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LINEAR OPERATORS ON NON-LOCALLY CONVEX ORLICZ SPACES

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Abstract. We study linear operators from a non-locally convex Orlicz space L^Φ to a Banach space $(X, \|\cdot\|_X)$. Recall that a linear operator $T : L^\Phi \to X$ is said to be σ-smooth whenever $u_n \xrightarrow{(o)} 0$ in L^Φ implies $\|T(u_n)\|_X \to 0$. It is shown that every σ-smooth operator $T : L^\Phi \to X$ factors through the inclusion map $j : L^\Phi \to L^{\overline{\Phi}}$, where $\overline{\Phi}$ denotes the convex minorant of Φ. We obtain the Bochner integral representation of σ-smooth operators $T : L^\Phi \to X$. This extends some earlier results of J. J. Uhl concerning the Bochner integral representation of linear operators defined on a locally convex Orlicz space.

1. Introduction and preliminaries. The theory of linear operators on Banach function spaces (in particular, L^p-spaces and Orlicz spaces L^Φ) has been developed by many authors (see [D], [G], [DP], [Ph], [Z], [DS], [D_1], [D_2], [D_3], [U], [C], [W]). Linear operators on non-locally convex Orlicz spaces L^Φ have been studied in [P], [T_1], [T_2], [K].

We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on L with respect to the dual pair (L, K). Given a topological vector space (L, τ) we will denote by $(L, \tau)^*$ its topological dual. For terminology concerning vector lattices and function spaces we refer to [AB], [KA], [Z].

Let (Ω, Σ, μ) be a σ-finite atomless measure space, and let L^0 denote the set of μ-equivalence classes of real valued measurable functions defined on Ω. Then L^0 is a super Dedekind complete Riesz space under the ordering $u \leq v$ whenever $u(\omega) \leq v(\omega)$ μ-a.e. on Ω. By $\mathcal{S}(\Sigma)$ we will denote the set of all Σ-simple functions defined on Ω.

Now we recall notation and some basic results concerning Orlicz spaces (see [MO_1], [MaO], [M], [RR]). By an *Orlicz function* we mean here a mapping $\Phi : [0, \infty) \to [0, \infty)$ that is non-decreasing, left continuous, continuous at 0, vanishing only at 0 and

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$\liminf_{t\to\infty} \frac{\Phi(t)}{t} > 0$. By Φ^* we denote the convex Orlicz function complementary to Φ in the sense of Young, i.e., $\Phi^*(s) = \sup\{st - \Phi(t) : t \geq 0\}$ for $s \geq 0$. Note that Φ^* takes only finite values whenever $\liminf_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ and jumps to ∞ whenever $\liminf_{t\to\infty} \frac{\Phi(t)}{t} < \infty$ (see [N_3, Lemmas 2.2 and 2.3]). The function $\overline{\Phi}(t) = (\Phi^*)^*(t)$ for $t \geq 0$ is called the *convex minorant* of Φ, because it is the largest convex Orlicz function smaller than Φ on $[0,\infty)$. Recall that Φ satisfies the Δ_2-condition (in symb. $\Phi \in \Delta_2$) if $\Phi(2t) \leq c\Phi(t)$ for all $t \geq 0$ and some $c > 0$. An Orlicz function Φ determines a functional $\varrho_\Phi : L^0 \to [0,\infty]$ by

$$\varrho_\Phi(u) = \int_\Omega \Phi(|u(\omega)|)\, d\mu.$$

The *Orlicz space* L^Φ is an ideal of L^0 defined by

$$L^\Phi = \{u \in L^0 : \varrho_\Phi(\alpha u) < \infty \quad \text{for some } \alpha > 0\}$$

