FUNCTION SPACES VIII
BANACH CENTER PUBLICATIONS, VOLUME 79
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

THE STRONG UNICITY CONSTANT AND ITS APPLICATIONS

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Abstract. In this report we discuss the applications of the strong unicity constant and highlight its use in the minimal projection problem.

Let X be a Banach space and let $V \subset X$ be a nonempty subset. An element $v_0 \in V$ is called a *strongly unique best approximation* to $x \in X$ if there exists $r > 0$ such that for any $v \in V$

$$\|x - v\| \geq \|x - v_0\| + r\|v - v_0\|. \tag{1}$$

The biggest constant r satisfying (1) is called the *strong unicity constant*. This notation was introduced by D. J. Newman and H. S. Shapiro (see [20] and [21]).

Example 1. Let $X = l^2_\infty$ with unit ball U (pictured below). Let $x = (0, 1)$, $v_0 = (1, 0)$ and

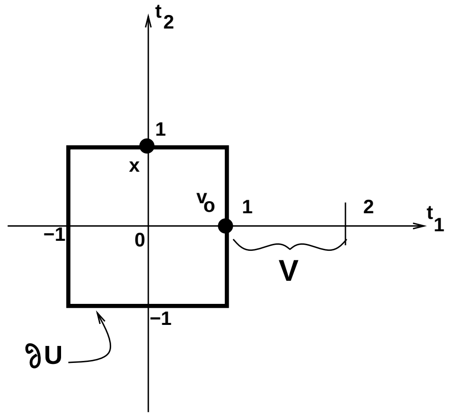
$$V = \{z \in X \mid z = (t, 0),\ 1 \leq t \leq 2\}.$$

Then it is easy to see that, in this case, the strong unicity constant is equal to 1. We note the constant r from (1) belongs to the interval $(0, 1]$ and our example attains this greatest value.

There exist two main applications of the strong unicity constant:

1. the error estimate of the Remez algorithm,

2000 *Mathematics Subject Classification*: Primary 47A58; Secondary 41A65.
Key words and phrases: best approximation, minimal projections.
The paper is in final form and no version of it will be published elsewhere.



2. the Lipschitz continuity of the best approximation mapping at x_0 (if there exists a strongly unique best approximation to x_0).

The error estimate of the Remez algorithm is based on an iteration process for finding the constant r from (1) (see [27], [28]). First this algorithm was used for Chebyshev approximation ([10], [11]). Now it is used for filter design in Digital Signal Processing (see [1]). Currently, in numerical methods of Chebyshev approximation, the strong unicity constant is used in conjunction with the metric projection with Lipschitz continuity property (see e.g., [7], [2], [6], [22], [8], [9], [16]).

The aim of this report is to present a third application of the strong unicity constant in the case of projections from l_∞^n onto some $n-k$ dimensional subspace ($n \geq 3,\ 1 \leq k \leq n-1$). In this case $X = \mathcal{L}(l_\infty^n)$, (equipped with the operator norm),

$$V = \mathcal{P}(l_\infty^n, W) = \{L \in \mathcal{L}(l_\infty^n, W)\ :\ L_{|W} = id_W\},$$

the set of all linear projections from l_∞^n onto $n-k$ dimensional subspace W, $x = 0$ and $v_0 = P_0 \in \mathcal{P}(l_\infty^n, W)$. Thus in the case of projections, (1) reduces to

$$\|P\| \geq \|P_0\| + r\|P - P_0\|. \tag{2}$$

The problem considered in our report may be treated as a development of the results initiated by G. Lewicki in [15].

A projection is called *minimal* if

$$\|P_0\| = \lambda(Y, X) = \inf\{\|P\| \mid P \in \mathcal{P}(Y, X)\}. \tag{3}$$

It is worth noting that there exist a large number of papers concerning minimal projections. Mainly the problems concern existence (see e.g., [5], [13]), uniqueness (see e.g., [4], [23]) and formulas for minimal projections (see e.g., [3], [24]).

