of M defined on E with $\dim E = +\infty$, the convex dual M^* is defined on $E^*_{w^*}$).

Let U be an open subset of a Banach space E. If $f\colon U \to \mathbb{R}$ is Lipschitz continuous on U, then f has *Clarke's generalized derivative* $f^\circ(x; \cdot)$:

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda} \quad (v \in E). \tag{1.3}$$

The set $\partial_C f(x) = \{\zeta \in E^* : \langle \zeta, v\rangle \leq f^\circ(x; v)\ (\forall v \in E)\}$ is called the *generalized gradient (Clarke's C-subgradient) of f at x* (then [12, 23] $\partial_C f(x) \in \operatorname{CvCp}(E^*_{w^*})$). A function f is said [12, Definition 2.3.4] to be *regular in Clarke's sense at x* if the directional derivative $f'(x; v)$ of f at x along v exists and $f'(x; v) = f^\circ(x, v)$ for every $v \in E$.

2. Auxiliary lemmas. If $g\colon \Omega \times E \to \mathbb{R}$ is locally Lipschitz continuous with respect to the second variable, then $g^\circ(s, u_0; v)$ denotes the Clarke derivative at u_0 in direction v of the function $u \mapsto g(s, u)$. By [12, 23] the function $g^\circ(s, u; v)$ is continuous in v. For the simplicity, $\partial_C g(s, u_0)$ denotes *the generalized gradient at u_0 of $g(s, \cdot)$*. The proofs of Lemmas 2.1, 2.3 are standard via the known measurable selection theorems [8] (for proving Lemma 2.3 one need else Lebourg's theorem [12, Theorem 2.3.7]).

LEMMA 2.1. *Let $g\colon \Omega \times E \to \mathbb{R}$ be a function such that $g(\cdot, u)$ is measurable for any $u \in E$ and $g(s, \cdot)$ is Lipschitz continuous on each ball of E for almost all $s \in \Omega$. Then, given any measurable functions $x, v : \Omega \to E$, the function $s \in \Omega \mapsto g^\circ(s, x(s); v(s))$ is measurable.*

THEOREM 2.2 (Lebourg [12, Theorem 2.3.7]). *Let* $f: U \to \mathbb{R}$ *be Lipschitz continuous on an open subset* U *of* E. *Assume that* U *contains the convex interval* $[x, y]$. *Then, there exists a point* $u \in (x, y)$ *such that* $f(x) - f(y) \in \langle \partial_C f(u), x - y \rangle$.

LEMMA 2.3. *Let* g *be as in Lemma 2.1 and* $x, y : \Omega \to E$ *be two measurable functions. Suppose that the multifunction* $\Lambda: \Omega \times [0,1] \to \mathrm{CvCp}(\mathbb{R})$, $(s, \alpha) \mapsto \Lambda(s, \alpha) := \langle \partial_C g(s, \alpha x(s) + (1-\alpha) y(s)), x(s) - y(s) \rangle$ *is* $(\mathfrak{A} \times \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))\,(\mathrm{mod}\ 0)$*-measurable. Then there exist measurable functions* $u: \Omega \to E$ *and* $\xi: \Omega \to E^*_{w^*}$ *such that* $u(s) \in [x(s), y(s)]$, $\xi(s) \in \partial_C g(s, u(s))$ *and* $g(s, x(s)) - g(s, y(s)) = \langle \xi(s), x(s) - y(s) \rangle$ *a.e.*

Proof. We divide the proof into Steps 2.1-2.3.

Step 2.1: Given a fixed measurable function $z: \Omega \to E^*_{\omega^*}$, the multifunction $M: \Omega \to \mathrm{CvCp}(E^*_{\omega^*}) \cup \{\emptyset\}, M(s) := \partial_C g(s, z(s))$ has $\mathrm{dom}\, M = \Omega\,(\mathrm{mod}\ 0)$. By Lemma 2.1, the function $s \mapsto g^\circ(s, z(s); v) = \sup\{\langle u^*, v \rangle : u^* \in \partial_C g(s, z(s))\}$ is measurable for all $v \in E$, and so, by [8], M is measurable.

Step 2.2: Let

$$H(s) := \{u \in [x(s), y(s)] : g(s, x(s)) - g(s, y(s)) \in \langle \partial_C g(s, u), x(s) - y(s) \rangle\},$$
$$F: \Omega \times [0,1] \to \mathrm{CvCp}(\mathbb{R}), F(s, \alpha) := \{g(s, x(s)) - g(s, y(s)) - r : r \in \Lambda(s, \alpha)\}.$$

By the assumption, we deduce that F is $(\mathfrak{A} \times \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))\,(\mathrm{mod}\ 0)$-measurable and so $F^-(\{0\}) \in (\mathfrak{A} \times \mathcal{B}(\mathbb{R}))\,(\mathrm{mod}\ 0)$. On other hand, by Lebourg's Theorem 2.2 for a.a. $s \in \Omega$ there exists $\alpha \in [0,1]$ such that $0 \in F(s, \alpha)$, i.e. there exists $\Omega_0 \subset \Omega$ such that $\mu(\Omega \backslash \Omega_0) = 0$ and $\mathrm{Proj}_{\Omega_0} F^-(\{0\}) = \Omega_0$. Hence, by the von Neumann–Aumann selection theorem [8, Theorem III.22] the multifunction $\widetilde{H}: \Omega_0 \to 2^{[0,1]} \backslash \{\emptyset\}$, $\widetilde{H}(s) := \{\alpha : 0 \in F(s, \alpha)\}$ (whose graph coincides with $F^-(\{0\})$) has a measurable selector $a: \Omega_0 \to [0,1]$. Then $s \mapsto u(s) := a(s)x(s) + (1 - a(s))y(s)$ is a measurable selector of the multifunction H on Ω_0.

Step 2.3: By Lebourg's Theorem 2.2,

$$\triangle(s) := \{u^* \in E^* : g(s, x(s)) - g(s, y(s)) = \langle u^*, x(s) - y(s) \rangle\} \in \mathrm{CvCl}(E^*_{\omega^*})$$

for all $s \in \Omega_0$. It is an easy check that $\triangle$ is measurable. By Steps 2.1–2.2, we deduce that the multifunction $\Gamma: \Omega_0 \to 2^{E^*_{\omega^*}}$, $\Gamma(s) := \triangle(s) \cap \partial_C g(s, u(s)) \in \mathrm{Cl}(E^*_{\omega^*})$, is measurable. By [8], there exists a measurable selector ξ of Γ, i.e. $\xi(s) \in \partial_C g(s, u(s))$ and $g(s, x(s)) - g(s, y(s)) = \langle \xi(s), x(s) - y(s) \rangle$ on Ω_0. ∎

A function $f: \Omega \times F \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is called a normal integrand if $f(s, \cdot)$ is lower semicontinuous for almost all (a.a.) $s \in \Omega$ and f is $(\mathfrak{A} \times \mathcal{B}(F), \mathcal{B}(\overline{\mathbb{R}}))\,(\mathrm{mod}\ 0)$-measurable on $\Omega \times F$. If F is a Lusin space, then [7, Lemma 1.2.3] every Carathéodory function on $\Omega \times F$ is a normal integrand. If f is a normal integrand on $\Omega \times E$, then the dual convex normal integrand $f^*: \Omega \times E^*_{w^*} \to \overline{\mathbb{R}}$ is defined by $f^*(s, u^*) = \sup\{\langle u, u^* \rangle - f(s, u) : u \in E\}$. Denote by X^*_{s} the space of singular linear functionals on $X = X(\Omega, E)$ (see [20, 22, 33]). Lemma 2.4 is taken from V. Levin [20, Corollary 1 of Theorem 6.7, p. 216], A. Kozek [19] ($X = L^M(\Omega, E)$ with E^* being separable), [8] (see also the references cited therein; in [20] this result is valid for a more general space X). Given a Banach space E

and a convex function $\varphi : E \to \mathbb{R}$, the subdifferential of φ is defined by

$$\partial\varphi(u) = \{\xi \in E^* : \langle \xi, v - u\rangle \leq \varphi(v) - \varphi(u) \text{ for all } v \in E\rangle\}.$$

LEMMA 2.4. *Let f be a normal convex integrand on $\Omega \times E$ and the functional I_f, $I_f(x) := \int_\Omega f(s, x(s))d\mu(s)$, be considered on X with (M_m) or (M_∞). Suppose that the set $\operatorname{dom} I_f := \{x \in X : I_f(x) < \infty\}$ is nonempty. Then for every $x_0 \in \operatorname{dom} I_f$ the subdifferential $\partial I_f(x_0)$ consists of all linear functionals $\lambda \in (X(\Omega, E))^*$ of the form $\lambda(x) = \langle x, y\rangle + \lambda_s(x)\,(x \in X)$, where $y \in X'(\Omega, E^*_{w^*}) \cap \operatorname{Sel} \partial f_X(\cdot, x_0(\cdot))$, $\lambda_s \in (X(\Omega, E))^*_s$, $\lambda_s \in K(\operatorname{dom} I_f, x_0) := \{l \in (X(\Omega, E))^* : l(z - x_0) \leq 0\,(\forall z \in \operatorname{dom} I_f)\}$, and $\partial f_X(s, u_0)$ denotes the subdifferential at u_0 of the convex function $u \in \operatorname{vsupp} X(s) \mapsto f(s, u)$.*