and equipped with the complete topology $\mathcal{T}_\Phi$ of the F-Riesz norm

$$|u|_\Phi := \inf\{\alpha > 0 : \varrho_\Phi(u/\alpha) \leq \alpha\}.$$

The space $(L^\Phi, \mathcal{T}_\Phi)$ is locally convex if and only if $L^\Phi = L^{\Phi_0}$ for some convex Orlicz function Φ_0 (see [MaO]). In case Φ is a convex Orlicz function $\mathcal{T}_\Phi$ can be generated by two Riesz norms:

$$\|u\|_\Phi := \inf\,\{\alpha > 0 : \varrho_\Phi(u/\alpha) \leq 1\}$$

and

$$\|u\|_\Phi^0 := \sup\left\{\int_\Omega |u(\omega)v(\omega)|\, d\mu : v \in L^{\Phi^*},\ \varrho_{\Phi^*}(u) \leq 1\right\}.$$

Let $(L^\Phi)'$ stand for the *Köthe dual* of L^Φ. Then $(L^\Phi)' = L^{\Phi^*}$ (see [N_3, Theorem 3.3], [MW]). Let $(L^\Phi)_n^\sim$ denote the order continuous dual of L^Φ. Then $(L^\Phi)_n^\sim$ can be identified with L^{Φ^*} through the mapping: $L^{\Phi^*} \ni v \mapsto \varphi_v \in (L^\Phi)_n^\sim$, where

$$\varphi_v(u) = \int_\Omega u(\omega)v(\omega)\, d\mu \quad \text{for all } u \in L^\Phi.$$

The functional ϱ_Φ restricted to L^Φ is a *modular* (see [MO_1], [MO_2], [M]). Recall that a sequence (u_n) in L^Φ is said to be *modularly convergent* to $u \in L^\Phi$ (in symb. $u_n \xrightarrow{\varrho_\Phi} u$) if $\varrho_\Phi(\alpha(u_n - u)) \to 0$ for some $\alpha > 0$.

For $\varepsilon > 0$ let $U_\Phi(\varepsilon) = \{u \in L^\Phi : \varrho_\Phi(u) \leq \varepsilon\}$. Then the family of all sets of the form: $\bigcup_{n=1}^\infty(\sum_{i=1}^n U_\Phi(\varepsilon_i))$, where (ε_i) is a sequence of positive numbers, forms a local base at 0 (consisting of solid subsets of L^Φ) for a topology $\mathcal{T}_\Phi^\wedge$ on L^Φ, and called the *modular topology* (see [N_1], [N_2], [N_4]). The basic properties of $\mathcal{T}_\Phi^\wedge$ are included in the following theorem (see [N_1, Theorem 1.1], [N_2, Theorem 2.5 and 3.2], [N_4, Theorem 2.2]).

THEOREM 1.1. *Let Φ be an Orlicz function. Then the following statements hold:*

(i) *$\mathcal{T}_\Phi^\wedge$ is the finest of all linear topologies ξ on L^Φ for which $u_n \xrightarrow{\varrho_\Phi} 0$ implies $u_n \xrightarrow{\xi} 0$.*
(ii) *$\mathcal{T}_\Phi^\wedge$ is the finest Lebesgue topology on L^Φ.*
(iii) *$\mathcal{T}_\Phi^\wedge \subset \mathcal{T}_\Phi$, with equality if and only if $\Phi \in \Delta_2$.*
(iv) *$(L^\Phi, \mathcal{T}_\Phi^\wedge)^* = (L^\Phi)_n^\sim = \{\varphi_v : v \in L^{\Phi^*}\}$.*

(v) *$\tau(L^\Phi, L^{\Phi^*})$ is equal to the restriction of the modular topology $\mathcal{T}_{\overline{\Phi}}^{\wedge}$ i.e., $\tau(L^\Phi, L^{\Phi^*}) = \mathcal{T}_{\overline{\Phi}}^{\wedge} \restriction_{L^\Phi}$. In particular, $\tau(L^\Phi, L^{\Phi^*}) = \mathcal{T}_{\Phi}^{\wedge}$ whenever Φ is convex.*

