GENERALIZED GRADIENTS FOR LOCALLY LIPSCHITZ INTEGRAL FUNCTIONALS ON NON-$L^p$-TYPE SPACES OF MEASURABLE FUNCTIONS

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Abstract. Let $(\Omega, \mu)$ be a measure space, $E$ be an arbitrary separable Banach space, $E^*_w$ be the dual equipped with the weak$^*$ topology, and $g : \Omega \times E \to \mathbb{R}$ be a Carathéodory function which is Lipschitz continuous on each ball of $E$ for almost all $s \in \Omega$. Put $G(x) := \int_\Omega g(s, x(s))d\mu(s)$. Consider the integral functional $G$ defined on some non-$L^p$-type Banach space $X$ of measurable functions $x : \Omega \to E$. We present several general theorems on sufficient conditions under which any element $\gamma \in X^*$ of Clarke’s generalized gradient (multivalued $C$-subgradient) $\partial_C G(x)$ has the representation $\gamma(v) = \int_\Omega \langle \zeta(s), v(s) \rangle d\mu(s) (v \in X)$ via some measurable function $\zeta : \Omega \to E^*_w$ of the associate space $X'$ such that $\zeta(s) \in \partial_C g(s, x(s))$ for almost all $s \in \Omega$. Here, given a fixed $s \in \Omega$, $\partial_C g(s, w_0)$ denotes Clarke’s generalized gradient for the function $g(s, \cdot)$ at $w_0 \in E$. Concerning $X$, we suppose that it is either a so-called non-solid Banach $M$-space (in particular, non-solid generalized Orlicz space) or Köthe–Bochner space (solid space).

Introduction. The purpose of the present paper is to formulate and prove several general results (see Theorems 3.2–3.3, Remark 3.4 in Section 3, Theorems 4.2–4.5 in Section 4) for the calculation of Clarke’s generalized gradients (also called multivalued $C$-

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subgradients in nonsmooth analysis) of locally Lipschitz integral functionals. We succeed to obtain the results for these functionals defined on a non-$L^p$-type space $X$ of measurable functions which is either a so-called non-solid Banach $M$-space (in particular, non-solid generalized Orlicz space) or Köthe–Bochner space (solid space).

Theorem 4.2 in the case of the solid space $X$ is a generalization of the $L^p$-result of F. H. Clarke, J. P. Aubin/F. H. Clarke in 1976–1983 [5, 11, 12]. Note that this $L^p$-result is intensively used in the theory of partial differential inclusions and non-smooth mechanics (see, e.g., [9, 16, 23, 24]).

In Section 6 we show that Theorem 4.2 with its proof in a very special case for the solid regular Orlicz–Bochner space $X = L^\Phi(E)$ implies an alternative proof for the $L^\Phi$-result of R. Pluciennik/S. Tian/Y. Wang in 1990 [36, Theorem 2]. We observe that the $L^p$-result of F. Clarke under his condition (A) (see [12, Theorems 2.7.5, 2.7.3]) can be standardly generalized to the case of Lipschitz integral functionals defined on a Köthe–Bochner space $X$ (solid space), and so we shall not present this generalization herein. It turns out that the $L^p$-result of F. Clarke [12, Theorem 2.7.5] under his condition (B) together with its proof given in [12, Proofs of Theorems 2.7.5, 2.7.2] can be generalized in several directions via introducing the so-called $U$-property such as in Lemmas 3.1, 4.1 and via using verifiable “majorant” conditions such as in Theorems 3.2–3.3, 4.2–4.3. Analyzing the above conditions (A)-(B), we propose the “majorant” conditions $(K6)$–$(K7)$ and $(K8)$–$(K9)$ in Theorems 4.4–4.5 and we need not apply Lebourg’s mean value theorem [12, Theorem 2.3.7] for proving Theorems 4.4–4.5.

By the non-smooth variational methods [9, 24], the calculation formula for Clarke’s generalized gradient established in Corollary 6.2 of Theorems 4.3, 4.5, can be applied to the solvability problem for the Dirichlet elliptic inclusion in $\mathbb{R}^2$ involving exponential-growth-type multivalued right-hand side in the Orlicz space $L^{b_0}$, where $\Phi_0(\alpha) = \exp(\alpha^2) - 1$ (see details in our paper [32]). Observe that the above mentioned $L^\Phi$-result of [36, Theorem 2] cannot be applied to this problem, since the “exponential-type” function $\Phi_0$ does not satisfy the condition $(\Delta)$ of [36]. The above inclusion in $L^{b_0}$ was studied by topological methods in [1, 2, 30, 31].

In Section 1 we give some terminology and auxiliary facts for Banach lattices, Köthe–Bochner spaces, Banach $M$-spaces, and Clarke’s generalized gradients. In Section 2 we give the auxiliary Lemmas 2.1, 2.3, and then the known Lemma 2.4 for convex integral functionals defined on these spaces. In Section 5 we give Corollaries 5.1–5.2 of Theorems 3.2–3.3 for the case of the non-solid generalized Orlicz space $X = L^M(\Omega, \mathbb{R}^m)$ with $m \geq 2$. In Section 6 we give Corollary 6.2 of Theorems 4.3, 4.5 for the solid Orlicz–Bochner space $X = L^\Phi(E)$.

1. Some terminology and auxiliary facts. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a complete $\sigma$-finite $\sigma$-additive measure $\mu$ on some $\sigma$-algebra $\mathcal{A}$ of subsets of $\Omega$. Throughout this paper $E$ denotes a separable Banach space. Denote [3] by $\text{Cl}(E)$ (resp., $\text{Cp}(E)$, $\text{Cv}(E)$, $\text{CvCp}(E)$, etc.) the family of all nonempty closed (resp., compact, convex, convex compact, etc.) subsets of $E$. By $\text{bco} A$ we denote the balanced closed convex hull of $A \subset E$. Put $B_E(u, r) := \{\omega \in E : \|\omega - u\|_E \leq r\}$ for $r \in (0, \infty)$. Given a Suslin locally convex
space $F$, we denote by $\mathcal{B}(F)$ the $\sigma$-algebra of Borel subsets of $F$. Then a multifunction $\Gamma: \Omega \to 2^F$ is called (see, e.g., [8, 13, 15]) measurable if $\Gamma^-(C) := \{s : \Gamma(s) \cap C \neq \emptyset\} \in \mathcal{A}$ for $C \in \mathcal{B}(F)$. $\text{Sel}\Gamma$ denotes the set of all measurable selections of $\Gamma$. The theory of measurable selections is given in [8, 13, 15] (in particular, for $\Gamma: \Omega \to 2^{E^*_w}$, since (by [8, p. 198], [7, p. 4.11]) the dual space $E^*_w$ with the topology $w^* = \sigma(E^*, E)$ is a Lusin space for the case of separability of $E$).

Given a function $x: \Omega \to E$ and a multifunction $H: \Omega \times E \to \text{Cp}(E^*_w)$, define $N_H(x) := \text{Sel} H(\cdot, x(\cdot))$. General theorems on the boundedness of the (Nemytskij) multivalued superposition operator $N_H$ can be found in [3, 4, 26, 34, 39]. We say that the Filippov implicit function property is valid for the multifunction $H$ if given any measurable function $\xi: \Omega \to E^*_w$ and any measurable multifunction $b: \Omega \to \text{Cp}(E)$ with $\xi(s) \in H(s, b(s))$ almost everywhere (a.e.), there exists a measurable function $z: \Omega \to E$ with $z(s) \in b(s)$ and $\xi(s) \in H(s, z(s))$ a.e. The multifunction $H$ is called $(\mathcal{A} \times B(E), B(E^*_w))$ (mod 0)-measurable if exists $\Omega_0$ with $\mu(\Omega \setminus \Omega_0) = 0$ such that $H$ is $(\mathcal{A} \times B(E), B(E^*_w))$-measurable on $\Omega_0 \times E$. The multifunction $H$ is called multi-superpositionally measurable if the multifunction $s \mapsto N_H(c)(s) := H(s, c(s))$ is measurable for any measurable multifunction $s \mapsto c(s) \in \text{Cp}(E)$. Many sufficient conditions for the above properties for $H$ can be found e.g. in [2, 8, 16, 42]. By Filippov’s theorem [2, Theorem 6.1] all the above properties are valid for any Carathéodory multifunction $H: \Omega \times \mathbb{R}^m \to \text{Cp}(\mathbb{R}^m)$.

Further, $L^0(\Omega, F)$ denotes the space of all (equivalent classes of) measurable functions $x: \Omega \to F$. A Banach space $K \subset L^0(\Omega, \mathbb{R})$ with norm $\| \cdot \|_K$ is called [4, 22, 41] a Banach lattice with monotone norm (also under the name, Köthe space, Banach ideal space), if $x \in K$ and $y \in L^0(\Omega, \mathbb{R})$ and $|y(s)| \leq |x(s)|$ a.e., then $y \in K$ and $\|y\|_K \leq \|x\|_K$. Put $K_+ := \{x \in K : x(s) \geq 0 \text{ a.e.}\}$. We shall use only $K$ with $\text{supp} \ K = \Omega$. Given a Banach lattice $K \subset L^0(\Omega, \mathbb{R})$, define [21, 41] the Köthe–Bochner space $X = K(E) \subset L^0(\Omega, E)$ as the Banach space of all measurable functions $x: \Omega \to E$ such that $\|x(\cdot)\|_E \in K$, with norm $\|x\|_X := \|\|x(\cdot)\|\|_K$. Let $L^\infty$ be the Banach algebra of all essentially bounded measurable scalar-valued functions defined on $\Omega$. A Banach space $X \subset L^0(\Omega, E)$ with norm $\|\cdot\|_X$ is called [29] a Banach $M$-space (or Banach $L^\infty$-module [27, 28, 33], or ideal* space [41]) if $x \in X$ and $\alpha \in L^\infty$ imply that $\alpha x \in X$ and $\|\alpha x\|_X \leq \|\alpha\|_{L^\infty} \|x\|_X$. If the above condition is valid only for $\alpha \in (L^\infty)_+$, then $X$ is called a Banach $M_+$-space.

The spaces $K$ and $K(E)$ are called solid spaces (see, e.g., [35] and references cited therein). Spaces of measurable functions which are not from the classes of the above spaces $K$ and $K(E)$ are usually called non-solid spaces. Historical comments on different classes of non-solid spaces can be found in [6, 20, 25, 35].

A prominent example of a Banach $M$-space is the generalized (non-solid in general for the case $m \geq 2$) Orlicz space $L^M = L^M(\Omega, \mathbb{R}^m) \subset L^0(\Omega, \mathbb{R}^m)$ with the Luxemburg norm $\|x\|_M = \inf \{\lambda > 0 : \int_\Omega M(x(s)/\lambda) d\mu(s) \leq 1\} < \infty$, where $M: \mathbb{R}^m \to [0, \infty)$ is a given Young (even, convex) function (see more general cases of $M$ defined on $\Omega \times E$ with $\text{dim} E \geq 2$, e.g. in [10, 14, 17, 18, 19, 20, 25, 26, 35, 40]).

From now on, we suppose that $X = X(\Omega, E)$ satisfies one of the following conditions (cf. [6, 20, 35]):
\[(M_m)\] \(E = \mathbb{R}^m, m \in \mathbb{N}, X \subset L^0(\Omega, E)\) is a Banach \(M\)-space with \(\text{supp} X = \Omega;\)

\[(M_\infty)\] \(\text{dim} E = +\infty, X \subset L^0(\Omega, E)\) is a Banach \(M\)-space and exist \(\alpha, \beta \in L^1(\Omega, (0, \infty))\) such that \(L^\infty(\Omega, E; \alpha) \subset X \subset L^1(\Omega, E; \beta)\) continuously, where \(L^\infty(\Omega, E; \alpha)\) is equipped with norm \(\|x\|_{L^\infty(\alpha)} := \frac{\|x\|}{\alpha(\cdot)}\) and \(L^1(\Omega, E; \beta)\) is equipped with norm \(\|x\|_{L^1(\beta)} := \frac{\|x\|}{\beta(\cdot)}\).