3. Lipschitz integral functionals on non-solid Banach M-spaces. Given a Banach M_+-space $X \subset L^0(\Omega, E)$, we say that a multivalued operator $M\colon X \to 2^{L^1(\Omega,\mathbb{R})}\backslash\{\emptyset\}$ has the *U-property* if given any $x \in X$ for each sequence $x_k \subset X$ with $\sum_{k=1}^\infty \|x_k - x\|_X < \infty$ there exists $\beta \in L^1(\Omega, \mathbb{R})$ such that $|\alpha_k(s)| \leq \beta(s)$ a.e. for every $\alpha_k \in M(x_k)$ and for every $k \in \mathbb{N}$. Given a function $g\colon \Omega \times E \to \mathbb{R}$, define the integral functional

$$G(x) := \int_\Omega g(s, x(s))d\mu(s). \tag{3.1}$$

LEMMA 3.1. *Let $g\colon \Omega \times E \to \mathbb{R}$ be a Carathéodory function such that $g(s, \cdot)$ is Lipschitz continuous on each ball of E for almost all $s \in \Omega$, and $X \subset L^0(\Omega, E)$ be a non-solid Banach M_+-space with either (M_∞) or (M_m) with $m \geq 2, \operatorname{vsupp} X(s) = \mathbb{R}^m$. Suppose the following conditions:*

$(N1)$ $\partial_C g\colon \Omega \times E \to \operatorname{CvCp}(E^*_{\omega^*})$ is $(\mathfrak{A} \times \mathcal{B}(E), \mathcal{B}(E^*_{w^*}))\,(\mathrm{mod}\ 0)$-measurable;

$(N2)$ The multivalued superposition operator $N_{\partial_C g}\colon X \to 2^{X'}\backslash\{\emptyset\}$ is bounded on each ball of X, and:

1) for each $x, v \in X$ the function $s \in \Omega \mapsto g^\circ(s, x(s); v(s))$ belongs to $L^1(\Omega, \mathbb{R})$,

2) the operator $M_v\colon X \to 2^{L^1(\Omega,\mathbb{R})} \setminus \{\emptyset\}$ has the U-property, where $M_v(x) := \{\alpha \in L^1(\Omega, \mathbb{R}) : \alpha(\cdot) = \langle \zeta(\cdot), v(\cdot)\rangle, \zeta \in N_{\partial_C g}(x)\}$.

If G is finite at least for one $x_* \in X$, then G is Lipschitz on each ball of X and

$$\partial_C G(x) \subset N_{\partial_C g}(x), \tag{3.2}$$

i.e. $\gamma \in \partial_C G(x) \Rightarrow \gamma(\cdot) = \langle \zeta, \cdot\rangle$ for some $\zeta \in X'$ with $\zeta(s) \in \partial_C g(s, x(s))$ a.e. If additionally the function $g(s, \cdot)$ is regular (in Clarke's sense) at $x(s)$ for almost all $s \in \Omega$, then the functional G is regular at x and $\partial_C G(x) = N_{\partial_C g}(x)$.

Proof. We shall mimic Clarke's proof of his Theorem 2.7.5/(B) in [12] but new moments in our proof will be emphasized and given in detail. We divide our proof into Steps 3.1-3.4.

Step 3.1: We claim that the functional G is Lipschitz continuous on every ball of X. To prove this, we fix $y, z \in B_X(0, r)$. By the condition $(N1)$, $\partial_C g\colon \Omega \times E \to \operatorname{CvCp}(E^*_{w^*})$ is $(\mathfrak{A} \times \mathcal{B}(E), \mathcal{B}(E^*_{w^*}))\,(\mathrm{mod}\ 0)$-measurable, and so we can apply Lemma 2.3 (the parametric version of Lebourg's theorem). We can then find some measurable functions $\xi_0\colon \Omega \to E^*_{w^*}$,

$u_0 \colon \Omega \to E$ and $\theta_0 \colon \Omega \to [0,1]$ such that

$$u_0(s) = \theta_0(s)z(s) + (1-\theta_0(s))y(s),$$
$$g(s,z(s)) - g(s,y(s)) = \langle \xi_0(s), z(s)-y(s)\rangle,$$
$$\xi_0(s) \in \partial_C g(s,u_0(s)) \text{ a.e.}$$

Since X is a Banach M_+-space, we have that $u \in X$ and

$$\|u_0\|_X \le \|\theta_0 z\|_X + \|(1-\theta_0)y\|_X \le \|z\|_X + \|y\|_X \le 2r. \tag{3.3}$$

By the condition $(N2)$, $C(2r) := \sup\{\|v\|_{X'} : v \in N_{\partial_C g}(w), w \in B_X(0,2r)\} < \infty$. Therefore $\|\xi_0\|_{X'} \le C(2r)$ follows. Hence, by the definition of the associate space X', we deduce that

$$\begin{aligned} |G(z)-G(y)| &\le \int_\Omega |g(s,z(s)) - g(s,y(s))| d\mu(s) \\ &\le \int_\Omega |\langle \xi_0(s), z(s)-y(s)\rangle| d\mu(s) \le \|\xi_0\|_{X'}\|z-y\|_X \le C(2r)\|z-y\|_X \end{aligned}$$

for $z,y \in B_X(0,r)$. Since $G(x_*) \in \mathbb{R}$, the claim of Step 3.1 follows.

Step 3.2: We shall prove that $G^\circ(x;v) \le \int_\Omega g^\circ(s,x(s);v(s))d\mu(s)$ for $x,v \in X$. From the definition of the Clarke derivative in the direction $v \in X$ we have

$$G^\circ(x;v) = \limsup_{\substack{y\to x\\ \lambda\downarrow 0}} \int_\Omega \frac{g(s,y(s)+\lambda v(s)) - g(s,y(s))}{\lambda} d\mu(s). \tag{3.4}$$

Let us choose arbitrary sequences λ_k in $\mathbb{R}$ and y_k in X such that $\lambda_k \downarrow 0$, $\|y_k - x\|_X \to 0$ and the limit

$$b := \lim_{k\to\infty} \int_\Omega F_k(s) d\mu(s) \tag{3.5}$$

exists, where $F_k(s) := \frac{g(s,y_k(s)+\lambda_k v(s)) - g(s,y_k(s))}{\lambda_k}$. Under (M_∞) or (M_m), by [22, 41] for $m=1$ and by [29, Theorem 2.1] for $m>1$ together with Riesz's theorem, we can choose a subsequence k_j and D_0 with $\mu(\Omega\backslash D_0) = 0$ such that $y_{k_j}(s) \to x(s)$ as $j\to\infty$ $(\forall s \in D_0)$, $\|y_{k_j} - x\|_X \le 1/2^j$ and $\lambda_{k_j} \le 1/2^j$.

We claim the existence of $\beta \in L^1(\Omega,\mathbb{R})$ *and of* $D_1 \subset D_0$ *with* $\mu(\Omega\backslash D_1) = 0$ *such that* $|F_{k_j}(s)| \le \beta(s)$ *on* D_1 *for all* $j \in \mathbb{N}$. To prove this, by the condition $(N1)$ together with Lemma 2.3 there exist measurable functions $\xi_k \in N_{\partial_C g}(u_k)$ and u_k such that $u_k(s) \in [y_k(s)+\lambda_k v(s), y_k(s)]$ and $F_k(s) = \langle \xi_k(s), v(s)\rangle$ a.e., and so $F_k \in M_v(u_k)$. Since X is a Banach M_+-space, we get, by an analogous argument to that for (3.3) in Step 3.1, that $u_k \in X$ and

$$\|u_{k_j} - x\|_X \le \|y_{k_j} - x\|_X + \lambda_{k_j}\|v\|_X \le 1/2^j + (1/2^j)\|v\|_X \Rightarrow \sum_{j=1}^\infty \|u_{k_j} - x\|_X < \infty. \tag{3.6}$$

Since M_v has the *U-property* due to the condition $(N2)$, there exist $\beta \in L^1(\Omega,\mathbb{R})$ and $D_1 \subset D_0$ with $\mu(\Omega\backslash D_1) = 0$ such that $|F_{k_j}(s)| \le \beta(s)$ $(\forall s \in D_1)$.

Using the above claim together with the measurability of the function $s \mapsto g^\circ(s,x(s);$ $v(s))$ (see Lemma 2.1), we *can apply the Fatou lemma for the functions* $s \in D_1 \mapsto$

$\beta(s) - F_{k_j}(s) \in [0, \infty)$, and by $(N2)$ we deduce then

$$\begin{aligned} b &= \limsup_{j\to\infty} \int_{D_1} F_{k_j}(s)d\mu(s) \le \int_{D_1} \limsup_{j\to\infty} F_{k_j}(s)d\mu(s) \\ &\le \int_{D_1} \limsup_{\substack{u\to x(s)\\ \lambda\downarrow 0}} \frac{g(s,u+\lambda v(s)) - g(s,u)}{\lambda} d\mu(s) = \int_\Omega g^\circ(s,x(s);v(s))d\mu(s) < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} G^\circ(x;v) &= \sup\{b = \lim_{k\to\infty} \int_\Omega F_k(s)d\mu(s) : \|y_k - x\|_X \to 0, \lambda_k \downarrow 0\} \\ &\le \int_\Omega g^\circ(s,x(s);v(s))d\mu(s) < \infty. \end{aligned}$$

Step 3.3: We mimic Clarke's argument in the proof of [12, Theorem 2.7.2]. We know (see Lemma 2.1) that the function $(s,u) \in \Omega \times E \mapsto g^\circ(s,x(s);u)$ is a Carathéodory convex integrand, and $v \in X \mapsto \hat{G}(v) := \int_\Omega g^\circ(s,x(s);v(s))d\mu(s)$ is a convex functional on X such that $\hat{G}(0) = 0$. If $\gamma \in \partial_C G(x)$, then by Step 3.2 for every $v \in X$ we have $\gamma(v) \le G^\circ(x;v) \le \int_\Omega g^\circ(s,x(s);v(s))d\mu(s) = \hat{G}(v) - \hat{G}(0)$, and so γ is an element of the subdifferential $\partial\hat{G}(0)$. By Lemma 2.4 together with $\operatorname{dom} I_{\hat{G}} = X$ from the condition $(N2)$, we can deduce that $\partial\hat{G}(0)$ consists of linear functionals $\gamma \in X^*$ of the form $\langle\gamma, v\rangle = \int_\Omega \langle\zeta(s), v(s)\rangle d\mu(s)$ $(v \in X)$ with $\zeta \in X'$ and $\zeta(s) \in \partial_C g(s,x(s))$ a.e. on Ω. Hence, the inclusion (3.2) follows.