In view of [O] the dual space $(L^\Phi)^*(= (L^\Phi, \mathcal{T}_\Phi)^*)$ is a Banach space under the norm

$$\|\varphi\|_\Phi = \sup\{|\varphi(u)| : u \in L^\Phi, \varrho_\Phi(u) \le 1\}$$

for $\varphi \in (L^\Phi)^*$. Moreover, by [O, 1.31] the following inequality holds:

$$|\varphi(u)| \le \|\varphi\|_\Phi(\varrho_\Phi(u) + 1) \quad \text{for all } u \in L^\Phi. \tag{1.1}$$

From now on we assume that $(X, \|\cdot\|_X)$ is a real Banach space, and X^* stands for its Banach dual. We distinguish two classes of linear operators $T : L^\Phi \to X$ (see [OW]).

DFINITION 1.1. A linear operator $T : L^\Phi \to X$ is said to be *σ-smooth* (resp. *modularly continuous*) if $u_n \xrightarrow{(o)} 0$ (resp. $u_n \xrightarrow{\varrho_\Phi} 0$) in L^Φ implies $\|T(u_n)\|_X \to 0$.

In Section 2, we study a relationship between σ-smooth operators, modularly continuous operators and $(\mathcal{T}_\Phi^\wedge, \|\cdot\|_X)$-continuous linear operators $T : L^\Phi \to X$. It is shown that every σ-smooth linear operator $T : L^\Phi \to X$ factors through the inclusion map $j : L^\Phi \to L^{\overline{\Phi}}$, where $\overline{\Phi}$ stands for the convex minorant of Φ. In Section 3, we obtain a Bochner integral representation of σ-smooth operators $T : L^\Phi \to X$. This extends some earlier results due to J. J. Uhl [U, Theorem 1], where Φ is supposed to be convex and $\Phi \in \Delta_2$.

2. Smooth operators.

We first establish a relationship between different classes of linear operators $T : L^\Phi \to X$.

THEOREM 2.1. *Let Φ be an Orlicz function. Then for a linear operator $T : L^\Phi \to X$ the following statements are equivalent:*

(i) *T is modularly continuous.*
(ii) *T is σ-smooth.*
(iii) *$x^* \circ T \in (L^\Phi)^\sim_n$ for all $x^* \in X^*$.*
(iv) *T is $(\sigma(L^\Phi, L^{\Phi^*}), \sigma(X, X^*))$-continuous.*
(v) *T is $(\tau(L^\Phi, L^{\Phi^*}), \|\cdot\|_X)$-continuous.*
(vi) *T is $(\mathcal{T}_{\overline{\Phi}}^{\wedge} \restriction_{L^\Phi}, \|\cdot\|_X)$-continuous.*
(vii) *T is $(\mathcal{T}_\Phi^\wedge, \|\cdot\|_X)$-continuous.*

Proof. (i)⇒(ii). Assume that T is modularly continuous and let $u_n \xrightarrow{(o)} 0$ in L^Φ. Then by the Lebesgue dominated convergence theorem $u_n \xrightarrow{\varrho_\Phi} 0$, so $\|T(u_n)\|_X \to 0$. This means that T is σ-smooth.

(ii)⇒(iii). Assume that T is σ-smooth. Hence $x^* \circ T \in (L^\Phi)^\sim_c = (L^\Phi)^\sim_n$ for every $x^* \in X^*$.

(iii)⇔(iv). See [AB, Theorem 9.26].

(iv)⇔(v). See [Wi, Corollary 11-1-3, Corollary 11-2-6].

(v)⇔(vi). It is obvious, because $\tau(L^\Phi, L^{\Phi^*}) = \mathcal{T}_{\overline{\Phi}}^{\wedge} \restriction_{L^\Phi}$ (see [N_4, Theorem 2.2]).

(vi)⇒(vii). Clear, because $\mathcal{T}_{\overline{\Phi}}^{\wedge} \restriction_{L^\Phi} \subset \mathcal{T}_\Phi^\wedge$.