Under the condition \((M_m)\) with \(m \geq 2\), the associate space \(X' = X'(\Omega, \mathbb{R}^m)\) is defined in \([27, 28, 33]\) by

\[
X' := \{x' \in L^0(\Omega, \mathbb{R}^m) : x'(s) \in \text{vsupp} X(s) \text{ a.e., } (x, x') \in \mathbb{R} (\forall x \in X)\},
\]

where \((x, x') := \int_\Omega (x(s), x'(s))d\mu(s)\) and the vector support \(\text{vsupp} X\) (always exists) can be defined by \(\text{vsupp} X(s) := \text{the closure of} \{x_1(s), x_2(s), \ldots \} \text{ a.e. for some sequence} x_n \in X\) such that \(x \in X \Rightarrow x(s) \in \text{the closure of} \{x_1(s), x_2(s), \ldots \} \text{ a.e.}\). Under the condition \((M_\infty)\), the associate space \(X' = X'((\Omega, E_{w_*}^\infty))\) is defined as

\[
X' := \{x' \in L^0(\Omega, E_{w_*}^\infty) : (x, x') \in \mathbb{R} (\forall x \in X)\}.
\]

It is known that \(X'\) can be interpreted as a Banach subspace of the dual space \(X^*\) by the injection \(x' \mapsto \langle \cdot, x' \rangle\) with norm \(\|x'\|_{X'} := \sup\{\langle x, x' \rangle : \|x\|_X \leq 1\} < \infty\). If \(X = K(E)\) is the Köthe–Bochner space, then \(X' = (K(E))^\prime = K'(E_{w_*}^\infty)\), where \(K'(E_{w_*}^\infty)\) is the Köthe–Bochner space (modelled by means of \(K')\) of measurable functions \(y : \Omega \rightarrow E_{w_*}^\infty\) (if in addition \(E^*\) is separable, then \(X' = K'(E^*)\)). If \(X = L^M\), then \(X' = L^{M^*}\) with equivalent norms (in the case of \(M\) defined on \(E\) with \(\text{dim} E = +\infty\), the convex dual \(M^*\) is defined on \(E_{w_*}^\infty\)).

Let \(U\) be an open subset of a Banach space \(E\). If \(f : U \rightarrow \mathbb{R}\) is Lipschitz continuous on \(U\), then \(f\) has Clarke’s generalized derivative \(f^\circ(x; \cdot)\):

\[
f^\circ(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda} \quad (v \in E).
\]

The set \(\partial_C f(x) = \{\zeta \in E^* : \langle \zeta, v \rangle \leq f^\circ(x; v) \text{ for all } v \in E\}\) is called the generalized gradient (Clarke’s C-subgradient) of \(f\) at \(x\) (then \([12, 23]\) \(\partial_C f(x) \in \text{CvCp}(E_{w_*}^\infty)\)). A function \(f\) is said \([12, \text{Definition 2.3.4}]\) to be regular in Clarke’s sense at \(x\) if the directional derivative \(f'(x; v)\) of \(f\) at \(x\) along \(v\) exists and \(f'(x; v) = f^\circ(x; v)\) for every \(v \in E\).

2. Auxiliary lemmas. If \(g : \Omega \times E \rightarrow \mathbb{R}\) is locally Lipschitz continuous with respect to the second variable, then \(g^\circ(s, u_0; v)\) denotes the Clarke derivative at \(u_0\) in direction \(v\) of the function \(u \mapsto g(s, u)\). By \([12, 23]\) the function \(g^\circ(s, u; v)\) is continuous in \(v\). For the simplicity, \(\partial_C g(s, u_0)\) denotes the generalized gradient at \(u_0\) of \(g(s, \cdot)\). The proofs of Lemmas 2.1, 2.3 are standard via the known measurable selection theorems \([8]\) (for proving Lemma 2.3 one need else Lebourg’s theorem \([12, \text{Theorem 2.3.7}]\)).

**Lemma 2.1.** Let \(g : \Omega \times E \rightarrow \mathbb{R}\) be a function such that \(g(\cdot, u)\) is measurable for any \(u \in E\) and \(g(s, \cdot)\) is Lipschitz continuous on each ball of \(E\) for almost all \(s \in \Omega\). Then, given any measurable functions \(x, v : \Omega \rightarrow E\), the function \(s \in \Omega \mapsto g^\circ(s, x(s); v(s))\) is measurable.
THEOREM 2.2 (Lebourg [12, Theorem 2.3.7]). Let \( f : U \to \mathbb{R} \) be Lipschitz continuous on an open subset \( U \) of \( E \). Assume that \( U \) contains the convex interval \([x, y]\). Then, there exists a point \( u \in (x, y) \) such that \( f(x) - f(y) \in \langle \partial C f(u), x - y \rangle \).

LEMMA 2.3. Let \( g \) be as in Lemma 2.1 and \( x, y : \Omega \to E \) be two measurable functions. Suppose that the multifunction \( \Lambda : \Omega \times [0, 1] \to \text{CvCp}(\mathbb{R}) \), \((s, \alpha) \mapsto \Lambda(s, \alpha) := \langle \partial C g(s, \alpha x(s) + (1 - \alpha) y(s)), x(s) - y(s) \rangle \) is \((\mathfrak{A} \times \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})) \) (mod 0)-measurable. Then there exist measurable functions \( u : \Omega \to E \) and \( \xi : \Omega \to E^*_w \) such that \( u(s) \in [x(s), y(s)] \), \( \xi(s) \in \partial C g(s, u(s)) \) and \( g(s, x(s)) - g(s, y(s)) = \langle \xi(s), x(s) - y(s) \rangle \) a.e.

Proof. We divide the proof into Steps 2.1-2.3.

Step 2.1: Given a fixed measurable function \( z : \Omega \to E^*_w \), the multifunction \( M : \Omega \to \text{CvCp}(E^*_w) \cup \{\emptyset\} \), \( M(s) := \partial C g(s, z(s)) \) has dom \( M = \Omega \) (mod 0). By Lemma 2.1, the function \( s \mapsto g^*(s, z(s); v) = \sup \{\langle u^*, v \rangle : u^* \in \partial C g(s, z(s)) \} \) is measurable for all \( v \in E \) and so, by [8], \( M \) is measurable.

Step 2.2: Let
\[
H(s) := \{ u \in [x(s), y(s)] : g(s, x(s)) - g(s, y(s)) \in \langle \partial C g(s, u), x(s) - y(s) \rangle \},
\]
\[
F : \Omega \times [0, 1] \to \text{CvCp}(\mathbb{R}), F(s, \alpha) := \{ g(s, x(s)) - g(s, y(s)) - r : r \in \Lambda(s, \alpha) \}.
\]
By the assumption, we deduce that \( F \) is \((\mathfrak{A} \times \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})) \) (mod 0)-measurable and so \( F^-(\{0\}) \in (\mathfrak{A} \times \mathcal{B}(\mathbb{R})) \) (mod 0). On other hand, by Lebourg’s Theorem 2.2 for a.a. \( s \in \Omega \) there exists \( \alpha \in [0, 1] \) such that \( 0 \in F(s, \alpha) \), i.e. there exists \( \Omega_0 \subset \Omega \) such that \( \mu(\Omega \setminus \Omega_0) = 0 \) and \( \text{Proj}_\Omega F^-(\{0\}) = \Omega_0 \). Hence, by the von Neumann–Aumann selection theorem [8, Theorem III.22] the multifunction \( \tilde{H} : \Omega_0 \to \tilde{2}^{[0, 1] \setminus \{0\}}, \tilde{H}(s) := \{ \alpha : 0 \in F(s, \alpha) \} \) (whose graph coincides with \( F^-(\{0\}) \)) has a measurable selector \( a : \Omega_0 \to [0, 1] \). Then \( s \mapsto u(s) := a(s)x(s) + (1 - a(s))y(s) \) is a measurable selector of the multifunction \( H \) on \( \Omega_0 \).

Step 2.3: By Lebourg’s Theorem 2.2,
\[
\triangle(s) := \{ u^* \in E^* : g(s, x(s)) - g(s, y(s)) = \langle u^*, x(s) - y(s) \rangle \} \in \text{CvCl}(E^*_w)
\]
for all \( s \in \Omega_0 \). It is an easy check that \( \triangle \) is measurable. By Steps 2.1-2.2, we deduce that the multifunction \( \Gamma : \Omega_0 \to \tilde{2}^{E^*_w}, \Gamma(s) := \triangle(s) \setminus \partial C g(s, u(s)) \in \text{CvCl}(E^*_w) \), is measurable. By [8], there exists a measurable selector \( \xi \) of \( \Gamma \), i.e. \( \xi(s) \in \partial C g(s, u(s)) \) and \( g(s, x(s)) - g(s, y(s)) = \langle \xi(s), x(s) - y(s) \rangle \) on \( \Omega_0 \). \( \blacksquare \)

A function \( f : \Omega \times F \to \mathbb{R} := \mathbb{R} \cup \{\pm \infty\} \) is called a normal integrand if \( f(s, \cdot) \) is lower semicontinuous for almost all (a.a.) \( s \in \Omega \) and \( f \) is \((\mathfrak{A} \times \mathcal{B}(F), \mathcal{B}(\mathbb{R})) \) (mod 0)-measurable on \( \Omega \times F \). If \( F \) is a Lusin space, then [7, Lemma 1.2.3] every Carathéodory function on \( \Omega \times F \) is a normal integrand. If \( f \) is a normal integrand on \( \Omega \times E \), then the dual convex normal integrand \( f^* : \Omega \times E^*_w \to \mathbb{R} \) is defined by \( f^*(s, u^*) = \sup \{ \langle u, u^* \rangle - f(s, u) : u \in \in E \} \). Denote by \( X^*_f \) the space of singular linear functionals on \( X = X(\Omega, E) \) (see [20, 22, 33]). Lemma 2.4 is taken from V. Levin [20, Corollary 1 of Theorem 6.7, p. 216], A. Kozek [19] \((X = L^M(\Omega, E) \) with \( E^* \) being separable), [8] (see also the references cited therein; in [20] this result is valid for a more general space \( X \)). Given a Banach space \( E \)
and a convex function $\varphi : E \to \mathbb{R}$, the subdifferential of $\varphi$ is defined by
\[
\partial \varphi(u) = \{ \xi \in E^* : \langle \xi, v - u \rangle \leq \varphi(v) - \varphi(u) \text{ for all } v \in E \}.
\]

**Lemma 2.4.** Let $f$ be a normal convex integrand on $\Omega \times E$ and the functional $I_f$, $I_f(x) := \int_\Omega f(s, x(s))d\mu(s)$, be considered on $X$ with $(M_m)$ or $(M_\infty)$. Suppose that the set $\text{dom} I_f := \{ x \in X : I_f(x) < \infty \}$ is nonempty. Then for every $x_0 \in \text{dom} I_f$ the subdifferential $\partial I_f(x_0)$ consists of all linear functionals $\lambda \in (X(\Omega, E))^*$ of the form $\lambda(x) = \langle x, y \rangle + \lambda_s(x)(x \in X)$, where $y \in X'(\Omega, E^*_w) \cap \text{Sel} \partial f_x(\cdot, x_0(\cdot))$, $\lambda_s \in (X(\Omega, E))^*$, $\lambda_s \in K(\text{dom} I_f, x_0) := \{ l \in (X(\Omega, E))^* : l(\cdot - x_0) \leq 0 (\forall z \in \text{dom} I_f) \}$, and $\partial f_x(s, u_0)$ denotes the subdifferential at $u_0$ of the convex function $u \in \text{vsupp} X(s) \mapsto f(s, u)$.