Step 3.4: Suppose that $g(s,\cdot)$ is regular (in Clarke's sense [12, Section 2.3]) at $x(s)$ for almost all $s \in \Omega$. Fix $v \in X$. For an analogous reason and by an analogous argument as in Step 3.2, we *can apply the Fatou lemma for the functions* $s \mapsto \beta(s) + F_{k_j}(s) \in [0,\infty)$ and deduce

$$\begin{aligned} &\liminf_{\lambda\downarrow 0} \frac{G(x+\lambda v) - G(x)}{\lambda} \ge \int_\Omega \liminf_{\lambda\downarrow 0} \frac{g(s,x(s)+\lambda v(s)) - g(s,x(s))}{\lambda} d\mu(s) \\ &= \int_\Omega g'(s,x(s);v(s))d\mu(s) = \int_\Omega g^\circ(s,x(s);v(s))d\mu(s) \ge G^\circ(x;v). \end{aligned}$$

Now we can deduce by Clarke's argument in the proof of [12, Theorem 2.7.3, p. 87] that G is regular at x and $N_{\partial_C g}(x) \subset \partial_C G(x)$. ∎

By Lemma 3.1 we can prove the following Theorems 3.2-3.3.

THEOREM 3.2. *Let* $g\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ *be a Carathéodory function such that* $g(s,\cdot)$ *is Lipschitz continuous on each ball of* $\mathbb{R}^m$ *for almost all* $s \in \Omega$, *and* $X \subset L^0(\Omega, \mathbb{R}^m)$ *be a non-solid Banach* M*-space with* $m \ge 2, \operatorname{vsupp} X(s) = \mathbb{R}^m$. *Suppose that* $\partial_C g$ *satisfies* $(N1)$ *and there exists* $H\colon \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^m)$ *satisfying the following conditions:*

$(N3)$ *The multivalued superposition operator* $N_H\colon X \to 2^{X'}\backslash\{\emptyset\}$ *is bounded on each ball of* X*;*

$(N4)$ *There exists* $\Omega_0 \in \mathfrak{A}$ *with* $\mu(\Omega\backslash\Omega_0) = 0$ *such that*

1) $\partial_C g(s,u) \subset \overline{\operatorname{bco}}\, H(s,[-1,1]u)$ *for all* $s \in \Omega_0$ *and for all* $u \in \mathbb{R}^m$,

2) $H(s,C) \in \mathrm{Cp}(\mathbb{R}^m)$ *for* $C \in \mathrm{Cp}(\mathbb{R}^m)$ *for all* $s \in \Omega_0$, *3) H is multi-superpositionally measurable,*

4) H has the Filippov implicit function property.

Then, the statement of Lemma 3.1 is valid for G defined on X.

Proof. It suffices to check the condition $(N2)$, and then Theorem 3.2 follows from Lemma 3.1. We divide this proof into Steps 3.5-3.9. We need the following technical notion from [27, 28]. Given $Y \subset L^0(\Omega, \mathbb{R}^m)$ with $m \geq 2$, denote by $\mathrm{Arm}(Y)$ the set of all infra-semi-units for Y, i.e. measurable multifunctions $\mathbf{b}\colon \Omega \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ such that $\mathbf{b}(s) \in \mathrm{CvCp}(\mathbb{R}^m)$ is balanced in $\mathbb{R}^m$ a.e. and $\mathrm{Sel}\,\mathbf{b} \subset Y$.

Step 3.5: We claim that $\mathbf{a} \in \mathrm{Arm}(X')$ for each $\mathbf{b} \in \mathrm{Arm}(X)$, where

$$\mathbf{a}(s) := \overline{\mathrm{bco}}\, H(s, \mathbf{b}(s)).$$

To prove this claim, by the technical theorem 3 of [28] (see also [27, Theorem 4.2]), there exist $\tilde{x}_1, \ldots, \tilde{x}_m \in X$ such that

$$\sum_{j=1}^{m} [-1,1] \tilde{x}_j(s) \subset \mathbf{b}(s) \subset D(s) := (m + 1/2) \sum_{j=1}^{m} [-1,1] \tilde{x}_j(s) \ \text{ a.e.} \tag{3.7}$$

Put $\mathbf{c}(s) := \overline{\mathrm{bco}}\, H(s, D(s))$. Fix $y \in \mathrm{Sel}\,\mathbf{c} = \mathrm{Sel}\,\overline{\mathrm{bco}}\, H(\cdot, D(\cdot))$. By the condition $(N4)$ together with the parametric version [8, Theorem IV.11] of Carathéodory's theorem, the multifunctions $s \mapsto H(s, D(s)) \in \mathrm{Cp}(\mathbb{R}^m)$, $s \mapsto \mathbf{c}(s) \in \mathrm{Cp}(\mathbb{R}^m)$ are measurable and there exist measurable functions α_i, ξ_i such that $y(s) = \sum_{i=1}^{m+1} \alpha_i(s) \xi_i(s)$, $\sum_{i=1}^{m+1} |\alpha_i(s)| = 1$, $\xi_i(s) \in H(s, D(s))$ a.e.; then there exist some measurable functions z_i such that $z_i(s) \in D(s)$ and $\xi_i(s) \in H(s, z_i(s))$ a.e.

Then (e.g., by [29, Lemma 3.1/(2)]), for $z_i \in \mathrm{Sel}\, D$, there exist functions $\alpha_{ij} \in L^\infty$ such that $z_i(s) = (m + 1/2) \sum_{j=1}^{m} \alpha_{ij}(s) \tilde{x}_j(s)$ and $\|\alpha_{ij}\|_{L^\infty} \leq 1$. Since X is a Banach M-space, we obtain

$$\|z_i\|_X \leq (m + 1/2) \sum_{j=1}^{m} \|\alpha_{ij} \tilde{x}_j\|_X \leq (m + 1/2) \sum_{j=1}^{m} \|\tilde{x}_j\|_X := \kappa < \infty. \tag{3.8}$$

By the condition $(N3)$, $r(\kappa) := \sup\{\|\xi\|_{X'} : \xi \in N_H(z), \|z\|_X \leq \kappa\} < \infty$ for $\kappa \in (0, \infty)$. Since X' is a Banach M-space and $\xi_i \in N_H(z_i)$, (3.8) implies that

$$\|y\|_{X'} \leq \sum_{i=1}^{m+1} \|\alpha_i \xi_i\|_{X'} \leq \sum_{i=1}^{m+1} \|\xi_i\|_{X'} \leq (m+1)\, r(\kappa) < \infty. \tag{3.9}$$

Hence $\mathrm{Sel}\,\mathbf{c} \subset X'$, and so $\mathrm{Sel}\,\mathbf{a} \subset X'$. By the condition $(N4)$, $\mathbf{a}(s)$ is a balanced convex compact set and $\mathbf{a}$ is measurable, and hence the claim follows.

Step 3.6: We claim that given $\mathbf{a} \in \mathrm{Arm}(X')$ and $v \in X$, there exists $\beta_{\mathbf{a}} \in L^1(\Omega, (0, \infty))$ such that $|\langle v(s), d(s) \rangle| \leq \beta_{\mathbf{a}}(s)$ a.e. for any $d \in \mathrm{Sel}\,\mathbf{a}$. By the technical theorem 3 of [28] (see also [27, Theorem 4.2]), there exist $\tilde{y}_1, \ldots, \tilde{y}_m \in X'$ such that

$$\sum_{j=1}^{m} [-1,1] \tilde{y}_j(s) \subset \mathbf{a}(s) \subset (m + 1/2) \sum_{j=1}^{m} [-1,1] \tilde{y}_j(s) \ \text{ a.e.}$$

Then (e.g., by [29, Lemma 3.1/(2)]), for a fixed $d \in \mathrm{Sel}\,\mathbf{a}$, there exist m functions $\eta_j \in L^\infty$ such that $d(s) = (m + 1/2) \sum_{j=1}^{m} \eta_j(s) \tilde{y}_j(s)$ a.e. and $\eta_j(s) \in [-1, 1]$. Then, for a.a. $s \in \Omega$

we get

$$\begin{aligned}|\langle v(s), d(s)\rangle| &\le (m+1/2)\sum_{j=1}^{m}|\eta_j(s)||\langle v(s), \tilde{y}_j(s)\rangle| \\ &\le (m+1/2)\sum_{j=1}^{m}|\langle v(s), \tilde{y}_j(s)\rangle| := \beta_{\mathbf{a}}(s).\end{aligned}$$

Since $v \in X$ and $\tilde{y}_j \in X'$, we have $\beta_{\mathbf{a}} \in L^1(\Omega, \mathbb{R})$.

Step 3.7: We claim that for each $\delta \in (0,\infty)$ there exists $\tilde{r}(\delta) \in (0,\infty)$ such that $\|x\|_X \le \delta$ implies $\|y\|_{X'} \le \tilde{r}(\delta)$ for $y \in N_{\partial_C g}(x)$. We deduce this from the proof of Step 3.5. Fix $x \in X$ with $\|x\|_X \le \delta$. Put $\mathbf{b}(s) = [-1,1]x(s)$. Then $\mathbf{b}(s) = \sum_{j=1}^{m}[-1,1]\tilde{x}_j(s) \subset D(s) = [-1,1]D(s)$, where $\tilde{x}_1(s) = x(s), \tilde{x}_j(s) = 0$ $(j \ge 2)$. By the conditions $(N3)$–$(N4)$ together with (3.8)–(3.9), for a fixed $y \in N_{\partial_C g}(x) \subset \operatorname{Sel}\mathbf{c}$ we have that

$$\|z_i\|_X \le (m+1/2)\|\tilde{x}_1\|_X \le (m+1/2)\,\delta := k(\delta) < \infty$$

and $\|y\|_{X'} \le (m+1/2)\, r(k(\delta)) := \tilde{r}(\delta) < \infty$.