(vii)⇒(i). It is obvious, because $u_n \xrightarrow{\varrho_\Phi} 0$ in L^Φ implies $u_n \to 0$ for $\mathcal{T}_\Phi^\wedge$. ∎

Now, we consider the problem of extension of linear operators $T : L^\Phi \to X$.

THEOREM 2.2. *Let Φ be an Orlicz function. Assume that $T : L^\Phi \to X$ is a $(\mathcal{T}_\Phi^\wedge, \|\cdot\|_X)$-continuous linear operator. Then there exists a $(\mathcal{T}_{\overline{\Phi}}^\wedge, \|\cdot\|_X)$-continuous linear operator $\overline{T} : L^{\overline{\Phi}} \to X$ such that $\overline{T}(u) = T(u)$ for all $u \in L^\Phi$.*

Proof. In view of Theorem 2.1, T is $(\mathcal{T}_{\overline{\Phi}}^\wedge \restriction_{L^\Phi}, \|\cdot\|_X)$-continuous. Now let $u \in L^{\overline{\Phi}}$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $s_n(\omega) \to u(\omega)$ μ-a.e., and $|s_n(\omega)| \le |u(\omega)|$ μ-a.e., that is, $s_n \xrightarrow{(\mathrm{o})} 0$ in $L^{\overline{\Phi}}$. Hence $s_n \to u$ for $\mathcal{T}_{\overline{\Phi}}^\wedge$, because $\mathcal{T}_{\overline{\Phi}}^\wedge$ is a Lebesgue topology on $L^{\overline{\Phi}}$. Then (s_n) is a Cauchy sequence in $(L^\Phi, \mathcal{T}_{\overline{\Phi}}^\wedge \restriction_{L^\Phi})$, so $(T(s_n))$ is a Cauchy sequence in $(X, \|\cdot\|_X)$. Let us put $\overline{T}(u) := \lim T(s_n)$ in $(X, \|\cdot\|_X)$. Note that if $u \in L^\Phi$, then $T(u) = \lim T(s_n)$ in $(X, \|\cdot\|_X)$ and $\overline{T}(u) = T(u)$.

Now we shall show that if (s_n^1) and (s_n^2) are sequences in $\mathcal{S}(\Sigma)$ such that $s_n^1 \xrightarrow{(\mathrm{o})} u$ and $s_n^2 \xrightarrow{(\mathrm{o})} u$ in $L^{\overline{\Phi}}$, then $\lim T(s_n^1) = \lim T(s_n^2)$ in $(X, \|\cdot\|_X)$. Indeed, we have $s_n^1 \to u$ for $\mathcal{T}_{\overline{\Phi}}^\wedge$ and $s_n^2 \to u$ for $\mathcal{T}_{\overline{\Phi}}^\wedge$, so $s_n^1 - s_n^2 \to 0$ for $\mathcal{T}_{\overline{\Phi}}^\wedge \restriction_{L^\Phi}$. Hence $\|T(s_n^1) - T(s_n^2)\|_X \to 0$. Set $x_1 = \lim T(s_n^1)$ and $x_2 = \lim T(s_n^2)$ in $(X, \|\cdot\|_X)$. Then

$$\|x_1 - x_2\|_X \le \|x_1 - T(s_n^1)\|_X + \|T(s_n^1) - T(s_n^2)\|_X + \|T(s_n^2) - x_2\|_X,$$

and it follows that $\|x_1 - x_2\|_X = 0$, so $x_1 = x_2$.