### 3. Lipschitz integral functionals on non-solid Banach $M$-spaces.

Given a Banach $M_\lambda$-space $X \subset L^0(\Omega, E)$, we say that a multivalued operator $M : X \to 2^{L^1(\Omega, \mathbb{R})} \setminus \{ \emptyset \}$ has the $U$-property if for any given $x \in X$ for each sequence $x_k \subset X$ with $\sum_{k=1}^\infty \| x_k - x \|_X < \infty$ there exists $\beta \in L^1(\Omega, \mathbb{R})$ such that $|\alpha_k(s)| \leq \beta(s)$ a.e. for every $\alpha_k \in M(x_k)$ and for every $k \in \mathbb{N}$. Given a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, define the integral functional
\[
G(x) := \int_\Omega g(s, x(s))d\mu(s).
\]

**Lemma 3.1.** Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $g(s, \cdot)$ is Lipschitz continuous on each ball of $E$ for almost all $s \in \Omega$, and $X \subset L^0(\Omega, E)$ be a non-solid Banach $M_\lambda$-space with either $(M_\infty)$ or $(M_m)$ with $m \geq 2$, $\text{vsupp} X(s) = \mathbb{R}^m$. Suppose the following conditions:

1. $\partial_C g : \Omega \times E \to \text{CvCp}(E^*_w)$ is $(\mathcal{A} \times \mathcal{B}(E), \mathcal{B}(E_w^*))$ (mod 0)-measurable;
2. The multivalued superposition operator $N_{\partial_C g} : X \to 2^{X^*} \setminus \{ \emptyset \}$ is bounded on each ball of $X$, and:

1) for each $x, v \in X$ the function $s \in \Omega \mapsto g^0(s, x(s); v(s))$ belongs to $L^1(\Omega, \mathbb{R})$,
2) the operator $M_v : X \to 2^{L^1(\Omega, \mathbb{R})} \setminus \{ \emptyset \}$ has the $U$-property, where $M_v(x) := \{ \alpha \in L^1(\Omega, \mathbb{R}) : \alpha(\cdot) = \langle \zeta(\cdot), v(\cdot) \rangle, \zeta \in N_{\partial_C g}(x) \}$.

If $G$ is finite at least for one $x_* \in X$, then $G$ is Lipschitz on each ball of $X$ and
\[
\partial_C G(x) \subset N_{\partial_C g}(x),
\]

i.e. $\gamma \in \partial_C G(x) \Rightarrow \gamma(\cdot) = \langle \zeta(\cdot), \cdot \rangle$ for some $\zeta \in X'$ with $\zeta(s) \in \partial_C g(s, x(s))$ a.e. If additionally the function $g(s, \cdot)$ is regular (in Clarke’s sense) at $x(s)$ for almost all $s \in \Omega$, then the functional $G$ is regular at $x$ and $\partial_C G(x) = N_{\partial_C g}(x)$.

**Proof.** We shall mimic Clarke’s proof of his Theorem 2.7.5/(B) in [12] but new moments in our proof will be emphasized and given in detail. We divide our proof into Steps 3.1-3.4.

**Step 3.1:** We claim that the functional $G$ is Lipschitz continuous on every ball of $X$. To prove this, we fix $y, z \in B_X(0, r)$. By the condition $(N1)$, $\partial_C g : \Omega \times E \to \text{CvCp}(E^*_w)$ is $(\mathcal{A} \times \mathcal{B}(E), \mathcal{B}(E_w^*))$ (mod 0)-measurable, and so we can apply Lemma 2.3 (the parametric version of Lebourg’s theorem). We can then find some measurable functions $\xi_0 : \Omega \to E^*_w$,
$u_0: \Omega \to E$ and $\theta_0: \Omega \to [0, 1]$ such that
\[
u_0(s) = \theta_0(s)z(s) + (1 - \theta_0(s))y(s),
\]
\[
g(s, z(s)) - g(s, y(s)) = \langle \xi_0(s), z(s) - y(s) \rangle,
\]
\[
\xi_0(s) \in \partial C g(s, u_0(s)) \text{ a.e.}
\]
Since $X$ is a Banach $M_+$-space, we have that $u \in X$ and
\[
\|u_0\|_X \leq \|\theta_0z\|_X + \|(1 - \theta_0)y\|_X \leq \|z\|_X + \|y\|_X \leq 2r.
\] (3.3)
By the condition (N2), $C(2r) := \sup\{\|v\|_{X'} : v \in N_{\partial g}(w), w \in B_X(0, 2r)\} < \infty$. Therefore $\|\xi_0\|_{X'} \leq C(2r)$ follows. Hence, by the definition of the associate space $X'$, we deduce that
\[
|G(z) - G(y)| \leq \int_{\Omega} |g(s, z(s)) - g(s, y(s))|d\mu(s)
\]
\[
\leq \int_{\Omega} |\langle \xi_0(s), z(s) - y(s) \rangle|d\mu(s) \leq \|\xi_0\|_{X'}\|z - y\|_X \leq C(2r)\|z - y\|_X
\]
for $z, y \in B_X(0, r)$. Since $G(x_s) \in \mathbb{R}$, the claim of Step 3.1 follows.

**Step 3.2:** We shall prove that $G^o(x; v) \leq \int_{\Omega} g^o(s, x(s); v(s))d\mu(s)$ for $x, v \in X$. From the definition of the Clarke derivative in the direction $v \in X$ we have
\[
G^o(x; v) = \limsup_{y \to x} \lim_{\lambda \to 0} \int_{\Omega} \frac{g(s, y(s) + \lambda v(s)) - g(s, y(s))}{\lambda} d\mu(s).
\] (3.4)
Let us choose arbitrary sequences $\lambda_k$ in $\mathbb{R}$ and $y_k$ in $X$ such that $\lambda_k \downarrow 0$, $\|y_k - x\|_X \to 0$ and the limit
\[
b := \lim_{k \to \infty} \int_{\Omega} F_k(s)d\mu(s)
\] (3.5)
exists, where $F_k(s) := \frac{g(s, y_k(s) + \lambda_k v(s)) - g(s, y_k(s))}{\lambda_k}$. Under $(M_\infty)$ or $(M_m)$, by [22, 41] for $m = 1$ and by [29, Theorem 2.1] for $m > 1$ together with Riesz’s theorem, we can choose a subsequence $k_j$ and $D_0$ with $\mu(\Omega \setminus D_0) = 0$ such that $y_{kj}(s) \to x(s)$ as $j \to \infty$ ($\forall s \in D_0$), $\|y_{kj} - x\|_X \leq 1/2^j$ and $\lambda_{kj} \leq 1/2^j$.

We claim the existence of $\beta \in L^1(\Omega, \mathbb{R})$ and of $D_1 \subset D_0$ with $\mu(\Omega \setminus D_1) = 0$ such that $|F_{kj}(s)| \leq \beta(s)$ on $D_1$ for all $j \in \mathbb{N}$. To prove this, by the condition (N1) together with Lemma 2.3 there exist measurable functions $\xi_k \in N_{\partial g}(u_k)$ and $u_k$ such that $u_k(s) \in [y_k(s) + \lambda_k v(s), y_k(s)]$ and $F_k(s) = \langle \xi_k(s), v(s) \rangle$ a.e., and so $F_k \in M_v(u_k)$. Since $X$ is a Banach $M_+$-space, we get, by an analogous argument to that for (3.3) in Step 3.1, that $u_k \in X$ and
\[
\|u_{kj} - x\|_X \leq \|y_{kj} - x\|_X + \lambda_{kj}\|v\|_X \leq 1/2^j + (1/2^j)|v|_X \Rightarrow \sum_{j=1}^{\infty} \|u_{kj} - x\|_X < \infty.
\] (3.6)
Since $M_v$ has the $U$-property due to the condition (N2), there exist $\beta \in L^1(\Omega, \mathbb{R})$ and $D_1 \subset D_0$ with $\mu(\Omega \setminus D_1) = 0$ such that $|F_{kj}(s)| \leq \beta(s)$ ($\forall s \in D_1$).

Using the above claim together with the measurability of the function $s \mapsto g^o(s, x(s); v(s))$ (see Lemma 2.1), we can apply the Fatou lemma for the functions $s \in D_1 \mapsto g^o(s, x(s); v(s))$. 

\[
\|u_{kj} - x\|_X \leq \|y_{kj} - x\|_X + \lambda_{kj}\|v\|_X \leq 1/2^j + (1/2^j)|v|_X \Rightarrow \sum_{j=1}^{\infty} \|u_{kj} - x\|_X < \infty.
\] (3.6)
\[\beta(s) - F_{k_j}(s) \in [0, \infty),\] and by (N2) we deduce then
\[
b = \limsup_{j \to \infty} \int_{D_1} F_{k_j}(s) d\mu(s) \leq \int_{D_1} \limsup_{j \to \infty} F_{k_j}(s) d\mu(s) \leq \int_{D_1} \limsup_{j \to \infty} g(s, u + \lambda v(s)) - g(s, u) d\mu(s) = \int_{\Omega} g^\circ(s, x(s); v(s)) d\mu(s) < \infty.
\]

Therefore,
\[
G^\circ(x; v) = \sup \{b = \lim_{k \to \infty} \int_{\Omega} F_k(s) d\mu(s) : \|y_k - x\| \to 0, \lambda_k \downarrow 0\} \\
\leq \int_{\Omega} g^\circ(s, x(s); v(s)) d\mu(s) < \infty.
\]

**Step 3.3:** We mimic Clarke's argument in the proof of [12, Theorem 2.7.2]. We know (see Lemma 2.1) that the function \((s, u) \in \Omega \times E \mapsto g^\circ(s, x(s); u)\) is a Carathéodory convex integrand, and \(v \in X \mapsto \tilde{G}(v) := \int_{\Omega} g^\circ(s, x(s); v(s)) d\mu(s)\) is a convex functional on \(X\) such that \(\tilde{G}(0) = 0\). If \(\gamma = \partial C G(x)\), then by Step 3.2 for every \(v \in X\) we have \(\gamma(v) \leq G^\circ(x; v) \leq \int_{\Omega} \tilde{G}(v) = \tilde{G}(v) - \tilde{G}(0)\), and so \(\gamma\) is an element of the subdifferential \(\partial G(0)\). By Lemma 2.4 together with \(\text{dom} I_{\tilde{G}} = X\) from the condition (N2), we can deduce that \(\partial G(0)\) consists of linear functionals \(\gamma \in X^*\) of the form \((\gamma, v) = \int_{\Omega} (\zeta(s), v(s)) d\mu(s) (v \in X)\) with \(\zeta \in X'\) and \(\zeta(s) \in \partial C g(s, x(s))\) a.e. on \(\Omega\). Hence, the inclusion (3.2) follows.

**Step 3.4:** Suppose that \(g(s, \cdot)\) is regular (in Clarke's sense [12, Section 2.3]) at \(x(s)\) for almost all \(s \in \Omega\). Fix \(v \in X\). For an analogous reason and by an analogous argument as in Step 3.2, we can apply the Fatou lemma for the functions \(s \mapsto \beta(s) + F_{k_j}(s) \in [0, \infty)\) and deduce
\[
\liminf_{\lambda \downarrow 0} \frac{G(x + \lambda v) - G(x)}{\lambda} \geq \int_{\Omega} \liminf_{\lambda \downarrow 0} \frac{g(s, x(s) + \lambda v(s)) - g(s, x(s))}{\lambda} d\mu(s) \\
= \int_{\Omega} g'(s, x(s); v(s)) d\mu(s) = \int_{\Omega} g^\circ(s, x(s); v(s)) d\mu(s) \geq G^\circ(x; v).
\]

Now we can deduce by Clarke's argument in the proof of [12, Theorem 2.7.3, p. 87] that \(G\) is regular at \(x\) and \(N_{\partial C g(x)} \subset \partial C G(x)\).