Step 3.8: We shall check the U-property for the multivalued operator $M_v\colon X \to 2^{L^1(\Omega,\mathbb{R})} \setminus \{\emptyset\}$ defined such as in condition $(N2)$ at a fixed $x \in X$. Fix $v \in X$ and a sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ such that $\sum_{i=1}^{\infty}\|x_i - x\|_X < \infty$. Fix $\alpha_i \in M_v(x_i)$; then $\alpha_i(s) = \langle v(s), \xi_i(s)\rangle$ a.e. on Ω for some $\xi_i \in N_{\partial_C g}(x_i)$. By the technical theorem 7 of [28] (see also [27, Theorem 4.1]) we get some $\Omega_1 \subset \Omega_0$ with $\mu(\Omega\backslash\Omega_1) = 0$ such that the series of compact sets

$$\mathbf{b}(s) := x(s) + \sum_{i=1}^{\infty}[-1,1](x_i(s) - x(s))$$

converges in the space $\mathrm{Cp}(\mathbb{R}^m)$ for all $s \in \Omega_1$; moreover, putting $\mathbf{b}(s) := \{0\}$ for $s \in \Omega\backslash\Omega_1$, we get $\mathbf{b} \in \mathrm{Arm}(X)$. By Step 3.5 we get that $\mathbf{a}(\cdot) := \overline{\mathrm{bco}}\, H(\cdot, \mathbf{b}(\cdot)) \in \mathrm{Arm}(X')$. Due to the condition $(N4)$ we get that

$$x_i(s) \in \mathbf{b}(s) = [-1,1]\mathbf{b}(s), \quad \partial_C g(s, x_i(s)) \subset \overline{\mathrm{bco}}\, H(s, \mathbf{b}(s))\; (\forall s \in \Omega_1)$$

and $\xi_i \in N_{\partial_C g}(x_i) \subset \operatorname{Sel}\mathbf{a}$. Hence, by Step 3.6, the U-property for M_v follows.

Step 3.9: We claim that the function $s \in \Omega \mapsto g^\circ(s, x(s); v(s))$ belongs to $L^1(\Omega, \mathbb{R})$ for any $x, v \in X$. Since $X \subset L^0(\Omega, \mathbb{R}^m)$ is a Banach M-space, by [29, Theorem 3.2, Lemma 3.2], there exist $\alpha \in L^0(\Omega, (0,\infty))$ and m functions $g_1, \ldots, g_m \in X$ such that the linear hull of $\{g_1(s), \ldots, g_m(s)\} = \operatorname{vsupp} X(s) = \mathbb{R}^m$ a.e., and $\mathbf{v}(s) := \sum_{j=1}^{m}[-1,1]g_j$ satisfies $\mathbf{v} \in \mathrm{Arm}(X)$ with $\alpha(s)B_{\mathbb{R}^m}(0,1) \subset \mathbf{v}(s)$ a.e. Then,

$$\begin{aligned}&|g^\circ(s, x(s); v(s))| \\ &\le \sup\left\{\left|\frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda}\right| : \lambda \in (0,1], \|\bar{u} - x(s)\|_{\mathbb{R}^m} \le \alpha(s)\right\} \\ &= \sup\left\{\left|\frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda}\right| : \lambda \in (0,1], \bar{u} \in x(s) + \alpha(s)B_{\mathbb{R}^m}(0,1)\right\} \\ &\le \sup\left\{\left|\frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda}\right| : \lambda \in (0,1], \bar{u} \in x(s) + \mathbf{v}(s)\right\}.\end{aligned}$$

By Lebourg's Theorem 2.2 and $(N4)$, we deduce that

$$\begin{aligned}
&|g^\circ(s,x(s);v(s))| \\
&\le \sup\{|\langle v(s),u^*\rangle| : u^*\in\partial_C g(s,u), u\in[\bar u,\bar u+\lambda v(s)], \lambda\in(0,1], \bar u\in x(s)+\mathbf{v}(s)\} \\
&\le \sup\{|(v(s),u^*)| : u^*\in\partial_C g(s,u), u\in[-1,1]v(s)+x(s)+\mathbf{v}(s)\} \\
&\le \sup\{|\langle v(s),u^*\rangle| : u^*\in\partial_C g(s,u), u\in\mathbf{b}_0(s)\} \\
&\le \sup\{|\langle v(s),u^*\rangle| : u^*\in\mathbf{a}_0(s)\} \ \text{ a.e.},
\end{aligned}$$

where

$$\mathbf{b}_0(s) := v(s)+[-1,1]x(s)+\mathbf{v}(s), \quad \mathbf{a}_0(s) := \overline{\operatorname{bco}}\, H(s,\mathbf{b}_0(s)).$$

Then, $\mathbf{b}_0\in\operatorname{Arm}(X)$, and so by Step 3.5, $\mathbf{a}_0\in\operatorname{Arm}(X')$. Since $\mathbf{a}_0$ is measurable, by [8] there exists some Castaing representation $\{u_q^* : q\in\mathbb{N}\}$ for $\mathbf{a}_0$, i.e. $u_q^*\in\operatorname{Sel}\mathbf{a}_0$ and $\mathbf{a}_0(s)=$ the closure of $\{u_q^*(s) : q\in\mathbb{N}\}$ a.e. Hence,

$$|g^\circ(s,x(s);v(s))| \le \sup\{|\langle v(s),u_q^*(s)\rangle| : q\in\mathbb{N}\} \ \text{ a.e.}$$

By Step 3.6 for $\mathbf{a}_0$, there exists $\beta_{\mathbf{a}_0}\in L^1(\Omega,\mathbb{R})$ with $|\langle v(s),u_q^*(s)\rangle| \le \beta_{\mathbf{a}_0}(s)$ a.e. So, the above claim follows. ∎

THEOREM 3.3. *Let $g\colon\Omega\times\mathbb{R}^m\to\mathbb{R}$, a non-solid Banach M-space $X\subset L^0(\Omega,\mathbb{R}^m)$ with $m\ge 2$ be such as in Theorem 3.2. Suppose that $\partial_C g$ satisfies $(N1)$ and for every $R\in(0,\infty)$ there exists $H_R\colon\Omega\times\mathbb{R}^m\to\operatorname{Cp}(\mathbb{R}^m)$ satisfying the following conditions:*

$(N5)$ *The multivalued superposition operator $N_{H_R}\colon B_X(0,R)\to 2^{X'}\backslash\{\emptyset\}$ is bounded;*

$(N6)$ *$\partial_C g(s,x(s))\subset\overline{\operatorname{bco}}\, H_R(s,[-1,1]x(s))$ a.e. for each $x\in B_X(0,R)$, and there exists $\Omega_0\in\mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0)=0$ such that*

1) *$H_R(s,C)\in\operatorname{Cp}(\mathbb{R}^m)$ for $C\in\operatorname{Cp}(\mathbb{R}^m)$ for all $s\in\Omega_0$,*
2) *H_R is multi-superpositionally measurable,*
3) *the Filippov implicit function property is valid for H_R.*

Then, the statement of Lemma 3.1 is valid for G defined on X.

Proof. It suffices to check the condition $(N2)$, and then Theorem 3.3 follows from Lemma 3.1. Hence, it suffices to modify Steps 3.5, 3.7–3.9.

Modification of Step 3.5: Let $\mathbf{b}\in\operatorname{Arm}(X)$ with $\|x\|_X\le\delta$ $(\forall x\in\operatorname{Sel}\mathbf{b})$ for some $\delta\in(0,\infty)$. Then we claim that $\mathbf{a}_{R_1}\in\operatorname{Arm}(X')$, where $\mathbf{a}_{R_1}(s) := \overline{\operatorname{bco}}\, H_{R_1}(s,\mathbf{b}(s))$ and $R_1 := (m+1/2)m\delta$. The proof of this claim is analogous to the proof of Step 3.5. Here, (3.7)–(3.8) imply that $\|\tilde x_j\|\le\delta$ and $\|z_i\|\le R_1$. So, it suffices to substitute H_{R_1} for H, $\mathbf{a}_{R_1}$ for $\mathbf{a}$, and $\mathbf{c}_{R_1}(s) := \overline{\operatorname{bco}}\, H_{R_1}(s,D(s))$ for $\mathbf{c}(s)$.

Modification of Step 3.7: It suffices to substitute $\mathbf{c}_{R_1}$ for $\mathbf{c}$, where $R_1=(m+1/2)m\delta$, then given $y\in N_{\partial_C g}(x)$ with $\|x\|_X\le\delta$ we get $y\in N_{\partial g}(x)\subset\operatorname{Sel}\mathbf{c}_{R_1}$.

Modification of Step 3.8: We observe that by the technical theorem 7 of [28] (see also [27, Theorem 4.1]) together with [29, Lemma 3.1] we get

$$\|\tilde x\|_X \le \delta_2 := \|x\|_X + \sum_{i=1}^{\infty}\|x_i-x\|_X < \infty$$

for every $\tilde{x} \in \operatorname{Sel}\mathbf{b}$. Then, it suffices to substitute H_{R_2} for H and $\mathbf{a}_{R_2}$ for $\mathbf{a}$ with $R_2 := (m+1/2)m\delta_2$.

Modification of Step 3.9: Observe that by [29, Lemma 3.1], we have

$$\tilde{x} \in \operatorname{Sel}\mathbf{b}_0 \Longrightarrow \|\tilde{x}\|_X \le \delta_3 := \|v\|_X + \|x\|_X + \sum_{i=1}^{m} \|g_i\|_X < \infty.$$

It suffices to substitute H_{R_3} for H and $\mathbf{a}_{R_3}$ for $\mathbf{a}$, where $R_3 := (m+1/2)m\delta_3$. ∎

REMARK 3.4. The statements of Lemma 3.1 and Theorems 3.2–3.3 remain valid for the non-solid Banach space $X \subset L^0(\Omega,\mathbb{R})$ with the arbitrary vector support $\operatorname{vsupp} X$, but then we need to substitute $g^\circ_X(s,u_0;v)$ for $g^\circ(s,u_0;v)$ as well as $\partial_C g_X(s,u_0)$ for $\partial_C g(s,u_0)$, where $g^\circ_X(s,u_0;v)$ denotes the Clarke derivative at u_0 in direction $v \in \operatorname{vsupp} X(s)$ for the function $u \in \operatorname{vsupp} X(s) \mapsto g(s,u)$, and $\partial_C g_X(s,u_0)$ denote the generalized gradient at u_0 of the function $u \in \operatorname{vsupp} X(s) \mapsto g(s,u)$.