We shall now show that a linear operator $\overline{T} : L^{\overline{\Phi}} \to X$ is $(\mathcal{T}_{\overline{\Phi}}^\wedge, \|\cdot\|_X)$-continuous. Indeed, let $B_{\mathcal{T}_{\overline{\Phi}}^\wedge}$ stand for the local base at 0 for $\mathcal{T}_{\overline{\Phi}}^\wedge$, and let $\varepsilon > 0$ be given. Since T is $(\mathcal{T}_{\overline{\Phi}}^\wedge \restriction_{L^\Phi}, \|\cdot\|_X)$-continuous, there exists $W \in B_{\mathcal{T}_{\overline{\Phi}}^\wedge}$ such that $T(L^\Phi \cap W) \subset B_X(\varepsilon)$ $(= \{x \in X : \|x\|_X \le \varepsilon\})$. It is enough to show that $\overline{T}(W) \subset B_X(\varepsilon)$. In fact, let $w \in W$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $s_n \to w$ for $\mathcal{T}_{\overline{\Phi}}^\wedge$. Hence there exists $n_0 \in \mathbb{N}$ such that $s_n \in L^\Phi \cap W$ for all $n \ge n_0$; so $T(s_n) \in B_X(\varepsilon)$ for $n \ge n_0$. It follows that $\overline{T}(w) \in B_X(\varepsilon)$, as desired. ∎

As a consequence of Theorem 2.1 and Theorem 2.2 we obtain the following factorization of σ-smooth operators $T : L^\Phi \to X$.

COROLLARY 2.3. *Let Φ be an Orlicz function and let $T : L^\Phi \to X$ be a σ-smooth linear operator. Then T may be factorized: $T = \overline{T} \circ j$, where $j : L^\Phi \to L^{\overline{\Phi}}$ is the inclusion map and $\overline{T} : L^{\overline{\Phi}} \to X$ is a σ-smooth linear operator.*

3. Integral representation of smooth operators. In this section we obtain a Bochner integral representation of σ-smooth linear operators $T : L^\Phi \to X$, where Φ is an Orlicz function (not necessarily convex) and X has the Radon-Nikodym Property. We extend some earlier results due to J. J. Uhl (see [U, Theorem 1]), where Φ is supposed to be convex and $\Phi \in \Delta_2$. The problem of Bochner integral representation of linear operators $T : L^p \to X$ $(p > 1)$ has been studied in [DU, Theorem 3.4.8], [D_1], [D_2], [D_3].

For terminology concerning vector measures and Banach-space valued function spaces we refer to [DU, Chap. 3.1], [L]. Denote by $L^0(X)$ the set of μ-equivalence classes of all strongly Σ-measurable functions $g : \Omega \to X$. For $g : \Omega \to X$ let us put $\widetilde{g}(\omega) = \|g(\omega)\|_X$ for $\omega \in \Omega$. For an Orlicz function Φ the *Orlicz-Bochner space* $L^\Phi(X)$ is defined by

$$L^\Phi(X) = \{g \in L^0(X) : \widetilde{g} \in L^\Phi\}.$$

A linear operator $T : L^\Phi \to X$ is said to be *regular* if there exists $0 \le v \in L^{\Phi^*}$ such that $\|T(u)\|_X \le \varphi_v(|u|) = \int_\Omega |u(\omega)|v(\omega)\,d\mu$ for all $u \in L^\Phi$ (see [Bu, Def. 1.2]).

From now on we will assume that (Ω, Σ, μ) is a finite atomless measure space. Recall that a Banach space X has the *Radon-Nikodym property* (with respect to μ) (briefly $X \in RNP(\mu)$) if for each μ-continuous vector measure $m : \Sigma \to X$ of bounded variation (i.e., $|m|(\Omega) < \infty$) there exists $g \in L^1(X)$ such that

$$m(A) = \int_A g(\omega)\,d\mu \quad \text{for all } A \in \Sigma.$$

Then $|m|(A) = \int_A \|g(\omega)\|_X\,d\mu$ for all $A \in \Sigma$. Motivated by the variation $|m|(\Omega)$ and following [D_1], [D_3] we can define a norm functional of operators $T : L^\Phi \to X$ by

$$|\!|\!|T|\!|\!|_\Phi := \sup\Big\{\sum_{i=1}^n \|\alpha_i T(1_{A_i})\|_X : s = \sum_{i=1}^n \alpha_i 1_{A_i} \in \mathcal{S}(\Sigma),\ \varrho_\Phi(s) \le 1\Big\}.$$

Now we are in a position to state our main result.