By Lemma 3.1 we can prove the following Theorems 3.2-3.3.

**Theorem 3.2.** Let \(g : \Omega \times \mathbb{R}^m \to \mathbb{R}\) be a Carathéodory function such that \(g(s, \cdot)\) is Lipschitz continuous on each ball of \(\mathbb{R}^m\) for almost all \(s \in \Omega\), and \(X \subset L^0(\Omega, \mathbb{R}^m)\) be a non-solid Banach \(M\)-space with \(m \geq 2\), \(\text{vsupp} X(s) = \mathbb{R}^m\). Suppose that \(\partial C g\) satisfies (N1) and there exists \(H : \Omega \times \mathbb{R}^m \to \mathbb{Cp}(\mathbb{R}^m)\) satisfying the following conditions:

(N3) The multivalued superposition operator \(N_H : X \to 2^{X'} \setminus \emptyset\) is bounded on each ball of \(X\);

(N4) There exists \(\Omega_0 \in \mathcal{A}\) with \(\mu(\Omega \setminus \Omega_0) = 0\) such that
1) \(\partial C g(s, u) \subset \text{bco} H(s, [-1, 1]u)\) for all \(s \in \Omega_0\) and for all \(u \in \mathbb{R}^m\),
2) \(H(s, C) \in \text{Cp}(\mathbb{R}^m)\) for \(C \in \text{Cp}(\mathbb{R}^m)\) for all \(s \in \Omega_0\).
3) \( H \) is multi-superpositionally measurable,

4) \( H \) has the Filippov implicit function property.

Then, the statement of Lemma 3.1 is valid for \( G \) defined on \( X \).

\[ \text{Proof.} \] It suffices to check the condition (N2), and then Theorem 3.2 follows from Lemma 3.1. We divide this proof into Steps 3.5-3.9. We need the following technical notion from [27, 28]. Given \( Y \subset L^0(\Omega, \mathbb{R}^m) \) with \( m \geq 2 \), denote by \( \text{Arm}(Y) \) the set of all infra-semi-units for \( Y \), i.e. measurable multifunctions \( b: \Omega \to 2^\mathbb{R}^m \setminus \{\emptyset\} \) such that \( b(s) \in \text{CvCp}(\mathbb{R}^m) \) is balanced in \( \mathbb{R}^m \) a.e. and \( \text{Sel} b \subset Y \).

**Step 3.5:** We claim that \( a \in \text{Arm}(X') \) for each \( b \in \text{Arm}(X) \), where

\[ a(s) := \overline{\text{co}} H(s, b(s)). \]

To prove this claim, by the technical theorem 3 of [28] (see also [27, Theorem 4.2]), there exist \( \tilde{x}_1, \ldots, \tilde{x}_m \in X \) such that

\[ \sum_{j=1}^{m} [-1, 1] \tilde{x}_j(s) \subset b(s) \subset D(s) := (m + 1/2) \sum_{j=1}^{m} [-1, 1] \tilde{x}_j(s) \text{ a.e.} \quad (3.7) \]

Put \( c(s) := \overline{\text{co}} H(s, D(s)) \). Fix \( y \in \text{Sel} c = \text{Sel} \overline{\text{co}} H(., D(\cdot)) \). By the condition (N4) together with the parametric version [8, Theorem IV.11] of Carathéodory’s theorem, the multifunctions \( s \mapsto H(s, D(s)) \in \text{Cp}(\mathbb{R}^m) \), \( s \mapsto c(s) \in \text{Cp}(\mathbb{R}^m) \) are measurable and there exist measurable functions \( \alpha_i, \xi_i \) such that \( y(s) = \sum_{i=1}^{m} \alpha_i(s) \xi_i(s) \), \( \sum_{i=1}^{m} |\alpha_i(s)| = 1 \), \( \xi_i(s) \in H(s, D(s)) \) a.e.; then there exist some measurable functions \( z_i \) such that \( z_i(s) \in D(s) \) and \( \xi_i(s) \in H(s, z_i(s)) \) a.e.

Then (e.g., by [29, Lemma 3.1/(2)]), for \( z_i \in \text{Sel} D \), there exist functions \( \alpha_{ij} \in L^\infty \) such that \( z_i(s) = (m + 1/2) \sum_{j=1}^{m} \alpha_{ij}(s) \tilde{x}_j(s) \) and \( \|\alpha_{ij}\|_{L^\infty} \leq 1 \). Since \( X \) is a Banach \( M \)-space, we obtain

\[ \|z_i\|_X \leq (m + 1/2) \sum_{j=1}^{m} \|\alpha_{ij}\|_X \|\tilde{x}_j\|_X \leq (m + 1/2) \sum_{j=1}^{m} \|\tilde{x}_j\|_X := \kappa < \infty. \quad (3.8) \]

By the condition (N3), \( r(\kappa) := \sup \{\|\xi\|_{X'} : \xi \in N_H(z), \|z\|_X \leq \kappa \} < \infty \) for \( \kappa \in (0, \infty) \). Since \( X' \) is a Banach \( M \)-space and \( \xi_i \in N_H(z_i) \), (3.8) implies that

\[ \|y\|_{X'} \leq \sum_{i=1}^{m+1} \|\alpha_i\|_{X'} \|\xi_i\|_{X'} \leq \sum_{i=1}^{m+1} \|\xi_i\|_{X'} \leq (m + 1) r(\kappa) < \infty. \quad (3.9) \]

Hence \( \text{Sel} c \subset X' \), and so \( \text{Sel} a \subset X' \). By the condition (N4), \( a(s) \) is a balanced convex compact set and \( a \) is measurable, and hence the claim follows.

**Step 3.6:** We claim that given \( a \in \text{Arm}(X') \) and \( v \in X \), there exists \( \beta_a \in L^1(\Omega, (0, \infty)) \) such that \( |\langle v(s), d(s) \rangle| \leq \beta_a(s) \) a.e. for any \( d \in \text{Sel} a \). By the technical theorem 3 of [28] (see also [27, Theorem 4.2]), there exist \( \bar{y}_1, \ldots, \bar{y}_m \in X' \) such that

\[ \sum_{j=1}^{m} [-1, 1] \bar{y}_j(s) \subset a(s) \subset (m + 1/2) \sum_{j=1}^{m} [-1, 1] \bar{y}_j(s) \text{ a.e.} \]

Then (e.g., by [29, Lemma 3.1/(2)]), for a fixed \( d \in \text{Sel} a \), there exist \( m \) functions \( \eta_j \in L^\infty \) such that \( d(s) = (m + 1/2) \sum_{j=1}^{m} \eta_j(s) \bar{y}_j(s) \) a.e. and \( \eta_j(s) \in [-1, 1] \). Then, for a.a. \( s \in \Omega \)
we get
\[ |\langle v(s), d(s) \rangle| \leq (m + 1/2) \sum_{j=1}^{m} |\eta_j(s)\rangle |\langle v(s), \tilde{y}_j(s) \rangle| \]
\[ \leq (m + 1/2) \sum_{j=1}^{m} |\langle v(s), \tilde{y}_j(s) \rangle| := \beta_a(s). \]

Since \( v \in X \) and \( \tilde{y}_j \in X' \), we have \( \beta_a \in L^1(\Omega, \mathbb{R}) \).

**Step 3.7:** We claim that for each \( \delta \in (0, \infty) \) there exists \( \tilde{r}(\delta) \in (0, \infty) \) such that \( \|x\|_{X} \leq \delta \) implies \( \|y\|_{X'} \leq \tilde{r}(\delta) \) for \( y \in N_{\partial g}(x) \). We deduce this from the proof of Step 3.5. Fix \( x \in X \) with \( \|x\|_{X} \leq \delta \). Put \( b(s) = [-1, 1]|x(s)|. \) Then \( b(s) = \sum_{j=1}^{m} [-1, 1]\tilde{x}_j(s) \subset D(s) = [-1, 1]D(s) \), where \( \tilde{x}_1(s) = x(s), \tilde{x}_j(s) = 0 \) \((j \geq 2)\). By the conditions \((N3)-(N4)\) together with \((3.8)-(3.9)\), for a fixed \( y \in N_{\partial g}(x) \subset \text{Sel} c \) we have that
\[ \|z_i\|_X \leq (m + 1/2)\|\tilde{x}_1\|_X \leq (m + 1/2)\delta := k(\delta) < \infty \]
and \( \|y\|_{X'} \leq (m + 1/2)\|k(\delta)\| := \tilde{r}(\delta) < \infty. \)

**Step 3.8:** We shall check the \( U \)-property for the multivalued operator \( M_v : X \to 2^{L^1(\Omega, \mathbb{R})} \setminus \{\emptyset\} \) defined as in condition \((N2)\) at a fixed \( x \in X \). Fix \( v \in X \) and a sequence \( \{x_i\}_{i \in \mathbb{N}} \subset X \) such that \( \sum_{i=1}^{\infty} \|x_i - x\|_X < \infty \). Fix \( \alpha_i \in M_v(x_i); \) then \( \alpha_i(s) = \langle v(s), \xi_i(s) \rangle \) a.e. on \( \Omega \) for some \( \xi_i \in N_{\partial g}(x_i) \). By the technical theorem 7 of [28] (see also [27, Theorem 4.1]) we get some \( \Omega_1 \subset \Omega_0 \) with \( \mu(\Omega \setminus \Omega_1) = 0 \) such that the series of compact sets
\[ b(s) := x(s) + \sum_{i=1}^{\infty} [-1, 1](x_i(s) - x(s)) \]
converges in the space \( C_p(\mathbb{R}^m) \) for all \( s \in \Omega_1 \); moreover, putting \( b(s) := \{0\} \) for \( s \in \Omega \setminus \Omega_1 \), we get \( b \in \text{Arm}(X) \). By Step 3.5 we get that \( a(\cdot) := \text{bco} \ H(\cdot, b(\cdot)) \in \text{Arm}(X') \). Due to the condition \((N4)\) we get that
\[ x_i(s) \in b(s) = [-1, 1]b(s), \quad \partial g(s, x_i(s)) \subset \text{bco} \ H(s, b(s)) \quad (\forall s \in \Omega_1) \]
and \( \xi_i \in N_{\partial g}(x_i(s)) \subset \text{Sel} a. \) Hence, by Step 3.6, the \( U \)-property for \( M_v \) follows.