4. Lipschitz integral functionals on Köthe–Bochner spaces. Given a Banach lattice $K \subset L^0(\Omega,\mathbb{R})$, we say that an operator $Q\colon K_+ \to K'$ has the *U-property* if given any $a \in K_+$ for each sequence $a_k \subset K_+$ with $\sum_{k=1}^\infty \|a_k - a\|_K < \infty$ there exists $d \in K'$ such that $|Q(a_k)(s)| \le d(s)$ a.e. on Ω for every $k \in \mathbb{N}$.

LEMMA 4.1. *Let $g\colon \Omega \times E \to \mathbb{R}$ be a Carathéodory function such that $g(s,\cdot)$ is Lipschitz continuous on each ball of E for almost all $s \in \Omega$, and $K \subset L^0(\Omega,\mathbb{R})$ be a Banach lattice. Suppose the following conditions hold:*

($K1$) *There exists $h\colon \Omega \times [0,\infty) \to [0,\infty)$ such that*

$$\sup\{\|u^*\|_{E^*} : u^* \in \partial_C g(s,u), \|u\|_E \le \alpha\} \le h(s,\alpha)$$

for almost all $s \in \Omega$ and for all $\alpha \in [0,\infty)$;

($K2$) *The superposition operator $N_h\colon K_+ \to K'$ is bounded on each ball of K_+ and has the U-property.*

Then, the statement of Lemma 3.1 is valid for G defined on $X = K(E)$.

Proof. We shall mimic Clarke's proof of his Theorem 2.7.5/(B) [12] but new moments in our proof will be emphasized and given in detail. We divide our proof into Steps 4.1-4.5.

Step 4.1: We claim that the functional G is Lipschitz continuous on every ball of X. In fact, let $y, z \in B_X(0,r)$. By Lebourg's Theorem 2.2 for $g(s,\cdot)$ on some open ball containing the convex interval $[z(s),y(s)]$, we can find $\xi_0(s) \in E^*_{w^*}$, $u_0(s) \in E$ and $\theta_0(s) \in [0,1]$ such that

$$\begin{gathered} u_0(s) = \theta_0(s)z(s) + (1-\theta_0(s))y(s), \\ g(s,z(s)) - g(s,y(s)) = \langle \xi_0(s), z(s) - y(s)\rangle, \\ \xi_0(s) \in \partial_C g(s,u_0(s)) \ \text{ a.e.} \end{gathered}$$

We point out that the functions ξ_0, u_0, θ_0 are *not*, in general, all measurable. Since X is a Köthe–Bochner space, we get

$$\big\| \|z(\cdot)\|_E + \|y(\cdot)\|_E \big\|_K \le \big\| \|z(\cdot)\|_E \big\|_K + \big\| \|y(\cdot)\|_E \big\|_K = \|z\|_X + \|y\|_X \le 2r.$$

Note that

$$\|u_0(s)\|_E \le \|\theta_0(s)z(s)\|_E + \|(1-\theta_0(s))y(s)\|_E \le \|z(s)\|_E + \|y(s)\|_E \text{ a.e.}$$

By $\xi_0(s) \in \partial_C g(s, u_0(s))$ a.e., from the condition $(K1)$ we obtain

$$\|\xi_0(s)\|_{E^*} \le h(s, \|z(s)\|_E + \|y(s)\|_E) := \gamma_0(s) \text{ a.e.}$$

Due to the condition $(K2)$, we obtain $\widetilde{C}(2r) := \sup\{\|N_h(\tilde{\alpha})\|_{K'} : \|\tilde{\alpha}\|_K \le 2r\} < \infty$. Since $\gamma_0(s) = N_h(\|z(\cdot)\|_E + \|y(\cdot)\|_E)(s)$, hence $\|\gamma_0\|_{K'} \le \widetilde{C}(2r)$ follows. Since

$$|g(s,z(s)) - g(s,y(s))| \le \|\xi_0(s)\|_{E^*}\|z(s) - y(s)\|_E \text{ a.e.,}$$

we get

$$\begin{aligned}|G(z) - G(y)| &\le \int_\Omega |g(s,z(s)) - g(s,y(s))|\,d\mu \le \int_\Omega \gamma_0(s)\|z(s)-y(s)\|_E\,d\mu \\ &\le \|\gamma_0\|_{K'}\|\|z(\cdot) - y(\cdot)\|_E\|_K \le \widetilde{C}(2r)\|z-y\|_X.\end{aligned}$$

for $z, y \in B_X(0,r)$. Since $G(x_*) \in \mathbb{R}$, the claim of Step 4.1 follows.

Step 4.2: We shall use the notations from the beginning of Step 3.2 of the proof of Theorem 3.2, in particular, $F_k(s) := \frac{g(s,y_k(s)+\lambda_k v(s)) - g(s,y_k(s))}{\lambda_k}$. We can choose k_j and D_0 with $\mu(\Omega \backslash D_0) = 0$ such that $y_{k_j}(s) \to x(s)$ as $j \to \infty$ $(\forall s \in D_0)$, $\|y_{k_j} - x\|_X \le 1/2^j$ and $\lambda_{k_j} \le 1/2^j$.

We claim the existence of $\beta \in L^1(\Omega, \mathbb{R})$ and of $D_1 \subset D_0$ with $\mu(\Omega\backslash D_1) = 0$ such that $|F_{k_j}(s)| \le \beta(s)$ on D_1 for all $j \in \mathbb{N}$. To prove this, by Lebourg's Theorem 2.2 for $g(s,\cdot)$ on some open ball containing the convex interval $[y_k(s), y_k(s) + \lambda_k v(s)]$, there exist $\xi_k(s) \in E^*_{\omega^*}$ and $u_k(s) \in E$ and $\alpha_k(s) \in [0,1]$ such that

$$\begin{gathered}u_k(s) = \alpha_k(s)[y_k(s) + \lambda_k v(s)] + (1-\alpha_k(s))y_k(s),\\ g(s, y_k(s) + \lambda_k v(s)) - g(s, y_k(s)) = \langle \xi_k(s), \lambda_k v(s)\rangle,\\ \xi_k(s) \in \partial_C g(s, u_k(s)) \text{ a.e.}\end{gathered}$$

So $F_k(s) = \langle \xi_k(s), v(s)\rangle$. We point out that the functions ξ_k, u_k and α_k are *not*, in general, all measurable. We have that

$$\begin{aligned}\|u_{k_j}(s)\|_E &\le \|x(s)\|_E + \|\alpha_{k_j}(s)[y_{k_j}(s) + \lambda_{k_j}v(s)] + (1-\alpha_{k_j}(s))y_{k_j}(s) - x(s)\|_E \\ &\le \|x(s)\|_E + \|y_{k_j}(s) - x(s)\|_E + \lambda_{k_j}\|v(s)\|_E := a_j(s) \text{ a.e.}\end{aligned}$$

Hence we obtain, by the condition $(K1)$,

$$\|\xi_{k_j}(s)\|_{E^*} \le h(s, a_j(s)) = N_h(a_j)(s) \text{ a.e.} \tag{4.1}$$

Since X is a Köthe–Bochner space, for the sequence a_j and a, $a(s) := \|x(s)\|_E$, we get

$$\begin{aligned}\|a_j - a\|_K &= \|\|y_{k_j}(\cdot) - x(\cdot)\|_E + |\lambda_{k_j}|\,\|v(\cdot)\|_E\|_K \\ &\le \|\|y_{k_j}(\cdot) - x(\cdot)\|_E\|_K + |\lambda_{k_j}|\,\|\|v(\cdot)\|_E\|_K \\ &= \|y_{k_j} - x\|_{K(E)} + |\lambda_{k_j}|\|v\|_{K(E)} \le 1/2^j + (1/2^j)\|\|v(\cdot)\|_E\|_K,\end{aligned}$$

and so

$$\sum_{j=1}^\infty \|a_j - a\|_K < \infty.$$

By the condition $(K1)$ the operator N_h has the *U-property*, and hence we can find $d \in K'$ such that $|N_h(a_j)(s)| \le d(s)$ a.e. Since $|F_{k_j}(s)| \le \|\xi_{k_j}(s)\|_{E^*}\|v(s)\|_E$, by (4.1) we deduce then the existence of $D_1 \subset D_0$ with $\mu(\Omega\backslash D_1) = 0$ such that $|F_{k_j}(s)| \le d(s)\|v(s)\|_E := \beta(s)(\forall s \in D_1)$. By $\|v(\cdot)\|_E \in K$, hence $\beta \in L^1(\Omega, \mathbb{R})$ follows.