THEOREM 3.1. *Let Φ be an Orlicz function and let $X \in RNP(\mu)$. Then for a linear operator $T : L^\Phi \to X$ the following statements are equivalent:*

(i) $|\!|\!|T|\!|\!|_\Phi < \infty$ *and T is modularly continuous.*
(ii) $|\!|\!|T|\!|\!|_\Phi < \infty$ *and T is σ-smooth.*
(iii) $|\!|\!|T|\!|\!|_\Phi < \infty$ *and T is $(\tau(L^\Phi, L^{\Phi^*}), \|\cdot\|_X)$-continuous.*
(iv) $|\!|\!|T|\!|\!|_\Phi < \infty$ *and T is $(\mathcal{T}_\Phi^\wedge, \|\cdot\|_X)$-continuous.*
(v) *There exists $g \in L^{\Phi^*}(X)$ such that*

$$T(u) = T_g(u) = \int_\Omega u(\omega)g(\omega)\,d\mu \quad \textit{for all } u \in L^\Phi$$

and

$$|\!|\!|T_g|\!|\!|_\Phi = \|\varphi_{\widetilde g}\|_\Phi = \sup\Big\{\Big|\int_\Omega u(\omega)\widetilde g(\omega)\,d\mu\Big| : u \in L^\Phi,\ \varrho_\Phi(u) \le 1\Big\}.$$

In particular, if Φ is a convex Orlicz function, then

$$|\!|\!|T_g|\!|\!|_\Phi = \|\widetilde g\|^0_{\Phi^*} = \|g\|^0_{L^{\Phi^*}(X)}.$$

(vi) *T is regular.*

Proof. (i)⇔(ii)⇔(iii)⇔(iv) follow from Theorem 2.1.

(i)⇒(v). Assume that $|\!|\!|T|\!|\!|_\Phi < \infty$ and T is modular continuous.

Define a vector measure $m_T : \Sigma \to X$ by $m_T(A) = T(1_A)$ for $A \in \Sigma$. We shall now show that m_T is μ-continuous. Indeed, let $\mu(A_n) \to 0$ with $A_n \in \Sigma$. Then

$$\varrho_\Phi(1_{A_n}) = \int_\Omega \Phi(1_{A_n}(\omega))\,d\mu = \Phi(1)\mu(A_n) \to 0,$$

so

$$\|m_T(A_n)\|_X = \|T(1_{A_n})\|_X \to 0.$$

It follows that m_T is countably additive and μ-continuous. Now, choose $\alpha > 0$ such that $\varrho_\Phi(\alpha 1_\Omega) \le 1$. For any finite Σ-partition $\{A_i : 1 \le i \le n\}$ of Ω we have $\alpha 1_\Omega = \sum_{i=1}^n \alpha 1_{A_i}$, so

$$\alpha\sum_{i=1}^{n}\|m_T(A_i)\|_X=\sum_{i=1}^{n}\|\alpha T(1_{A_i})\|_X\le \|T\|_\Phi .$$

Hence $|m_T|(\Omega)<\infty$, and since $X\in RNP(\mu)$ there exists $g\in L^1(X)$ such that

$$m_T(A)=\int_A g(\omega)\,d\mu \quad\text{and}\quad |m_T|(A)=\int_A\|g(\omega)\|_X\,d\mu \qquad\text{for } A\in\Sigma.$$

Then for $s=\sum_{i=1}^{n}\alpha_i 1_{A_i}\in\mathcal{S}(\Sigma)$ we have

$$\begin{aligned}T(s)&=\sum_{i=1}^{n}\alpha_i T(1_{A_i})=\sum_{i=1}^{n}\alpha_i m_T(A_i)\\&=\sum_{i=1}^{n}\alpha_i\int_{A_i} g(\omega)\,d\mu=\int_\Omega s(\omega)g(\omega)\,d\mu.\end{aligned}\tag{3.1}$$