**Step 3.9:** We claim that the function \( s \in \Omega \mapsto g^\circ(s, x(s); v(s)) \) belongs to \( L^1(\Omega, \mathbb{R}) \) for any \( x, v \in X \). Since \( X \subset L^0(\Omega, \mathbb{R}^m) \) is a Banach \( M \)-space, by [29, Theorem 3.2, Lemma 3.2], there exist \( \alpha \in L^0(\Omega, (0, \infty)) \) and \( m \) functions \( g_1, \ldots, g_m \in X \) such that the linear hull of \( \{g_1(s), \ldots, g_m(s)\} = \text{vsupp} X(s) = \mathbb{R}^m \) a.e., and \( v(s) := \sum_{j=1}^{m} [-1, 1]g_j \) satisfies \( v \in \text{Arm}(X) \) with \( \alpha(s)B_{\mathbb{R}^m}(0, 1) \subset v(s) \) a.e. Then,
\[ |g^\circ(s, x(s); v(s))| \]
\[ \leq \sup \left\{ \frac{g(s, \tilde{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda} : \lambda \in (0, 1], \|\bar{u} - x(s)\|_{\mathbb{R}^m} \leq \alpha(s) \right\} \]
\[ = \sup \left\{ \frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda} : \lambda \in (0, 1], \bar{u} \in x(s) + \alpha(s)B_{\mathbb{R}^m}(0, 1) \right\} \]
\[ \leq \sup \left\{ \frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda} : \lambda \in (0, 1], \bar{u} \in x(s) + v(s) \right\}. \]
By Lebourg’s Theorem 2.2 and (N4), we deduce that

$$|g^o(s,x(s);v(s))|$$

$$\leq \sup\{|\langle v(s), u^* \rangle| : u^* \in \partial_C g(s,u), u \in [\bar{u}, \bar{u} + \lambda v(s)], \lambda \in (0,1], \bar{u} \in x(s) + v(s)\}$$

$$\leq \sup\{|\langle v(s), u^* \rangle| : u^* \in \partial_C g(s,u), u \in [-1,1]v(s) + x(s) + v(s)\}$$

$$\leq \sup\{|\langle v(s), u^* \rangle| : u^* \in \partial_C g(s,u), u \in b_0(s)\}$$

$$\leq \sup\{|\langle v(s), u^* \rangle| : u^* \in a_0(s)\} \ a.e.,$$

where

$$b_0(s) := v(s) + [-1,1]x(s) + v(s), \quad a_0(s) := \overline{bco} H(s, b_0(s)).$$

Then, $b_0 \in \text{Arm}(X)$, and so by Step 3.5, $a_0 \in \text{Arm}(X')$. Since $a_0$ is measurable, by [8] there exists some Castaing representation $\{u_q^* : q \in \mathbb{N}\}$ for $a_0$, i.e. $u_q^* \in \text{Sel} a_0$ and $a_0(s) = \text{the closure of } \{u_q^*(s) : q \in \mathbb{N}\}$ a.e. Hence,

$$|g^o(s,x(s);v(s))| \leq \sup\{|\langle v(s), u_q^*(s) \rangle| : q \in \mathbb{N}\} \ a.e.$$

By Step 3.6 for $a_0$, there exists $\beta_{a_0} \in L^1(\Omega, \mathbb{R})$ with $|\langle v(s), u_q^*(s) \rangle| \leq \beta_{a_0}(s) \ a.e.$ So, the above claim follows. $\blacksquare$

**Theorem 3.3.** Let $g : \Omega \times \mathbb{R}^m \to \mathbb{R}$, a non-solid Banach $M$-space $X \subset L^0(\Omega, \mathbb{R}^m)$ with $m \geq 2$ be such as in Theorem 3.2. Suppose that $\partial_C g$ satisfies (N1) and for every $R \in (0, \infty)$ there exists $H_R : \Omega \times \mathbb{R}^m \to \text{Cp}(\mathbb{R}^m)$ satisfying the following conditions:

(N5) The multivalued superposition operator $N_{H_R} : B_X(0,R) \to 2^{X'} \setminus \{\emptyset\}$ is bounded;

(N6) $\partial_C g(s,x(s)) \subset \overline{bco} H_R(s, [-1,1]x(s)) \ a.e. \ for \ each \ x \in B_X(0,R)$, and there exists $\Omega_0 \in \mathcal{A}$ with $\mu(\Omega \setminus \Omega_0) = 0$ such that

1. $H_R(s,C) \in \text{Cp}(\mathbb{R}^m)$ for $C \in \text{Cp}(\mathbb{R}^m)$ for all $s \in \Omega_0$,
2. $H_R$ is multi-superpositionally measurable,
3. the Filippov implicit function property is valid for $H_R$.

Then, the statement of Lemma 3.1 is valid for $G$ defined on $X$.

**Proof.** It suffices to check the condition (N2), and then Theorem 3.3 follows from Lemma 3.1. Hence, it suffices to modify Steps 3.5, 3.7–3.9.

**Modification of Step 3.5:** Let $b \in \text{Arm}(X)$ with $\|x\| \leq \delta (\forall x \in \text{Sel} b)$ for some $\delta \in (0, \infty)$. Then we claim that $a_{R_1} \in \text{Arm}(X')$, where $a_{R_1}(s) := \overline{bco} H_{R_1}(s,b(s))$ and $R_1 := (m + 1/2)\delta$. The proof of this claim is analogous to the proof of Step 3.5. Here, (3.7)–(3.8) imply that $\|\tilde{x}_j\| \leq \delta$ and $\|z_i\| \leq R_1$. So, it suffices to substitute $H_{R_1}$ for $H$, $a_{R_1}$ for $a$, and $c_{R_1}(s) := \overline{bco} H_{R_1}(s,D(s))$ for $c(s)$.

**Modification of Step 3.7:** It suffices to substitute $c_{R_1}$ for $c$, where $R_1 = (m + 1/2)\delta$, then given $y \in N_{\partial_C g}(x)$ with $\|x\| \leq \delta$ we get $y \in N_{\partial g}(x) \subset \text{Sel} c_{R_1}$.

**Modification of Step 3.8:** We observe that by the technical theorem 7 of [28] (see also [27, Theorem 4.1]) together with [29, Lemma 3.1] we get

$$\|\tilde{x}\| \leq \delta_2 := \|x\| + \sum_{i=1}^{\infty} \|x_i - x\| < \infty$$
for every \( \tilde{x} \in \text{Sel } b_0 \). Then, it suffices to substitute \( H_{R_2} \) for \( H \) and \( a_{R_2} \) for \( a \) with \( R_2 := (m + 1/2)m\delta_2 \).

**Modification of Step 3.9:** Observe that by [29, Lemma 3.1], we have

\[
\tilde{x} \in \text{Sel } b_0 \implies \|\tilde{x}\|_X \leq \delta_3 := \|v\|_X + \|x\|_X + \sum_{i=1}^{m} \|g_i\|_X < \infty.
\]

It suffices to substitute \( H_{R_3} \) for \( H \) and \( a_{R_3} \) for \( a \), where \( R_3 := (m + 1/2)m\delta_3 \). □

**Remark 3.4.** The statements of Lemma 3.1 and Theorems 3.2–3.3 remain valid for the non-solid Banach space \( X \subset L^0(\Omega, \mathbb{R}) \) with the arbitrary vector support \( \text{vsupp } X \), but then we need to substitute \( g_X^\circ(s, u_0; v) \) for \( g^\circ(s, u_0; v) \) as well as \( \partial_C g_X(s, u_0) \) for \( \partial_C g(s, u_0) \), where \( g_X^\circ(s, u_0; v) \) denotes the Clarke derivative at \( u_0 \) in direction \( v \in \text{vsupp } X(s) \) for the function \( u \in \text{vsupp } X(s) \mapsto g(s, u) \), and \( \partial_C g_X(s, u_0) \) denote the generalized gradient at \( u_0 \) of the function \( u \in \text{vsupp } X(s) \mapsto g(s, u) \).

**4. Lipschitz integral functionals on Köthe–Bochner spaces.** Given a Banach lattice \( K \subset L^0(\Omega, \mathbb{R}) \), we say that an operator \( Q : K_+ \to K' \) has the \( U \)-property if given any \( a \in K_+ \) for each sequence \( a_k \subset K_+ \) with \( \sum_{k=1}^{\infty} \|a_k - a\|_K < \infty \) there exists \( d \in K' \) such that \( |Q(a_k)(s)| \leq d(s) \) a.e. on \( \Omega \) for every \( k \in \mathbb{N} \).

**Lemma 4.1.** Let \( g : \Omega \times E \to \mathbb{R} \) be a Carathéodory function such that \( g(s, \cdot) \) is Lipschitz continuous on each ball of \( E \) for almost all \( s \in \Omega \), and \( K \subset L^0(\Omega, \mathbb{R}) \) be a Banach lattice. Suppose the following conditions hold:

(K1) There exists \( h : \Omega \times [0, \infty) \to [0, \infty) \) such that

\[
\sup\{\|u^*\|_{E^*} : u^* \in \partial_C g(s, u), \|u\|_E \leq \alpha\} \leq h(s, \alpha)
\]

for almost all \( s \in \Omega \) and for all \( \alpha \in [0, \infty) \);

(K2) The superposition operator \( N_h : K_+ \to K' \) is bounded on each ball of \( K_+ \) and has the \( U \)-property.

Then, the statement of Lemma 3.1 is valid for \( G \) defined on \( X = K(E) \).

**Proof.** We shall mimic Clarke’s proof of his Theorem 2.7.5/(B) [12] but new moments in our proof will be emphasized and given in detail. We divide our proof into Steps 4.1–4.5.

**Step 4.1:** We claim that the functional \( G \) is Lipschitz continuous on every ball of \( X \). In fact, let \( y, z \in B_X(0, r) \). By Lebourg’s Theorem 2.2 for \( g(s, \cdot) \) on some open ball containing the convex interval \([z(s), y(s)]\), we can find \( \xi_0(s) \in E_{w^*}^*, u_0(s) \in E \) and \( \theta_0(s) \in [0, 1] \) such that

\[
u_0(s) - \theta_0(s)z(s) + (1 - \theta_0(s))y(s),
g(s, z(s)) - g(s, y(s)) = \langle \xi_0(s), z(s) - y(s) \rangle,
\]

\[\xi_0(s) \in \partial_C g(s, u_0(s)) \text{ a.e.}\]

We point out that the functions \( \xi_0, u_0, \theta_0 \) are not, in general, all measurable. Since \( X \) is a Köthe–Bochner space, we get

\[
\|z(\cdot)\|_E + \|y(\cdot)\|_E \leq \|z(\cdot)\|_{\mathcal{K}} + \|y(\cdot)\|_{\mathcal{K}} = \|z\|_X + \|y\|_X \leq 2r.
\]
Note that

$$\|u_0(s)\|_E \leq \|\theta_0(s)z(s)\|_E + \|(1 - \theta_0(s))y(s)\|_E \leq \|z(s)\|_E + \|y(s)\|_E \text{ a.e.}$$

By $\xi_0(s) \in \partial_C g(s, u_0(s))$ a.e., from the condition (K1) we obtain

$$\|\xi_0(s)\|_{E^*} \leq h(s, \|z(s)\|_E + \|y(s)\|_E) := \gamma_0(s) \text{ a.e.}$$

Due to the condition (K2), we obtain $\tilde{C}(2r) := \sup\{\|N_h(\tilde{\alpha})\|_{K'} : \|\tilde{\alpha}\|_K \leq 2r\} < \infty$. Since $\gamma_0(s) = N_h(\|z(\cdot)\|_E + \|y(\cdot)\|_E)(s)$, hence $\|\gamma_0\|_{K'} \leq \tilde{C}(2r)$ follows. Since

$$|g(s, z(s)) - g(s, y(s))| \leq \|\xi_0(s)\|_{E^*}\|z(s) - y(s)\|_E \text{ a.e.,}$$

we get

$$|G(z) - G(y)| \leq \int_{\Omega} |g(s, z(s)) - g(s, y(s))| \, d\mu \leq \int_{\Omega} \gamma_0(s)\|z(s) - y(s)\|_E \, d\mu$$

$$\leq \|\gamma_0\|_{K'}\|z(\cdot) - y(\cdot)\|_K \leq \tilde{C}(2r)\|z - y\|_X.$$ 

for $z, y \in B_X(0, r)$. Since $G(x_* \in \mathbb{R}$, the claim of Step 4.1 follows.