Step 4.3: We claim that the function $s \in \Omega \mapsto g^\circ(s, x(s); v(s))$ belongs to $L^1(\Omega, \mathbb{R})$ for any $x, v \in X$. Since K is a Banach lattice, by [22], [41, Theorem 2.2.6], there exists $\alpha \in K$ with $\alpha(s) > 0$ for $s \in \operatorname{supp} K = \Omega$. Then, by Lebourg's Theorem 2.2, $(K1)$ implies that

$$\begin{aligned}
&|g^\circ(s, x(s); v(s))| \\
&\le \sup\left\{\left|\frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda}\right| : \lambda \in (0,1], \|\bar{u} - x(s)\|_E \le \alpha(s)\right\} \\
&\le \sup\{|\langle v(s), u^*\rangle| : u^* \in \partial_C g(s,u), u \in [\bar{u}, \bar{u} + \lambda v(s)], \lambda \in (0,1], \|\bar{u} - x(s)\|_E \le \alpha(s)\} \\
&\le \sup\{|\langle v(s), u^*\rangle| : u^* \in \partial_C g(s,u), \|u\|_E \le \|x(s)\|_E + \alpha(s) + \|v(s)\|_E\} \\
&\le \|v(s)\|_E \sup\{\|u^*\|_{E^*} : u^* \in \partial_C g(s,u), \|u\|_E \le p(s)\} \\
&\le \|v(s)\|_E\, h(s, p(s)) \ \text{ a.e.,}
\end{aligned}$$

where $p(s) := \|x(s)\|_E + \alpha(s) + \|v(s)\|_E$ satisfies $p \in K_+$. Hence, by $(K2)$ together with $\|v(\cdot)\|_E \in K$, the above claim follows.

The remaining part of Step 4.2 and Steps 4.4–4.5 for proving Lemma 4.1 are analogous to Step 3.2 and Steps 3.3-3.4 of the proof of Lemma 3.1. ∎

By Lemma 4.1 we can prove the following Theorem 4.2.

THEOREM 4.2. *Let $g\colon \Omega \times E \to \mathbb{R}$ be a Carathéodory function such that $g(s,\cdot)$ is Lipschitz continuous on each ball of E for almost all $s \in \Omega$, and $K \subset L^0(\Omega, \mathbb{R})$ be a Banach lattice. Suppose that g satisfies the condition $(K1)$ with respect to $h\colon \Omega \times [0,\infty) \to [0,\infty)$ and h satisfies the following condition:*

$(K3)$ *The superposition operator $N_h\colon K_+ \to K'$ is bounded on each ball of K_+ and $h(s,\cdot)$ is nondecreasing for almost all $s \in \Omega$.*

Then, the statement of Lemma 3.1 is valid for G defined on $X = K(E)$.

Proof. It suffices to check the U-property for N_h, and then Theorem 4.2 follows from Lemma 4.1. Fix $a \in K_+$ and a sequence $a_i \in K_+$ such that $\sum_{i=1}^\infty \|a_i - a\|_K < \infty$. Then by the Riesz–Fischer property for the Banach lattice K (see, e.g., [22], [41, Theorem 3.2.1]), there exists $\Omega_0 \in \mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0) = 0$ such that the series $a_\infty(s) := \sum_{i=1}^\infty |a_i(s) - a(s)|$ converges for $s \in \Omega_0$; moreover putting $a_\infty(s) := 0\,(s \in \Omega\backslash\Omega_0)$, we get $a_\infty \in K_+$. Note that $a_i(s) \le a_\infty(s) + a(s)$ a.e., and then by the condition $(K3)$ we have

$$N_h(a_i)(s) = h(s, a_i(s)) \le h(s, a_\infty(s) + a(s)) = N_h(a_\infty + a)(s) := d(s) \ \text{ a.e.}$$

Since N_h maps K_+ into K', we obtain $N_h(a_\infty + a) = d \in K'$. ∎

THEOREM 4.3. *Let $g\colon \Omega \times E \to \mathbb{R}$ and a Banach lattice K be as in Theorem 4.2. Suppose that for every $R \in (0,\infty)$ there exists a function $h_R\colon \Omega \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

$(K4)$ *$\sup\{\|u^*\|_{E^*} : u^* \in \partial_C g(s,u), \|u\|_E \le a(s)\} \le h_R(s,a(s))$ is valid a.e. for each $a \in B_{K_+}(0,R)$;*

$(K5)$ *The superposition operator $N_{h_R}\colon B_{K_+}(0,R) \to K'$ is bounded and there exists $\Omega_0 \in \mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0) = 0$ such that $h_R(s,\cdot)$ is nondecreasing for $s \in \Omega_0$.*

Then, the statement of Lemma 3.1 is valid for G defined on $X = K(E)$.

Proof. It suffices to modify Steps 4.1-4.3 (in the proof of Lemma 4.1) as well as the proof of Theorem 4.2.

Modification of Step 4.1: It suffices to substitute $h_{2r}(s,\cdot)$ for $h(s,\cdot)$.

Modification of Step 4.2 together with the proof of Theorem 4.2: Observe by the Riesz–Fischer property for the Banach lattice K that we get also $\|a_\infty\|_K \le \sum_{i=1}^\infty \|a_i - a\|_K$. Then, it suffices to substitute $h_{r_1}(s,\cdot)$ for $h(s,\cdot)$, where $r_1 := \|a\|_K + \sum_{i=1}^\infty \|a_i - a\|_K < \infty$.

Modification of Step 4.3: It suffices to substitute $h_{r_2}(s,\cdot)$ for $h(s,\cdot)$, where

$$r_2 := \|\|x(\cdot)\|_E\|_K + \|\alpha\|_K + \|\|v(\cdot)\|_E\|_K < \infty. \ \blacksquare$$

THEOREM 4.4. *Let $g\colon \Omega \times E \to \mathbb{R}$ be a Carathéodory function such that $g(s,\cdot)$ is locally Lipschitz on E for almost all $s \in \Omega$, and $K \subset L^0(\Omega,\mathbb{R})$ be a Banach lattice. Suppose the following conditions hold:*

$(K6)$ *There exists $\tilde{h}\colon \Omega \times [0,\infty) \to [0,\infty)$ such that*

$$|g(s,u) - g(s,v)| \le \tilde{h}(s, \|u\|_E + \|v\|_E)\|u - v\|_E$$

for almost all $s \in \Omega$ and for all $u, v \in E$;

$(K7)$ *The superposition operator $N_{\tilde{h}}\colon K_+ \to K'$ is bounded on each ball of K_+ and $\tilde{h}(s,\cdot)$ is nondecreasing for almost all $s \in \Omega$.*

Then, the statement of Lemma 3.1 is valid for G defined on $X = K(E)$.

Proof. Observe that if $g(s,\cdot)$ is Lipschitz continuous on each ball of E for almost all $s \in \Omega$, then by [12, Proposition 2.1.2/(a)], $(K6)$–$(K7)$ imply $(K1)$ and $(K3)$ for $h(s,\alpha) = \tilde{h}(s,2\alpha)$, and so Theorem 4.4 follows from Theorem 4.2. In the general case we can give another direct proof without using Lebourg's Theorem 2.2 as following.

Modification of Step 4.1: Let $y, z \in B_X(0,r)$. Then,

$$\begin{aligned}|G(z) - G(y)| &\le \int_\Omega \tilde{h}(s, \|z(s)\|_E + \|y(s)\|_E)\|z(s) - y(s)\|_E \, d\mu(s)\\ &\le \|N_{\tilde{h}}(\|z(\cdot)\|_E + \|y(\cdot)\|_E)\|_{K'} \ \|\|z(\cdot) - y(\cdot)\|_E\|_K \le \widetilde{C}(2r)\|z - y\|_X.\end{aligned}$$

Modification of Step 4.2: Let F_k, y_k, k_j be as in Step 4.2. Then,

$$\begin{aligned}|F_{k_j}(s)| &\le \tilde{h}(s, \|y_{k_j}(s) + \lambda_{k_j} v(s)\|_E + \|y_{k_j}(s)\|_E)\, \|v(s)\|_E\\ &\le \tilde{h}(s, 2\,\|y_{k_j}(s)\|_E + \|v(s)\|_E)\, \|v(s)\|_E.\end{aligned}$$

Since $\sum_{j=1}^\infty \|\|y_{k_j}(\cdot) - x(\cdot)\|_E\|_K \le \sum_{j=1}^\infty \frac{1}{2^{k_j}} < \infty$, by the Riesz–Fischer property for the Banach lattice K (see, e.g., [22], [41, Theorem 3.2.1]), for some $\Omega_0 \in \mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0) = 0$ the series $\tilde{a}_\infty(s) := \sum_{i=1}^\infty \|y_{k_j}(s) - x(s)\|_E$ converges for $s \in \Omega_0$; moreover putting

$\tilde{a}_\infty(s) := 0\,(s \in \Omega\backslash\Omega_0)$, we get $\tilde{a}_\infty \in K_+$. Observe that $\|y_{k_j}(s)\|_E \le \tilde{a}_\infty(s) + \|x(s)\|_E$ a.e., and hence

$$|F_{k_j}(s)| \le \tilde{d}(s)\|v(s)\|_E \text{ a.e.,}$$

where $\tilde{d}(s) := N_{\tilde{h}}(2\tilde{a}_\infty + 2\|x(\cdot)\|_E + \|v(\cdot)\|_E)(s)$ with $\tilde{d} \in K'$.

Modification of Step 4.3: Let α be as in Step 4.3. Then,

$$\begin{aligned}
&|g^\circ(s, x(s); v(s))| \\
&\le \sup\left\{\left|\frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda}\right| : \lambda \in (0,1], \|\bar{u} - x(s)\|_E \le \alpha(s)\right\} \\
&\le \sup\left\{\tilde{h}(s, 2\|\bar{u}\|_E + \lambda\|v(s)\|_E)\|v(s)\|_E : \lambda \in (0,1], \|\bar{u} - x(s)\|_E \le \alpha(s)\right\} \\
&\le \sup\left\{\tilde{h}(s, 2\|\bar{u}\|_E + \|v(s)\|_E)\|v(s)\|_E : \|\bar{u}\|_E \le \|x(s)\|_E + \alpha(s)\right\} \\
&\le \|v(s)\|_E\, \tilde{h}(s, \tilde{p}(s)) \text{ a.e.,}
\end{aligned}$$

where $\tilde{p}(s) := 2\|x(s)\|_E + 2\alpha(s) + \|v(s)\|_E$ with $\tilde{p} \in K_+$. Then, $g^\circ(\cdot, x(\cdot), v(\cdot)) \in L^1(\Omega, \mathbb{R})$. The remaining part of the proof is analogous to Steps 3.2-3.4 of the proof of Lemma 3.1. ∎

THEOREM 4.5. *Let $g\colon \Omega \times E \to \mathbb{R}$ and a Banach lattice K be as in Theorem 4.4. Suppose that for every $R \in (0,\infty)$ there exists $\tilde{h}_R\colon \Omega \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

($K8$) $|g(s, u(s)) - g(s, v(s))| \le \tilde{h}_R(s, \|u(s)\|_E + \|v(s)\|_E)\|u(s) - v(s)\|_E$ *is valid a.e. for each* $u, v \in B_{K(E)}(0, R)$*;*

($K9$) *The superposition operator* $N_{\tilde{h}_R}\colon B_{K_+}(0, R) \to K'$ *is bounded and there exists* $\Omega_0 \in \mathfrak{A}$ *with* $\mu(\Omega\backslash\Omega_0) = 0$ *such that* $\tilde{h}_R(s, \cdot)$ *is nondecreasing for* $s \in \Omega_0$*.*

Then, the statement of Lemma 3.1 is valid for G defined on $X = K(E)$.