We now show that for $s=\sum_{i=1}^{n}\alpha_i 1_{A_i}\in\mathcal{S}(\Sigma)$ with $\varrho_\Phi(s)\le 1$ we have

$$\sum_{i=1}^{n}|\alpha_i|\int_{A_i}\|g(\omega)\|_X\,d\mu=\sum_{i=1}^{n}|\alpha_i|\,|m_T|(A_i)\le\|T\|_\Phi .$$

Indeed, let $\varepsilon>0$ be given. Then for each $1\le i\le n$ there exists a Σ-partition $(A_{i,j})_{j=1}^{k_i}$ of A_i such that

$$|m_T|(A_i)\le\sum_{j=1}^{k_i}\|m_T(A_{i,j})\|_X+\frac{\varepsilon}{n|\alpha_i|}=\sum_{j=1}^{k_i}\|T(1_{A_{i,j}})\|_X+\frac{\varepsilon}{n|\alpha_i|}.$$

Hence

$$\sum_{i=1}^{n}|\alpha_i|\,|m_T|(A_i)\le\sum_{i=1}^{n}\Big(\sum_{j=1}^{k_i}\|\alpha_i T(1_{A_{i,j}})\|_X\Big)+\varepsilon\le\|T\|_\Phi+\varepsilon,$$

because

$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{k_i}\alpha_i 1_{A_{i,j}}\Big)=\sum_{i=1}^{n}\alpha_i\,1_{A_i}.$$

Then

$$\begin{aligned}\sum_{i=1}^{n}\|\alpha_i T(1_{A_i})\|_X&=\sum_{i=1}^{n}|\alpha_i|\|m_T(A_i)\|_X\le\sum_{i=1}^{n}|\alpha_i|\,|m_T|(A_i)\\&=\sum_{i=1}^{n}|\alpha_i|\int_{A_i}\|g(\omega)\|_X\,d\mu=\int_\Omega\Big(\sum_{i=1}^{n}|\alpha_i|1_{A_i}(\omega)\Big)\|g(\omega)\|_X\,d\mu\\&=\int_\Omega|s(\omega)|\widetilde{g}(\omega)\,d\mu\le\|T\|_\Phi .\end{aligned}$$

Taking suprema on the left, we get

$$\|T\|_\Phi=\sup\Big\{\int_\Omega|s(\omega)|\widetilde{g}(\omega)\,d\mu : s\in\mathcal{S}(\Sigma),\ \varrho_\Phi(s)\le 1\Big\}.\tag{3.2}$$

Now we are ready to show $u\widetilde{g}\in L^1$ for every $u\in L^\Phi$, i.e., $\widetilde{g}\in(L^\Phi)'=L^{\Phi^*}$. Indeed, let $u\in L^\Phi$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $0\le s_n(\omega)\uparrow|u(\omega)|$ for

$\omega \in \Omega$ (see [KA, Corollary I.6]). Choose $\alpha > 0$ such that $\varrho_\Phi(\alpha u) \le 1$. Then by Fatou's lemma and (3.2) we get

$$\int_\Omega \alpha |u(\omega)| \widetilde{g}(\omega)\, d\mu \le \sup_n \int_\Omega \alpha s_n(\omega) \widetilde{g}(\omega)\, d\mu \le \|\!|T|\!\|_\Phi,$$

and this means that $\widetilde{g} \in (L^\Phi)' = L^{\Phi^*}$ and $ug \in L^1(X)$. Thus we can define a linear operator $T_g : L^\Phi \to X$ by

$$T_g(u) = \int_\Omega u(\omega) g(\omega)\, d\mu \quad \text{for } u \in L^\Phi.$$

We shall now show that $T_g(u) = T(u)$ for $u \in L^\Phi$. Indeed, let $u \in L^\Phi$ and choose $\alpha > 0$ such that $\varrho_\Phi(2\alpha u) < \infty$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $s_n(\omega) \to u(\omega)$ μ-a.e. and $|s_n(\omega)| \le |u(\omega)|$ μ-a.e. ([KA, Corollary I.6]). By the dominated convergence theorem $\varrho_\Phi(\alpha(s_n - u)) \to 0$, and since T is modularly continuous, we get $\|T(s_n) - T(u)\|_X \to 0$.