**Step 4.2:** We shall use the notations from the beginning of Step 3.2 of the proof of Theorem 3.2, in particular, $F_k(s) := \frac{g(s, y_k(s) + \lambda_k v(s)) - g(s, y_k(s))}{\lambda_k}$. We can choose $k_j$ and $D_0$ with $\mu(\Omega \setminus D_0) = 0$ such that $y_{k_j}(s) \to x(s)$ as $j \to \infty$ (\forall s \in D_0), $\|y_{k_j} - x\|_X \leq 1/2^j$ and $\lambda_{k_j} \leq 1/2^j$.

We claim the existence of $\beta \in L^1(\Omega, \mathbb{R})$ and of $D_1 \subset D_0$ with $\mu(\Omega \setminus D_1) = 0$ such that $|F_{k_j}(s)| \leq \beta(s)$ on $D_1$ for all $j \in \mathbb{N}$. To prove this, by Lebourg’s Theorem 2.2 for $g(s, \cdot)$ on some open ball containing the convex interval $[y_k(s), y_k(s) + \lambda_k v(s)]$, there exist $\xi_k(s) \in E_{\omega^*}$ and $u_k(s) \in E$ and $\alpha_k(s) \in [0, 1]$ such that

$$u_k(s) = \alpha_k(s)[y_k(s) + \lambda_k v(s)] + (1 - \alpha_k(s))y_k(s),$$

$$g(s, y_k(s) + \lambda_k v(s)) - g(s, y_k(s)) = \langle \xi_k(s), \lambda_k v(s) \rangle,$$

$$\xi_k(s) \in \partial_C g(s, u_k(s)) \text{ a.e.}$$

So $F_k(s) = \langle \xi_k(s), v(s) \rangle$. We point out that the functions $\xi_k, u_k$ and $\alpha_k$ are not, in general, all measurable. We have that

$$\|u_{k_j}(s)\|_E \leq \|x(s)\|_E + \|\alpha_{k_j}(s)[y_{k_j}(s) + \lambda_{k_j} v(s)] + (1 - \alpha_{k_j}(s))y_{k_j}(s) - x(s)\|_E$$

$$\leq \|x(s)\|_E + \|y_{k_j}(s) - x(s)\|_E + \lambda_{k_j} \|v(s)\|_E := a_j(s) \text{ a.e.}$$

Hence we obtain, by the condition (K1),

$$\|\xi_{k_j}(s)\|_{E^*} \leq h(s, a_{j}(s)) = N_h(a_j(s)) \text{ a.e.} \quad (4.1)$$

Since $X$ is a Köthe–Bochner space, for the sequence $a_j$ and $a, a(s) := \|x(s)\|_E$, we get

$$\|a_j - a\|_K = \|\|y_{k_j}(\cdot) - x(\cdot)\|_E + \|\lambda_{k_j} \|v(\cdot)\|_E\|_K$$

$$\leq \|\|y_{k_j}(\cdot) - x(\cdot)\|_E\|_K + \|\lambda_{k_j} \|\|v(\cdot)\|_E\|_K$$

$$= \|y_{k_j} - x\|_{K(E)} + \|\lambda_{k_j} \|\|v\|_{K(E)} \leq 1/2^j + (1/2^j)\|\|v(\cdot)\|_E\|_K,$$

and so

$$\sum_{j=1}^{\infty} \|a_j - a\|_K < \infty.$$
By the condition \((K1)\) the operator \(N_h\) has the \(U\)-property, and hence we can find \(d \in K'\) such that \(|N_h(a_j)(s)| \leq d(s)\) a.e. Since \(|F_{k_j}(s)| \leq \|\xi_{k_j}(s)\|_{E^*}\|v(s)\|_E\), by (4.1) we deduce then the existence of \(D_1 \subset D_0\) with \(\mu(\Omega \setminus D_1) = 0\) such that \(|F_{k_j}(s)| \leq d(s)\|v(s)\|_E := \beta(s)(\forall s \in D_1)\). By \(\|v(\cdot)\|_E \in K\), hence \(\beta \in L^1(\Omega, \mathbb{R})\) follows.

\textbf{Step 4.3:} We claim that the function \(s \in \Omega \mapsto g^o(s, x(s); v(s))\) belongs to \(L^1(\Omega, \mathbb{R})\) for any \(x, v \in X\). Since \(K\) is a Banach lattice, by [22], [41, Theorem 2.2.6], there exists \(\alpha \in K\) with \(\alpha(s) > 0\) for \(s \in \text{supp } K = \Omega\). Then, by Lebourg’s Theorem 2.2, \((K1)\) implies that

\[
|g^o(s, x(s); v(s))| 
\leq \sup \left\{ \left| \frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda} \right| : \lambda \in (0, 1], \|\bar{u} - x(s)\|_E \leq \alpha(s) \right\}
\leq \sup \left\{ \|v(s, u^*)\| : u^* \in \partial_C g(s, u), u \in [\bar{u}, \bar{u} + \lambda v(s)], \lambda \in (0, 1], \|\bar{u} - x(s)\|_E \leq \alpha(s) \right\}
\leq \sup \left\{ \|v(s)\|_E \sup \left\{ \|u^*\|_{E^*} : u^* \in \partial_C g(s, u), \|u\|_E \leq p(s) \right\} \right\}
\leq \|v(\cdot)\|_E h(s, p(s)) \text{ a.e.,}
\]

where \(p(s) : = \|x(s)\|_E + \alpha(s) + \|v(s)\|_E\) satisfies \(p \in K_+\). Hence, by \((K2)\) together with \(\|v(\cdot)\|_E \in K\), the above claim follows.

The remaining part of Step 4.2 and Steps 4.4–4.5 for proving Lemma 4.1 are analogous to Step 3.2 and Steps 3.3–3.4 of the proof of Lemma 3.1.

By Lemma 4.1 we can prove the following Theorem 4.2.

\textbf{Theorem 4.2.} Let \(g: \Omega \times E \to \mathbb{R}\) be a Carathéodory function such that \(g(s, \cdot)\) is Lipschitz continuous on each ball of \(E\) for almost all \(s \in \Omega\), and \(K \subset L^0(\Omega, \mathbb{R})\) be a Banach lattice. Suppose that \(g\) satisfies the condition \((K1)\) with respect to \(h: \Omega \times [0, \infty) \to [0, \infty)\) and \(h\) satisfies the following condition:

\((K3)\) The superposition operator \(N_h: K_+ \to K'\) is bounded on each ball of \(K_+\) and \(h(s, \cdot)\) is nondecreasing for almost all \(s \in \Omega\).

Then, the statement of Lemma 3.1 is valid for \(G\) defined on \(X = K(E)\).

\textbf{Proof.} It suffices to check the \(U\)-property for \(N_h\), and then Theorem 4.2 follows from Lemma 4.1. Fix \(a \in K_+\) and a sequence \(a_i \in K_+\) such that \(\sum_{i=1}^{\infty} \|a_i - a\|_K < \infty\). Then by the Riesz–Fischer property for the Banach lattice \(K\) (see, e.g., [22], [41, Theorem 3.2.1]), there exists \(\Omega_0 \in \mathfrak{A}\) with \(\mu(\Omega \setminus \Omega_0) = 0\) such that the series \(a_\infty(s) := \sum_{i=1}^{\infty} |a_i(s) - a(s)|\) converges for \(s \in \Omega_0\); moreover putting \(a_\infty(s) := 0 (s \in \Omega \setminus \Omega_0)\), we get \(a_\infty \in K_+\). Note that \(a_i(s) \leq a_\infty(s) + a(s)\) a.e., and then by the condition \((K3)\) we have

\[
N_h(a_i(s)) = h(s, a_i(s)) \leq h(s, a_\infty(s) + a(s)) = N_h(a_\infty + a)(s) = d(s) \text{ a.e.}
\]

Since \(N_h\) maps \(K_+\) into \(K'\), we obtain \(N_h(a_\infty + a) = d \subset K'\).}

\textbf{Theorem 4.3.} Let \(g: \Omega \times E \to \mathbb{R}\) and a Banach lattice \(K\) be as in Theorem 4.2. Suppose that for every \(R \in (0, \infty)\) there exists a function \(h_R: \Omega \times [0, \infty) \to [0, \infty)\) satisfying the following conditions:

\[\text{...}\]
(K4) \( \sup\{\|u^*\|_{E^*} : u^* \in \partial_C g(s, u), \|u\|_E \leq a(s)\} \leq h_R(s, a(s)) \) is valid a.e. for each \( a \in B_{K^+}(0, R) \);

(K5) The superposition operator \( N_{h_R} : B_{K^+}(0, R) \to K' \) is bounded and there exists \( \Omega_0 \in \mathfrak{A} \) with \( \mu(\Omega \setminus \Omega_0) = 0 \) such that \( h_R(s, \cdot) \) is nondecreasing for \( s \in \Omega_0 \).

Then, the statement of Lemma 3.1 is valid for \( G \) defined on \( X = K(E) \).

**Proof.** It suffices to modify Steps 4.1-4.3 (in the proof of Lemma 4.1) as well as the proof of Theorem 4.2.

**Modification of Step 4.1:** It suffices to substitute \( h_{2r}(s, \cdot) \) for \( h(s, \cdot) \).

**Modification of Step 4.2 together with the proof of Theorem 4.2:** Observe by the Riesz–Fischer property for the Banach lattice \( K \) that we get also \( \|a_{\infty}\|_K \leq \sum_{i=1}^{\infty} \|a_i - a\|_K \).

Then, it suffices to substitute \( h_{r_1}(s, \cdot) \) for \( h(s, \cdot) \), where \( r_1 := \|a\|_K + \sum_{i=1}^{\infty} \|a_i - a\|_K < \infty \).

**Modification of Step 4.3:** It suffices to substitute \( h_{r_2}(s, \cdot) \) for \( h(s, \cdot) \), where

\[
r_2 := \|\|x(\cdot)\|_E\|_K + \|\alpha\|_K + \||v(\cdot)\|_E\|_K < \infty.
\]

**Theorem 4.4.** Let \( g : \Omega \times E \to \mathbb{R} \) be a Carathéodory function such that \( g(s, \cdot) \) is locally Lipschitz on \( E \) for almost all \( s \in \Omega \), and \( K \subset L^0(\Omega, \mathbb{R}) \) be a Banach lattice. Suppose the following conditions hold:

(K6) There exists \( \tilde{h} : \Omega \times [0, \infty) \to [0, \infty) \) such that

\[
|g(s, u) - g(s, v)| \leq \tilde{h}(s, \|u\|_E + \|v\|_E)\|u - v\|_E
\]

for almost all \( s \in \Omega \) and for all \( u, v \in E \);

(K7) The superposition operator \( N_{\tilde{h}} : K_+ \to K' \) is bounded on each ball of \( K_+ \) and \( \tilde{h}(s, \cdot) \) is nondecreasing for almost all \( s \in \Omega \).

Then, the statement of Lemma 3.1 is valid for \( G \) defined on \( X = K(E) \).

**Proof.** Observe that if \( g(s, \cdot) \) is Lipschitz continuous on each ball of \( E \) for almost all \( s \in \Omega \), then by [12, Proposition 2.1.2/(a)], (K6)–(K7) imply (K1) and (K3) for \( h(s, \alpha) = \tilde{h}(s, 2\alpha) \), and so Theorem 4.4 follows from Theorem 4.2. In the general case we can give another direct proof without using Lebourg’s Theorem 2.2 as following.