Proof. It suffices to modify the alternative proof of Theorem 4.4 in the same way as in the proof of Theorem 4.3. ∎

5. Lipschitz integral functionals on non-solid generalized Orlicz spaces. Let $m \ge 2$ and $M\colon \Omega \times \mathbb{R}^m \to [0,\infty)$ be some generalized Young function (see, e.g., [6, 17, 19, 25]), i.e. M is a normal integrand, $M(s,\cdot)$ is convex even, $M(s, 0) = 0$, and the set $\{u : M(s,u) = 0\}$ is bounded for a.a. $s \in \Omega$. The non-solid generalized Orlicz space is defined by $L^M(\Omega, \mathbb{R}^m) := \{x \in L^0(\Omega, \mathbb{R}^m) : \int_\Omega M(s, \alpha x(s))d\mu(s) < \infty \text{ for some } \alpha > 0\}$ with the Luxemburg norm.

We shall use the following conditions (Δ_m) and (B_m):

(Δ_m) There exist some measurable function $\delta\colon \Omega \to [0,\infty)$ and $a \in (1,\infty)$ such that $\int_\Omega \sup\{M(s,u) : \|u\| \le \delta(s)\}d\mu\,(s) < \infty$ and $M(s, 2u) \le aM(s,u)$ for a.a. $s \in \Omega$ and all $u \in \mathbb{R}^m$ with $\|u\| \ge \delta(s)$;

(B_m) There exist $c \in L^{M^*}(\Omega, \mathbb{R}^m)$ and $b > 0$ such that

$$\partial_C g(s,u) \subset [-1,1]\,c(s) + b\,\overline{\mathrm{bco}}\,\partial M(s, [-1,1]\,u)$$

for almost all $s \in \Omega$ and for all $u \in \mathbb{R}^m$.

COROLLARY 5.1 (of Theorem 3.2). *Let $m \geq 2$, $M\colon \Omega\times\mathbb{R}^m \to [0,\infty)$ be aYoung function, and g be as in Theorem 3.2. Suppose that the conditions $(N1)$, (B_m) are satisfied for $\partial_C g$ and (Δ_m) is valid for M. If the Filippov implicit function property is valid for ∂M (in particular, if* grad $M(s,\cdot)\colon \mathbb{R}^m \to \mathbb{R}^m$ *is continuous), then the statement of Lemma 3.1 is valid for G defined on $X = L^M(\Omega,\mathbb{R}^m)$.*

Proof. By the known equality $X' = L^{M^*}$ (with equivalent norms) the assertion of Corollary 5.1 follows immediately from Theorem 3.2 via using the results of the following Steps 5.1-5.3.

Step 5.1: We claim that if M satisfies the condition (Δ_m), then for every $\kappa \in (0,\infty)$ there exists $r(\kappa) \in (0,\infty)$ such that $\|x\|_{L^M} \leq \kappa \Rightarrow \|\xi\|_{L^{M^*}} \leq r(\kappa)$, where ξ is an arbitrary measurable selector of the multifunction $s \mapsto \partial M(s,x(s))$. To prove this claim, fix x and ξ such that $\|x\|_M \leq \kappa$ and $\xi(s) \in \partial M(s,x(s))$ a.e. (then $\int_\Omega M(s,x(s)/\kappa)d\mu(s) \leq 1$). By the definition of the subdifferential of the convex function $M(s,\cdot)$,

$$M(s,2x(s)) \geq M(s,2x(s)) - M(s,x(s)) \geq \langle \xi(s), x(s)\rangle \ \text{ a.e.}$$

Since $\partial M(s,x(s)) = \{\zeta \in \mathbb{R}^m : M^*(s,\zeta) + M(s,x(s)) = \langle \zeta, x(s)\rangle\}$, we have

$$\begin{aligned} M^*(s,\xi(s)) &= \langle \xi(s), x(s)\rangle - M(s,x(s)) \\ &\leq M(s,2x(s)) - M(s,x(s)) \leq M(s,2x(s)) \ \text{ a.e.} \end{aligned}$$

Let $l \in \mathbb{N}$ be such that $\kappa \leq 2^{l-1}$. Denote $A = \{s \in \Omega : \ \|x(s)\| \leq \kappa\delta(s), x(s) \neq 0\}$. By the condition (Δ_m) we deduce that:

$$\begin{aligned} \int_\Omega M^*(s,\xi(s))\,d\mu(s) &\leq \int_A M(s,2x(s))\,d\mu(s) + \int_{\Omega\setminus A} M(s,2x(s))\,d\mu(s) \\ &= \int_A M\left(s, 2\,\frac{\|x(s)\|}{\delta(s)}\,\frac{x(s)\delta(s)}{\|x(s)\|}\right) d\mu(s) + \int_{\Omega\setminus A} M\left(s, 2\kappa\frac{x(s)}{\kappa}\right) d\mu(s) \\ &\leq \int_A M\left(s, 2^l\,\frac{x(s)\delta(s)}{\|x(s)\|}\right) d\mu(s) + \int_{\Omega\setminus A} M\left(s, 2^l\,\frac{x(s)}{\kappa}\right) d\mu(s) \\ &\leq a^l \int_A M\left(s, \frac{x(s)\delta(s)}{\|x(s)\|}\right) d\mu(s) + a^l \int_{\Omega\setminus A} M\left(s, \frac{x(s)}{\kappa}\right) d\mu(s) \\ &\leq a^l \left(\int_\Omega \sup\{M(s,u) : \|u\| \leq \delta(s)\}\,d\mu(s) + 1\right) \leq a^l(I+1), \end{aligned}$$

where $I := \int_\Omega \sup\{M(s,u) : \|u\| \leq \delta(s)\}d\mu(s) \in (0,\infty)$. Since $a^l(I+1) > 1$, by convexity of M^* and $M^*(s,0) = 0$, $\int_\Omega M^*(\frac{\xi(s)}{a^l(I+1)})d\mu(s) \leq 1$ follows. Therefore $\|\xi\|_{M^*} \leq a^l(I+1) := r(\kappa)$.

Step 5.2: Put $H(s,u) := c(s) + b\partial M(s,u)$. Then by the condition (B_m), $\partial_C g(s,u) \subset \overline{\operatorname{bco}}\, H(s,[-1,1]u)$ and the Filippov implicit function property for H is valid as this valid for ∂M. Fix $y \in N_H(x)$ with $\|x\|_{L^M} \leq \kappa < \infty$. Then (e.g., by [29, Lemma 3.1]), there exist measurable functions α, d such that $y(s) = \alpha(s)c(s) + bd(s)$ a.e. with $\alpha(s) \in [-1,1]$ and $d(s) \in \partial M(s,[-1,1]x(s))$ a.e. By the Filippov implicit function property for ∂M, there exists some measurable function λ such that $\lambda(s) \in [-1,1]$ and $d(s) \in \partial M(s,\lambda(s)x(s))$ a.e. Observe that $\|\lambda x\|_{L^M} \leq \|x\|_{L^M} \leq \kappa < \infty$ and then by Step 5.1 for λx we get

$\|d\|_{L^{M^*}} \le r(\kappa) < \infty$. Hence,

$$\|y\|_{L^{M^*}} \le \|\alpha c\|_{L^{M^*}} + \|bd\|_{L^{M^*}} \le \|c\|_{L^{M^*}} + b\,\|d\|_{L^{M^*}} := \tilde{r}(k) < \infty$$

for $y \in N_H(x)$.

Step 5.3: By [38, Theorem 2W, Theorem 1N], $\partial M : \Omega \times \mathbb{R}^m \to \mathrm{Cp}(\mathbb{R}^m)$ is multi-superpositionally measurable. By [12, Proposition 2.2.6, Proposition 2.1.2/(a)] (for the convex function $M(s,\cdot)\colon \mathbb{R}^m \to \mathbb{R}$) together with [2, Lemma 2.9, p. 13], $\partial M(s,C) \in \mathrm{Cp}(\mathbb{R}^m)$ for $C \in \mathrm{Cp}(\mathbb{R}^m)$ a.e. Hence, the condition $(N4)$ follows. ∎

We shall use the following condition:

(NM) There exist $\Omega_0 \in \mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0) = 0$ and for every $R \in (0,\infty)$ exist $b_R, d_R \in (0,\infty)$ and $a_R \in L^1(\Omega,[0,\infty))$ such that

$$u^* \in \partial_C g(s,u) \Longrightarrow M^*(s,u^*/d_R) \le a_R(s) + b_R M(s,u/R)$$

for all $s \in \Omega_0, u \in \mathbb{R}^m$.

Corollary 5.2 (of Theorem 3.3). *Let $m \ge 2$, $M\colon \Omega\times\mathbb{R}^m \to [0,\infty)$ be a Young function with $M^*(s,u^*) \in [0,\infty)$, and g be as in Theorem 3.3. Suppose that $\partial_C g$ satisfies the conditions $(N1)$ and (NM). If the multifunction*

$$(s,u) \in \Omega \times \mathbb{R}^m \mapsto H_{MR}(s,u) := \{u^* \in \mathbb{R}^m : M^*(s, \tfrac{u^*}{d_R}) \le a_R(s) + b_R M(s, \tfrac{u}{R})\}$$

has the Filippov implicit function property, then the statement of Lemma 3.1 is valid for G defined on $X = L^M$.