On the other hand, $s_n(\omega)\widetilde{g}(\omega) \to u(\omega)\widetilde{g}(\omega)$ μ-a.e. and $|s_n(\omega)|\widetilde{g}(\omega) \le |u(\omega)|\widetilde{g}(\omega)$ μ-a.e., where $u\widetilde{g} \in L^1$. Using (3.1) we get

$$\begin{aligned}\|T(s_n) - T_g(u)\|_X &= \Big\| \int_\Omega s_n(\omega)\, g(\omega)\, d\mu - \int_\Omega u(\omega) g(\omega)\, d\mu \Big\|_X \\ &\le \int_\Omega |s_n(\omega) - u(\omega)| \widetilde{g}(\omega)\, d\mu \xrightarrow[n]{} 0.\end{aligned}$$

It follows that

$$T(u) = T_g(u) = \int_\Omega u(\omega) g(\omega)\, d\mu \quad \text{for } u \in L^\Phi.$$

Now assume that Φ is a convex Orlicz function. Then $\varrho_\Phi(u) \le 1$ if and only if $\|u\|_\Phi \le 1$ and it follows that $\|\!|T_g|\!\|_\Phi = \|\widetilde{g}\|_\Phi^0 = \|g\|^0_{L^{\Phi^*}(X)}$.

(v)⇒(vi). Assume that there exists $g \in L^{\Phi^*}(X)$ such that

$$T(u) = T_g(u) = \int_\Omega u(\omega) g(\omega)\, d\mu \quad \text{for all } u \in L^\Phi.$$

Then for $u \in L^\Phi$ we have

$$\|T(u)\|_X \le \int_\Omega |u(\omega)|\, \|g(\omega)\|_X\, d\mu = \varphi_{\widetilde{g}}(|u|),$$

where $\widetilde{g} \in L^{\Phi^*}$, i.e., T is regular.

(vi)⇒(ii). Assume that T is regular, i.e., there exists $0 \le v \in L^{\Phi^*}$ such that

$$\|T(u)\|_X \le \int_\Omega |u(\omega)| v(\omega)\, d\mu = \varphi_v(|u|) \quad \text{for all } u \in L^\Phi.$$

Let $s = \sum_{i=1}^n \alpha_i 1_{A_i} \in \mathcal{S}(\Sigma)$ with $\varrho_\Phi(s) \le 1$. Then using (3.1) we get

$$\begin{aligned}\sum_{i=1}^n \|\alpha_i T(1_{A_i})\|_X &= \sum_{i=1}^n |\alpha_i| \|T(1_{A_i})\|_X \le \sum_{i=1}^n |\alpha_i| \int_\Omega 1_{A_i}(\omega) v(\omega)\, d\mu \\ &= \int_\Omega \Big(\sum_{i=1}^n |\alpha_i| (1_{A_i})(\omega) \Big) v(\omega)\, d\mu = \varphi_v(|s|) \\ &\le \|\varphi_{\widetilde{g}}\|_\Phi (\varrho_\Phi(s) + 1) \le 2 \|\varphi_{\widetilde{g}}\|_\Phi.\end{aligned}$$

Hence $\|T_g\|_\Phi \le 2\|\varphi_{\widetilde{g}}\|_\Phi$. Now assume that $u_n \xrightarrow{(o)} 0$ in L^Φ. Since $\varphi_v \in (L^\Phi)^\sim_n$, we obtain that $\|T(u_n)\|_X \to 0$, i.e., T is σ-smooth.

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