**Modification of Step 4.1:** Let \( y, z \in B_X(0, r) \). Then,

\[
|G(z) - G(y)| \leq \int_{\Omega} \tilde{h}(s, \|z(s)\|_E + \|y(s)\|_E)\|z(s) - y(s)\|_E \, d\mu(s)
\]

\[
\leq \|N_{\tilde{h}}(\|z(\cdot)\|_E + \|y(\cdot)\|_E)\|_{K'} \|\|z(\cdot) - y(\cdot)\|_E\|_K \leq \tilde{C}(2r)\|z - y\|_X.
\]

**Modification of Step 4.2:** Let \( F_k, y_k, k_j \) be as in Step 4.2. Then,

\[
|F_{kj}(s)| \leq \tilde{h}(s, \|y_{kj}(s) + \lambda k_j v(s)\|_E + \|y_{kj}(s)\|_E)\|v(s)\|_E
\]

\[
\leq \tilde{h}(s, 2\|y_{kj}(s)\|_E + \|v(s)\|_E)\|v(s)\|_E.
\]

Since \( \sum_{j=1}^{\infty} \|y_{kj}(\cdot) - x(\cdot)\|_E \|K \leq \sum_{j=1}^{\infty} \|\lambda_j / 2^j\|_E < \infty \), by the Riesz–Fischer property for the Banach lattice \( K \) (see, e.g., [22], [41, Theorem 3.2.1]), for some \( \Omega_0 \in \mathfrak{A} \) with \( \mu(\Omega \setminus \Omega_0) = 0 \) the series \( \tilde{a}_{\infty}(s) := \sum_{j=1}^{\infty} \|y_{kj}(s) - x(s)\|_E \) converges for \( s \in \Omega_0 \); moreover putting
\(\tilde{a}_\infty(s) := 0 (s \in \Omega \setminus \Omega_0)\), we get \(\tilde{a}_\infty \in K_+\). Observe that \(\|y_{k_j}(s)\|_E \leq \tilde{a}_\infty(s) + \|x(s)\|_E\) a.e., and hence

\[|F_{k_j}(s)| \leq \tilde{d}(s)\|v(s)\|_E\] a.e.,

where \(\tilde{d}(s) := N_{\tilde{h}_j}(2\tilde{a}_\infty + 2\|x(\cdot)\|_E + \|v(\cdot)\|_E)\) with \(\tilde{d} \in K'\).

**Modification of Step 4.3:** Let \(\alpha\) be as in Step 4.3. Then,

\[
|g^\circ(s, x(s); v(s))|
\leq \sup \left\{ \left| \frac{g(s, \bar{u} + \lambda v(s)) - g(s, \bar{u})}{\lambda} \right| : \lambda \in (0, 1], \|\bar{u} - x(s)\|_E \leq \alpha(s) \right\}
\leq \sup \left\{ \tilde{h}(s, 2\|\bar{u}\|_E + \lambda \|v(s)\|_E)\|v(s)\|_E : \lambda \in (0, 1], \|\bar{u} - x(s)\|_E \leq \alpha(s) \right\}
\leq \sup \left\{ \tilde{h}(s, 2\|\bar{u}\|_E + \|v(s)\|_E)\|v(s)\|_E : \|\bar{u}\|_E \leq \|x(s)\|_E + \alpha(s) \right\}
\leq \|v(s)\|_E \tilde{h}(s, \tilde{p}(s))\] a.e.,

where \(\tilde{p}(s) := 2\|x(s)\|_E + 2\alpha(s) + \|v(s)\|_E\) with \(\tilde{p} \in K_+\). Then, \(g^\circ(\cdot, x(\cdot), v(\cdot)) \in L^1(\Omega, \mathbb{R})\). The remaining part of the proof is analogous to Steps 3.2-3.4 of the proof of Lemma 3.1. 

**Theorem 4.5.** Let \(g : \Omega \times E \to \mathbb{R}\) and a Banach lattice \(K\) be as in Theorem 4.4. Suppose that for every \(R \in (0, \infty)\) there exists \(\tilde{h}_R : \Omega \times [0, \infty) \to [0, \infty)\) satisfying the following conditions:

\((K8)\) \(|g(s, u(s)) - g(s, v(s))| \leq \tilde{h}_R(s, \|u(s)\|_E + \|v(s)\|_E)\|u(s) - v(s)\|_E\) is valid a.e. for each \(u, v \in B_{K(E)}(0, R)\);

\((K9)\) The superposition operator \(N_{\tilde{h}_R} : B_{K(\Omega)}(0, R) \to K'\) is bounded and there exists \(\Omega_0 \in \mathcal{A}\) with \(\mu(\Omega \setminus \Omega_0) = 0\) such that \(\tilde{h}_R(s, \cdot)\) is nondecreasing for \(s \in \Omega_0\).

Then, the statement of Lemma 3.1 is valid for \(G\) defined on \(X = K(E)\).

**Proof.** It suffices to modify the alternative proof of Theorem 4.4 in the same way as in the proof of Theorem 4.3.

5. Lipschitz integral functionals on non-solid generalized Orlicz spaces. Let \(m \geq 2\) and \(M : \Omega \times \mathbb{R}^m \to [0, \infty)\) be some generalized Young function (see, e.g., [6, 17, 19, 25]), i.e., \(M\) is a normal integrand, \(M(s, \cdot)\) is convex even, \(M(s, 0) = 0\), and the set \(\{u : M(s, u) = 0\}\) is bounded for a.a. \(s \in \Omega\). The non-solid generalized Orlicz space is defined by \(L^M(\Omega, \mathbb{R}^m) := \{x \in L^0(\Omega, \mathbb{R}^m) : \int_\Omega M(s, \alpha x(s))d\mu(s) < \infty\} \) for some \(\alpha > 0\) with the Luxemburg norm.

We shall use the following conditions \((\Delta_m)\) and \((B_m)\):

\((\Delta_m)\) There exist some measurable function \(\delta : \Omega \to [0, \infty)\) and \(a \in (1, \infty)\) such that \(\int_\Omega \sup \left\{ M(s, u) : \|u\| \leq \delta(s) \right\} d\mu(s) < \infty\) and \(M(s, 2u) \leq aM(s, u)\) for a.a. \(s \in \Omega\) and all \(u \in \mathbb{R}^m\) with \(\|u\| \geq \delta(s)\);

\((B_m)\) There exist \(c \in L^{M'}(\Omega, \mathbb{R}^m)\) and \(b > 0\) such that

\[
\partial_C g(s, u) \subset [-1, 1]c(s) + b \text{co} \partial M(s, [-1, 1] u)
\]

for almost all \(s \in \Omega\) and for all \(u \in \mathbb{R}^m\).
Corollary 5.1 (of Theorem 3.2). Let \( m \geq 2, M : \Omega \times \mathbb{R}^m \to [0, \infty) \) be a Young function, and \( g \) be as in Theorem 3.2. Suppose that the conditions (N1), (B\(_m\)) are satisfied for \( \partial_C g \) and \( (\Delta_m) \) is valid for \( M \). If the Filippov implicit function property is valid for \( \partial M \) (in particular, if \( \text{grad} M(s, \cdot) : \mathbb{R}^m \to \mathbb{R}^m \) is continuous), then the statement of Lemma 3.1 is valid for \( G \) defined on \( X = L^M(\Omega, \mathbb{R}^m) \).

Proof. By the known equality \( X' = L^{M^*} \) (with equivalent norms) the assertion of Corollary 5.1 follows immediately from Theorem 3.2 via using the results of the following Steps 5.1-5.3.

Step 5.1: We claim that if \( M \) satisfies the condition \( (\Delta_m) \), then for every \( \kappa \in (0, \infty) \) there exists \( r(\kappa) \in (0, \infty) \) such that \( \|x\|_{L^M} \leq \kappa \Rightarrow \|\xi\|_{L^{M^*}} \leq r(\kappa) \), where \( \xi \) is an arbitrary measurable selector of the multifunction \( s \mapsto \partial M(s, x(s)) \). To prove this claim, fix \( x \) and \( \xi \) such that \( \|x\|_M \leq \kappa \) and \( \xi(s) \in \partial M(s, x(s)) \) a.e. (then \( \int_\Omega M(s, x(s)/\kappa)d\mu(s) \leq 1 \). By the definition of the subdifferential of the convex function \( M(s, \cdot) \),

\[
M(s, 2x(s)) \geq M(s, 2x(s)) - M(s, x(s)) \geq \langle \xi(s), x(s) \rangle \quad \text{a.e.}
\]

Since \( \partial M(s, x(s)) = \{ \xi \in \mathbb{R}^m : M^*(s, \xi) + M(s, x(s)) = \langle \xi, x(s) \rangle \} \), we have

\[
M^*(s, \xi(s)) = \langle \xi(s), x(s) \rangle - M(s, x(s)) \leq M(s, 2x(s)) - M(s, x(s)) \leq M(s, 2x(s)) \quad \text{a.e.}
\]

Let \( l \in \mathbb{N} \) be such that \( \kappa \leq 2^{l-1} \). Denote \( A = \{ s \in \Omega : \|x(s)\| \leq \kappa \delta(s), x(s) \neq 0 \} \). By the condition \( (\Delta_m) \) we deduce that:

\[
\int_\Omega M^*(s, \xi(s))d\mu(s) \leq \int_A M(s, 2x(s))d\mu(s) + \int_{\Omega \setminus A} M(s, 2x(s))d\mu(s)
\]

\[
= \int_A M \left( s, 2 \frac{\|x(s)\|}{\delta(s)} \frac{x(s)\delta(s)}{\|x(s)\|} \right) d\mu(s) + \int_{\Omega \setminus A} M \left( s, 2 \kappa \frac{x(s)}{\kappa} \right) d\mu(s)
\]

\[
\leq \int_A M \left( s, 2^l \frac{x(s)\delta(s)}{\|x(s)\|} \right) d\mu(s) + \int_{\Omega \setminus A} M \left( s, 2 \kappa \frac{x(s)}{\kappa} \right) d\mu(s)
\]

\[
\leq a^l \int_A M \left( s, \frac{x(s)\delta(s)}{\|x(s)\|} \right) d\mu(s) + a^l \int_{\Omega \setminus A} M \left( s, \frac{x(s)}{\kappa} \right) d\mu(s)
\]

\[
\leq a^l \left( \int_\Omega \sup\{M(s, u) : \|u\| \leq \delta(s)\} d\mu(s) + 1 \right) \leq a^l(I + 1),
\]

where \( I := \int_\Omega \sup\{M(s, u) : \|u\| \leq \delta(s)\}d\mu(s) \in (0, \infty) \). Since \( a^l(I + 1) > 1 \), by convexity of \( M^* \) and \( M^*(s, 0) = 0 \), \( \int_\Omega M^*(\frac{\xi(s)}{a^l(I + 1)})d\mu(s) \leq 1 \) follows. Therefore \( \|\xi\|_{L^{M^*}} \leq a^l(I + 1) := r(\kappa) \).

Step 5.2: Put \( H(s, u) := c(s) + b\partial M(s, u) \). Then by the condition \( (B_m) \), \( \partial_C g(s, u) \subset \text{bco} H(s, [-1, 1]u) \) and the Filippov implicit function property for \( H \) is valid as this valid for \( \partial M \). Fix \( y \in N_H(x) \) with \( \|x\|_{L^M} \leq \kappa < \infty \). Then (e.g., by [29, Lemma 3.1]), there exist measurable functions \( \alpha, d \) such that \( y(s) = \alpha(s)c(s) + bd(s) \) a.e. with \( \alpha(s) \in [-1, 1] \) and \( d(s) \in \partial M(s, [-1, 1]x(s)) \) a.e. By the Filippov implicit function property for \( \partial M \), there exists some measurable function \( \lambda \) such that \( \lambda(s) \in [-1, 1] \) and \( d(s) \in \partial M(s, \lambda(s)x(s)) \) a.e. Observe that \( \|\lambda x\|_{L^M} \leq \|x\|_{L^M} \leq \kappa < \infty \) and then by Step 5.1 for \( \lambda x \) we get
\[ \|d\|_{L^{M^*}} \leq r(\kappa) < \infty. \text{ Hence,} \]
\[ \|y\|_{L^{M^*}} \leq \|\alpha c\|_{L^{M^*}} + \|bd\|_{L^{M^*}} \leq \|c\|_{L^{M^*}} + b\|d\|_{L^{M^*}} := \tilde{r}(k) < \infty \]
for \( y \in N_H(x) \).