A projection $\pi_0 \in \mathcal{P}(Y, X)$ is called the *strongly unique minimal projection* (or SUM-projection) if there exists a constant $s \in (0, 1]$ such that the inequality

$$\|\pi_0\| + s\|\pi - \pi_0\| \leq \|\pi\| \tag{4}$$

holds for each $\pi \in \mathcal{P}(Y, X)$ (see, for example, [12] for results involving SUM-projections onto hyperplanes).

It is easy to prove that the SUM-projection π_0 is the unique minimal projection in $\mathcal{P}(Y, X)$. The largest possible constant for which the inequality in (4) holds is called the *strongly unique projection constant* (or SUP-constant).

It is known (see for example [3]) that if $Y = l_\infty^n$ and $X \subset Y$ is of dimension $n-1$ $(n \geq 3)$ with $X = f^{-1}(0)$ where

$$f = (f_1, \ldots, f_n) \in Y^*, \quad \|f\|_1 = \sum_{i=1}^n |f_i| = 1$$

and

$$0 < f_1 < f_2 < \cdots < f_{n-1} < \frac{1}{2}, \quad f_n \geq \frac{1}{2} \tag{5}$$

then the minimal projection π_0 from l_∞^n onto X has norm one and is unique. Moreover, in this case, π_0 is the SUM-projection and the SUP-constant, $s_0 = s_0(\pi_0)$ is equal to $1 - 2f_{n-1}$ ([14], Theorem 2.3.1).

If a minimal projection π^{00} from l_∞^n onto $f^{-1}(0)$ has norm $u > 1$ then π^{00} is the SUM-projection and the SUP-constant is equal to

$$uf_1 \frac{1 - 2f_1}{1 - 2f_1 - uf_1} \tag{6}$$

where $f = (f_1, \ldots, f_n)$ and $0 < f_1 \leq f_2 \leq \cdots \leq f_n < 1/2$ (in this case we note that as $u \to 1$ we find that (6) approaches $f_1 \frac{1-2f_1}{1-3f_1}$, which in general is not equal to the above expression of $1 - 2f_1$, [17]).

In our report we consider subspaces $X = X_{n-k} \subset l_\infty^n$, $1 \leq k \leq n-1$, $n \geq 3$, such that $\dim X = n - k$. Note that this consideration is quite general due to the following proposition.

PROPOSITION 1. *Let B be an n-dimensional Banach space with unit ball U. Let U be a polytope with m $(n-1)$-dimensional faces. Then B is isometrically isomorphic to an n-dimensional subspace of l_∞^{n+m-1}.*

Proof. This follows immediately from Theorem 1 by G. V. Epifanov in [12]. ∎

Since we are interested in situations for which the minimal projection onto X_{n-k} is unique, we may assume without loss generality that (see [24])

$$X = \bigcap_{p=1}^{k} (f^{(p)})^{-1}(0) \tag{7}$$

where the hyperplanes $\{(f^{(p)})^{-1}(0)\}_{p=1}^k$ are given by the linearly independent functionals $\{f^{(p)}\}_{p=1}^k \in (l_\infty^n)^*$ such that, for $p = 1, \ldots, k$, we have

$$\|f^{(p)}\|_1 = 1, \; f^{(p)} = (f_1^{(p)}, \ldots, f_n^{(p)}) \tag{8}$$

$$0 < f_1^{(p)} < f_2^{(p)} < \cdots < f_{n-k}^{(p)} < \frac{1}{2}, \tag{9}$$

$$f_{n-k+1}^{(1)} \geq \frac{1}{2}, \; f_{n-k+2}^{(2)} \geq \frac{1}{2}, \ldots, f_n^{(k)} \geq \frac{1}{2}, \tag{10}$$

$$f_i^{(p)} = 0 \text{ if } p + i \neq n, \; i = n-k+1, \ldots, n. \tag{11}$$

Moreover, if conditions (8)–(11) hold then the unique minimal projection from l_∞^n onto X_{n-k} has norm one (see [3], Thm. 1; [26], Lemma 2.4.1 and [24], Chp. 2).