Proof. It suffices to check the conditions $(N5)$–$(N6)$ for $X = L^M(\Omega,\mathbb{R}^m)$ and $H_R := H_{MR}$ (then Corollary 5.2 follows from Theorem 3.3). Due to the above definition of M, it is known [17, 19, 10] and easy to check that $H_{MR}(s,u) \in \mathrm{Cp}(\mathbb{R}^m)$ and $H_{MR}(s,u)$ is symmetric and $H_{MR}(s,C) \in \mathrm{Cp}(\mathbb{R}^m)$ for $C \in \mathrm{Cp}(\mathbb{R}^m)$ and for all $s \in \Omega_0$. By [12, Proposition 2.2.6], both M and M^* are Carathéodory, and so (e.g., by [42, Theorem 1]) H_{MR} is multi-superpositionally measurable. Since $\overline{\mathrm{bco}}\, H_{MR}(s,[-1,1]u) = H_{MR}(s,u)$, by (NM) we deduce $(N6)$.

Fix $x \in X = L^M(\Omega,\mathbb{R}^m)$ with $\|x\|_{L^M} \le R$. Then for any $\xi \in \mathrm{Sel}\, H_{MR}(\cdot, x(\cdot))$ we get

$$\begin{aligned}\int_\Omega M^*\left(s,\xi(s)/d_R\right) d\mu(s) &\le \|a_R\|_{L^1} + b_R \int_\Omega M\left(s,x(s)/R\right) d\mu(s)\\ &\le \|a_R\|_{L^1} + b_R + 1 := \tilde{r}(R) \in (1,\infty).\end{aligned}$$

Hence, $\int_\Omega M^*(s, \frac{\xi(s)}{d_R\tilde{r}(R)})d\mu(s) \le 1$, and so $\|\xi\|_{L^{M^*}} \le d_R\tilde{r}(R)$. By $(L^M(\Omega,\mathbb{R}^m))' = L^{M^*}(\Omega,\mathbb{R}^m)$ with equivalent norms, $(N5)$ follows. ∎

Remark 5.3. If $M\colon \Omega\times\mathbb{R} \to [0,+\infty]$ is such that $\mathcal{L}(s) := \{u : M(s,u) < +\infty\}$ is a linear subspace of $\mathbb{R}$, then $X = L^M(\Omega,\mathbb{R})$ has $\mathrm{vsupp}\, X(s) = \mathcal{L}(s)$ and the statements of Corollaries 5.1-5.2 remain valid but in the form of Remark 3.4 together with substituting $M^*_{\mathcal{L}}$ for M^*, where $M^*_{\mathcal{L}}(s,\cdot)$ is the convex dual on $\mathcal{L}(s)$ of the function $u \in \mathcal{L}(s) \mapsto M(s,u)$, and $N_{\partial_C g}(x) \subset X' = L^{M^*_{\mathcal{L}}}(\Omega,\mathbb{R})\,(x \in X)$.

6. Lipschitz integral functionals on Orlicz–Bochner spaces. Let $\Phi\colon \Omega\times[0,\infty)\to[0,\infty)$ be some Musielak–Orlicz function (convex in the second variable) and $\Phi^*\colon \Omega\times[0,\infty)\to[0,\infty)$ be the convex dual to Φ (see [4, 25, 37]). The generalized Orlicz–Bochner (Musielak–Orlicz–Bochner) space is defined by $L^\Phi(E)=\{x\in L^0(\Omega,E): \int_\Omega \Phi(s,\alpha\|x(s)\|_E)d\mu(s)<\infty$ for some $\alpha>0\}$ with the Luxemburg norm.

The following conditions (Δ) and (B) are taken from [36]:

(Δ) There exist some measurable function $\delta\colon\Omega\to[0,\infty)$ and $a\in(1,\infty)$ such that $\int_\Omega\Phi(s,\delta(s))d\mu(s)<\infty$ and $\Phi(s,2\beta)\le a\Phi(s,\beta)$ a.e. for $\beta\ge\delta(s)$;

(B) There exist $c\in L^{\Phi^*}(\Omega,\mathbb{R})$ and $b\in(0,\infty)$ such that
$$u^*\in\partial_C g(s,u)\Rightarrow\ \|u^*\|_{E^*}\le c(s)+b\varphi(s,\|u\|_E)\ \text{ a.e. on }\Omega$$
for all $u\in E$, where $\varphi(s,\cdot)$ is the right derivative of the convex function $\Phi(s,\cdot)$.

Lemma 6.1. *Suppose that conditions* (Δ) *and* (B) *are satisfied. Then the conditions* $(K1)$ *and* $(K3)$ *in Theorem 4.2 are valid for* $h(s,\alpha):=|c(s)|+b|\varphi(s,\alpha)|$ *with respect to* $X=K(E)$ *with* $K:=L^\Phi(\Omega,\mathbb{R})$.

Proof. By the condition (B) the condition $(K1)$ follows. By the condition (Δ) together with [36, Lemma 1],
$$\kappa\in(0,\infty)\Rightarrow\tau(\kappa):=\sup\{\|\varphi(\cdot,|\alpha(\cdot)|)\|_{L^{\Phi^*}}:\|\alpha\|_{L^\Phi}\le\kappa\}<\infty.$$
By the equality $K'=L^{\Phi^*}$ with equivalent norms (see, e.g., [4, 25, 37]), $\|N_h(\alpha)\|_{K'}\le\|c\|_{K'}+b\tau(\kappa)<\infty$ follows for every $\alpha\in K_+$ with $\|\alpha\|_K\le\kappa<\infty$. By the nondecreasing property of $\varphi(s,\cdot)$, $h(s,\cdot)$ is nondecreasing for a.a. $s\in\Omega$. Hence $(K3)$ follows. ■

By Lemma 6.1 together with $(L^\Phi(E))'=L^{\Phi^*}(E^*_{\omega^*})$, *Theorem 4.2 with its proof implies an alternative proof for* [36, Theorem 2].

We shall use the following conditions $(S\Phi1)$ and $(S\Phi2)$:

($S\Phi1$) There exists $\Omega_0\in\mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0)=0$ and for every $R\in(0,\infty)$ there exist $b_R,d_R\in(0,\infty)$ and $a_R\in L^1(\Omega,[0,\infty))$ such that
$$u^*\in\partial_C g(s,u)\Longrightarrow\Phi^*(s,\|u^*\|_{E^*}/d_R)\le a_R(s)+b_R\Phi(s,\|u\|_E/R)$$
for all $s\in\Omega_0$, $u\in E$;

($S\Phi2$) There exists $\Omega_0\in\mathfrak{A}$ with $\mu(\Omega\backslash\Omega_0)=0$ and for every $R\in(0,\infty)$ there exist $b_R,d_R\in(0,\infty)$ and $a_R\in L^1(\Omega,[0,\infty))$ such that
$$|g(s,u)-g(s,v)|\le\tilde h_R(s,\|u\|_E+\|v\|_E)\|u-v\|_E$$
for all $s\in\Omega_0$ and for all $u,v\in E$, and
$$\Phi^*(s,\tilde h(s,\alpha)/d_R)\le a_R(s)+b_R\Phi(s,\alpha/R)$$
for all $s\in\Omega_0,\alpha\in[0,\infty)$.

Corollary 6.2 (of Theorems 4.3, 4.5). *Let* $g\colon\Omega\times E\to\mathbb{R}$ *be a Carathéodory function. Suppose one of the following conditions holds:*

1) $g(s,\cdot)$ is Lipschitz continuous on each ball of E for almost all $s \in \Omega$ and $\partial_C g$ satisfies $(S\Phi 1)$;

2) $g(s,\cdot)$ is locally Lipschitz on E for almost all $s \in \Omega$ and g satisfies $(S\Phi 2)$.

Then the statement of Lemma 3.1 is valid for G defined on $X = L^\Phi(E)$.

REMARK 6.3. It is an easy check that $(S\Phi 1)$ implies that $N_{\partial_C g}\colon L^\Phi(E) \to L^{\Phi^*}(E^*_{\omega^*})$ is bounded on $B_{L^\Phi(E)}(0,R)$. On the other hand, it is known (see e.g. [4, 26]) that if the measure μ is continuous and $\partial_C g$ is Carathéodory and $N_{\partial_C g}\colon L^\Phi(E) \to L^{\Phi^*}(E^*_{\omega^*})$ is bounded on $B_{L^\Phi(E)}(0,R)$, then $(S\Phi 1)$ is true. So, $(S\Phi 1)$ is natural for applications.

Proof of Corollary 6.2/$(S\Phi 1)$. Let $(\Phi^*)^{-1}(s,\cdot)$ be the right pre-image of $\Phi^*(s,\cdot)$. Put $h_R(s,\alpha) := d_R(\Phi^*)^{-1}(s, a_R(s) + b_R\Phi(s,\alpha/R))$. Then, $(K4)$ follows. It is easy to check (see, e.g., [4, 26]) that N_{h_R} maps boundedly $B_{(L^\Phi)_+}(0,R)$ into $L^{\Phi^*}(\Omega,\mathbb{R})$. Since $h_R(s,\cdot)$ is nondecreasing and $(L^\Phi(\Omega,\mathbb{R}))' = L^{\Phi^*}(\Omega,\mathbb{R})$ with equivalent norms, $(K5)$ follows. Therefore, Corollary 6.2/$(S\Phi 1)$ follows from Theorem 4.3. ■

Proof of Corollary 6.2/$(S\Phi 2)$. By analogous arguments to the proof of Corollary 6.2/ $(S\Phi 1)$, we deduce Corollary 6.2/$(S\Phi 2)$ from Theorem 4.5. ■

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