**Step 5.3:** By [38, Theorem 2W, Theorem 1N], \( \partial M : \Omega \times \mathbb{R}^m \to \text{Cp}(\mathbb{R}^m) \) is multi-superpositionally measurable. By [12, Proposition 2.2.6, Proposition 2.1.2/(a)] (for the convex function \( M(s, \cdot) : \mathbb{R}^m \to \mathbb{R} \)) together with [2, Lemma 2.9, p. 13], \( \partial M(s, C) \in \text{Cp}(\mathbb{R}^m) \) for \( C \in \text{Cp}(\mathbb{R}^m) \) a.e. Hence, the condition (N4) follows. \( \blacksquare \)

We shall use the following condition:

\((NM)\) There exist \( \Omega_0 \in \mathcal{A} \) with \( \mu(\Omega \setminus \Omega_0) = 0 \) and for every \( R \in (0, \infty) \) exist \( b_R, d_R \in (0, \infty) \) and \( a_R \in L^1(\Omega, [0, \infty)) \) such that
\[ u^* \in \partial_C g(s, u) \implies M^*(s, u^*/d_R) \leq a_R(s) + b_R M(s, u/R) \]
for all \( s \in \Omega_0, u \in \mathbb{R}^m \).

**Corollary 5.2** (of Theorem 3.3). Let \( m \geq 2, M : \Omega \times \mathbb{R}^m \to [0, \infty) \) be a Young function with \( M^*(s, u^*) \in [0, \infty) \), and \( g \) be as in Theorem 3.3. Suppose that \( \partial_C g \) satisfies the conditions (N1) and (NM). If the multifunction
\[ (s, u) \in \Omega \times \mathbb{R}^m \mapsto H_{MR}(s, u) := \{ u^* \in \mathbb{R}^m : M^*(s, u^*/d_R) \leq a_R(s) + b_R M(s, u/R) \} \]
has the Filippov implicit function property, then the statement of Lemma 3.1 is valid for \( G \) defined on \( X = L^M \).

**Proof.** It suffices to check the conditions (N5)-(N6) for \( X = L^M(\Omega, \mathbb{R}^m) \) and \( H_R := H_{MR} \) (then Corollary 5.2 follows from Theorem 3.3). Due to the above definition of \( M \), it is known [17, 19, 10] and easy to check that \( H_{MR}(s, u) \in \text{Cp}(\mathbb{R}^m) \) and \( H_{MR}(s, u) \) is symmetric and \( H_{MR}(s, C) \in \text{Cp}(\mathbb{R}^m) \) for \( C \in \text{Cp}(\mathbb{R}^m) \) and for all \( s \in \Omega_0 \). By [12, Proposition 2.2.6], both \( M \) and \( M^* \) are Carathéodory, and so (e.g., by [42, Theorem 1]) \( H_{MR} \) is multi-superpositionally measurable. Since \( \overline{\text{co}} H_{MR}(s, [-1, 1]u) = H_{MR}(s, u) \), by (NM) we deduce (N6).

Fix \( x \in X = L^M(\Omega, \mathbb{R}^m) \) with \( \|x\|_{L^M} \leq R \). Then for any \( \xi \in \text{Sel} H_{MR}(\cdot, x(\cdot)) \) we get
\[ \int_\Omega M^*(s, \xi(s)/d_R) \, d\mu(s) \leq \|a_R\|_{L^1} + b_R \int_\Omega M(s, x(s)/R) \, d\mu(s) \]
\[ \leq \|a_R\|_{L^1} + b_R + 1 := \tilde{r}(R) \in (1, \infty). \]
Hence, \( \int_\Omega M^*(s, \frac{\xi(s)}{d_R(\dot{R})}) \, d\mu(s) \leq 1 \), and so \( \|\xi\|_{L^{M^*}} \leq d_R(\tilde{r}(R)) \). By \( (L^M(\Omega, \mathbb{R}^m))' = L^{M^*}(\Omega, \mathbb{R}^m) \) with equivalent norms, (N5) follows. \( \blacksquare \)

**Remark 5.3.** If \( M : \Omega \times \mathbb{R} \to [0, +\infty] \) is such that \( \mathcal{L}(s) := \{ u : M(s, u) < +\infty \} \) is a linear subspace of \( \mathbb{R} \), then \( X = L^M(\Omega, \mathbb{R}) \) has vssup \( X(s) = \mathcal{L}(s) \) and the statements of Corollaries 5.1-5.2 remain valid but in the form of Remark 3.4 together with substituting \( M'_L \) for \( M^* \), where \( M'_L(s, \cdot) \) is the convex dual on \( \mathcal{L}(s) \) of the function \( u \in \mathcal{L}(s) \mapsto M(s, u) \), and \( N_{\partial_C g}(x) \subset X' = L^{M'_L}(\Omega, \mathbb{R}) \) (x \( \in X \)).
6. Lipschitz integral functionals on Orlicz–Bochner spaces. Let $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ be some Musielak–Orlicz function (convex in the second variable) and $\Phi^* : \Omega \times [0, \infty) \to [0, \infty)$ be the convex dual to $\Phi$ (see [4, 25, 37]). The generalized Orlicz–Bochner (Musielak–Orlicz–Bochner) space is defined by $L^\Phi(E) = \{ x \in L^0(\Omega, E) : \int_\Omega \Phi(s, \alpha\|x(s)\|_E)d\mu(s) < \infty \text{ for some } \alpha > 0 \}$ with the Luxemburg norm.

The following conditions $(\Delta)$ and $(B)$ are taken from [36]:

$(\Delta)$ There exist some measurable function $\delta : \Omega \to [0, \infty)$ and $a \in (1, \infty)$ such that

$$\int_\Omega \Phi(s, \delta(s))d\mu(s) < \infty \text{ and } \Phi(s, 2\beta) \leq a\Phi(s, \beta) \text{ a.e. for } \beta \geq \delta(s);$$

$(B)$ There exist $c \in L^{\Phi^*}(\Omega, \mathbb{R})$ and $b \in (0, \infty)$ such that

$$u^* \in \partial_C g(s, u) \Rightarrow \|u^*\|_{E^*} \leq c(s) + b\varphi(s, \|u\|_E) \text{ a.e. on } \Omega$$

for all $u \in E$, where $\varphi(s, \cdot)$ is the right derivative of the convex function $\Phi(s, \cdot)$.

Lemma 6.1. Suppose that conditions $(\Delta)$ and $(B)$ are satisfied. Then the conditions (K1) and (K3) in Theorem 4.2 are valid for $h(s, \alpha) := |c(s)| + b|\varphi(s, \alpha)|$ with respect to $X = K(E)$ with $K := L^\Phi(\Omega, \mathbb{R})$.

Proof. By the condition $(B)$ the condition (K1) follows. By the condition $(\Delta)$ together with [36, Lemma 1],

$$\kappa \in (0, \infty) \Rightarrow \tau(\kappa) := \sup\{\|\varphi(\cdot, |\alpha(\cdot)|)\|_{L^{\Phi^*}} : \|\alpha\|_{L^\Phi} \leq \kappa \} < \infty.$$ 

By the equality $K' = L^{\Phi^*}$ with equivalent norms (see, e.g., [4, 25, 37]), $\|N_h(\alpha)\|_{K'} \leq \|c\|_{K'} + b\tau(\kappa) < \infty$ follows for every $\alpha \in K_+$ with $\|\alpha\|_K \leq \kappa < \infty$. By the nondecreasing property of $\varphi(s, \cdot)$, $h(s, \cdot)$ is nondecreasing for a.a. $s \in \Omega$. Hence (K3) follows. 

By Lemma 6.1 together with $(L^\Phi(E))' = L^{\Phi^*}(E_{w^*}^*)$, Theorem 4.2 with its proof implies an alternative proof for [36, Theorem 2].

We shall use the following conditions $(S\Phi 1)$ and $(S\Phi 2)$:

$(S\Phi 1)$ There exists $\Omega_0 \in \mathfrak{A}$ with $\mu(\Omega \setminus \Omega_0) = 0$ and for every $R \in (0, \infty)$ there exist $b_R, d_R \in (0, \infty)$ and $a_R \in L^1(\Omega, [0, \infty))$ such that

$$u^* \in \partial_C g(s, u) \Rightarrow \Phi^*(s, \|u^*\|_{E^*}/d_R) \leq a_R(s) + b_R\Phi(s, \|u\|_E/R)$$

for all $s \in \Omega_0, u \in E$;

$(S\Phi 2)$ There exists $\Omega_0 \in \mathfrak{A}$ with $\mu(\Omega \setminus \Omega_0) = 0$ and for every $R \in (0, \infty)$ there exist $b_R, d_R \in (0, \infty)$ and $a_R \in L^1(\Omega, [0, \infty))$ such that

$$|g(s, u) - g(s, v)| \leq \tilde{h}_R(s, \|u\|_E + \|v\|_E)\|u - v\|_E$$

for all $s \in \Omega_0$ and for all $u, v \in E$, and

$$\Phi^*(s, \tilde{h}(s, \alpha)/d_R) \leq a_R(s) + b_R\Phi(s, \alpha/R)$$

for all $s \in \Omega_0, \alpha \in [0, \infty)$.

Corollary 6.2 (of Theorems 4.3, 4.5). Let $g : \Omega \times E \to \mathbb{R}$ be a Carathéodory function. Suppose one of the following conditions holds:
1) $g(s, \cdot)$ is Lipschitz continuous on each ball of $E$ for almost all $s \in \Omega$ and $\partial_C g$ satisfies (S$\Phi$1);
2) $g(s, \cdot)$ is locally Lipschitz on $E$ for almost all $s \in \Omega$ and $g$ satisfies (S$\Phi$2).

Then the statement of Lemma 3.1 is valid for $G$ defined on $X = L^\Phi(E)$.

Remark 6.3. It is an easy check that (S$\Phi$1) implies that $N_{\partial_C g} : L^\Phi(E) \to L^{\Phi^*}(E^*_\omega)$ is bounded on $B_{L^\Phi(E)}(0, R)$. On the other hand, it is known (see e.g. [4, 26]) that if the measure $\mu$ is continuous and $\partial_C g$ is Carathéodory and $N_{\partial_C g} : L^\Phi(E) \to L^{\Phi^*}(E^*_\omega)$ is bounded on $B_{L^\Phi(E)}(0, R)$, then (S$\Phi$1) is true. So, (S$\Phi$1) is natural for applications.

Proof of Corollary 6.2/(S$\Phi$1). Let $(\Phi^*)^{-1}(s, \cdot)$ be the right pre-image of $\Phi^*(s, \cdot)$. Put $h_R(s, \alpha) := d_R(\Phi^*)^{-1}(s, a_R(s) + b_R(\Phi(s, \alpha/R)))$. Then, (K4) follows. It is easy to check (see, e.g., [4, 26]) that $N_{h_R}$ maps boundedly $B_{L^\Phi(E)}(0, R)$ into $L^{\Phi^*}(\Omega, \mathbb{R})$. Since $h_R(s, \cdot)$ is nondecreasing and $(L^\Phi(\Omega, \mathbb{R}))' = L^{\Phi^*}(\Omega, \mathbb{R})$ with equivalent norms, (K5) follows. Therefore, Corollary 6.2/(S$\Phi$1) follows from Theorem 4.3.

Proof of Corollary 6.2/(S$\Phi$2). By analogous arguments to the proof of Corollary 6.2/(S$\Phi$1), we deduce Corollary 6.2/(S$\Phi$2) from Theorem 4.5.

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