Theorem 1. *Let $Y = l_\infty^n$ $(n \geq 3)$ and $X = X_{n-k} \subset Y$ be a subspace of dimension $n-k$ given by*

$$X_{n-k} = \bigcap_{p=1}^{k} (f^{(p)})^{-1}(0) \tag{12}$$

where $\{f^{(p)}\}_{p=1}^{k}$ satisfies (8)–(11). Let π_0 be the minimal projection from Y onto X. Then π_0 is the SUM-projection with norm one and for the SUP-constant $s_0 = s(\pi_0)$ we have the inequality

$$\min\left\{\frac{f^{(1)}_{n-k+1} - f^{(1)}_{n-k}}{f^{(1)}_{n-k+1} + f^{(1)}_{n-k}}, \frac{f^{(2)}_{n-k+2} - f^{(2)}_{n-k}}{f^{(2)}_{n-k+2} + f^{(2)}_{n-k}}, \ldots, \frac{f^{(k)}_{n} - f^{(k)}_{n-k}}{f^{(k)}_{n} + f^{(k)}_{n-k}}\right\} \leq s_o < 1. \tag{13}$$

Remark 1. This result extends the results of O. M. Martynov ([18] and [19]) regarding two and three dimensional subspaces of l_∞^4 and l_∞^6 respectively. (see Remark 2 below)

Remark 2. In general

$$\widehat{s} = \min\left\{\frac{f^{(1)}_{n-k+1} - f^{(1)}_{n-k}}{f^{(1)}_{n-k+1} + f^{(1)}_{n-k}}, \ldots, \frac{f^{(k)}_{n} - f^{(k)}_{n-k}}{f^{(k)}_{n} + f^{(k)}_{n-k}}\right\}$$

is not equal to the SUP-constant; indeed in the case $k = 1$, $n \geq 3$ we have

$$\widehat{s} = \frac{f^{(1)}_{n} - f^{(1)}_{n-1}}{f^{(1)}_{n} + f^{(1)}_{n-1}} < \frac{f^{(1)}_{n} - f^{(1)}_{n-1} + (f^{(1)}_{n-2} + \cdots + f^{(1)}_{1})}{f^{(1)}_{n} + f^{(1)}_{n-1} + (f^{(1)}_{n-2} + \cdots + f^{(1)}_{1})} = 1 - 2f^{(1)}_{n-1}$$

and, by [14] (Thm 2.3.1), $1 - 2f^{(1)}_{n-1}$ is the SUP-constant.

Corollary 1. *If $k = n-1$, $n \geq 3$, then under the hypotheses of Theorem 1 we have*

$$\widehat{s} = \min\{f^{(1)}_{2} - f^{(1)}_{1}, \ldots, f^{(n-1)}_{n} - f^{(n-1)}_{1}\}.$$

Conjecture 1. *Let $f^{(0)}, \ldots, f^{(n-k)} \in (l_\infty^n)^*$, where*

$$\begin{aligned}
f^{(0)} &= (f_1, \ldots, f_k, 0_{k+1}, \ldots, 0_n),\\
f^{(1)} &= (0_1, \ldots, 0_k, 1, 0_{k+2}, \ldots, 0_n),\\
f^{(2)} &= (0_1, \ldots, 0_{k+1}, 1, 0_{k+3}, \ldots, 0_n),\\
&\vdots\\
f^{(n-k)} &= (0_1, \ldots, 0_{n-1}, 1)
\end{aligned}$$

where $\sum_{i=1}^{k} f_i = 1$ and $0 < f_1 \leq f_2 \leq \cdots \leq f_{k-1} < f_k$. Let $\widehat{f} = (f_1, \ldots, f_k) \in (l_\infty^k)^*$ and let $\pi_{\widehat{f}}$ be the unique minimal projection onto $(\widehat{f})^{-1}(0)$ from l_∞^k. Let

$$H = \bigcap_{p=0}^{n-k} (f^{(p)})^{-1}(0)$$

and π_H be the unique minimal projection onto H from l_∞^n. Then the SUP-constant $s(\pi_H)$ is equal to the SUP-constant $s(\pi_{\widehat{f}})$.

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