THE STRONG UNICITY CONSTANT AND ITS APPLICATIONS

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Abstract. In this report we discuss the applications of the strong unicity constant and highlight its use in the minimal projection problem.

Let $X$ be a Banach space and let $V \subset X$ be a nonempty subset. An element $v_0 \in V$ is called a strongly unique best approximation to $x \in X$ if there exists $r > 0$ such that for any $v \in V$

\begin{equation}
\|x - v\| \geq \|x - v_0\| + r\|v - v_0\|.
\end{equation}

The biggest constant $r$ satisfying (1) is called the strong unicity constant. This notation was introduced by D. J. Newman and H. S. Shapiro (see [20] and [21]).

Example 1. Let $X = l^2_\infty$ with unit ball $U$ (pictured below). Let $x = (0, 1)$, $v_0 = (1, 0)$ and

$$V = \{z \in X \mid z = (t, 0), \; 1 \leq t \leq 2\}.$$ 

Then it is easy to see that, in this case, the strong unicity constant is equal to 1. We note the constant $r$ from (1) belongs to the interval $(0, 1]$ and our example attains this greatest value.

There exist two main applications of the strong unicity constant:

1. the error estimate of the Remez algorithm,
2. the Lipschitz continuity of the best approximation mapping at \( x_0 \) (if there exists a strongly unique best approximation to \( x_0 \)).

The error estimate of the Remes algorithm is based on an iteration process for finding
the constant \( r \) from (1) (see [27], [28]). First this algorithm was used for Chebyshev
approximation ([10], [11]). Now it is used for filter design in Digital Signal Processing (see
[1]). Currently, in numerical methods of Chebyshev approximation, the strong unicity
constant is used in conjunction with the metric projection with Lipschitz continuity
property (see e.g., [7], [2], [6], [22], [8], [9], [16]).

The aim of this report is to present a third application of the strong unicity constant
in the case of projections from \( l_\infty^n \) onto some \( n - k \) dimensional subspace (\( n \geq 3, 1 \leq k \leq n - 1 \)). In this case \( X = L(l_\infty^n) \), (equipped with the operator norm),
\[
V = P(l_\infty^n, W) = \{ L \in L(l_\infty^n, W) : L|_W = id_W \},
\]
the set of all linear projections from \( l_\infty^n \) onto \( n - k \) dimensional subspace \( W \), \( x = 0 \) and
\( v_0 = P_0 \in P(l_\infty^n, W) \). Thus in the case of projections, (1) reduces to
\[
\|P\| \geq \|P_0\| + r\|P - P_0\|.
\]

The problem considered in our report may be treated as a development of the results
initiated by G. Lewicki in [15].

A projection is called minimal if
\[
\|P_0\| = \lambda(Y, X) = \inf\{\|P\| \mid P \in P(Y, X)\}.
\]
It is worth noting that there exist a large number of papers concerning minimal projections. Mainly the problems concern existence (see e.g., [5], [13]), uniqueness (see e.g., [4],
[23]) and formulas for minimal projections (see e.g., [3], [24]).

A projection \( \pi_0 \in P(Y, X) \) is called the strongly unique minimal projection (or SUM-
projection) if there exists a constant \( s \in (0, 1] \) such that the inequality
\[
\|\pi_0\| + s\|\pi - \pi_0\| \leq \|\pi\|
\]
holds for each \( \pi \in P(Y, X) \) (see, for example, [12] for results involving SUM-projections
onto hyperplanes).
It is easy to prove that the SUM-projection \( \pi_0 \) is the unique minimal projection in \( P(Y, X) \). The largest possible constant for which the inequality in (4) holds is called the strongly unique projection constant (or SUP-constant).

It is known (see for example [3]) that if \( Y = l_\infty^n \) and \( X \subset Y \) is of dimension \( n - 1 \) \( (n \geq 3) \) with \( X = f^{-1}(0) \) where

\[
f = (f_1, \ldots, f_n) \in Y^*, \quad \|f\|_1 = \sum_{i=1}^n |f_i| = 1
\]

and

\[
0 < f_1 < f_2 < \cdots < f_{n-1} < \frac{1}{2}, \quad f_n \geq \frac{1}{2}
\]

then the minimal projection \( \pi_0 \) from \( l_\infty^n \) onto \( X \) has norm one and is unique. Moreover, in this case, \( \pi_0 \) is the SUM-projection and the SUP-constant, \( s_0 = s_0(\pi_0) \) is equal to \( 1 - 2f_{n-1} \) ([14], Theorem 2.3.1).

If a minimal projection \( \pi^{00} \) from \( l_\infty^n \) onto \( f^{-1}(0) \) has norm \( u > 1 \) then \( \pi^{00} \) is the SUM-projection and the SUP-constant is equal to

\[
u f_1 \frac{1 - 2f_1}{1 - 2f_1 - uf_1}
\]

where \( f = (f_1, \ldots, f_n) \) and \( 0 < f_1 \leq f_2 \leq \cdots \leq f_n < 1/2 \) (in this case we note that as \( u \to 1 \) we find that (6) approaches \( f_1 \frac{1 - 2f_1}{1 - 3f_1} \), which in general is not equal to the above expression of \( 1 - 2f_1 \), [17]).

In our report we consider subspaces \( X = X_{n-k} \subset l_\infty^n, 1 \leq k \leq n - 1, n \geq 3, \) such that \( \dim X = n - k \). Note that this consideration is quite general due to the following proposition.

**Proposition 1.** Let \( B \) be an \( n \)-dimensional Banach space with unit ball \( U \). Let \( U \) be a polytope with \( m \) \((n - 1)\)-dimensional faces. Then \( B \) is isometrically isomorphic to an \( n \)-dimensional subspace of \( l_\infty^{n+m-1} \).

**Proof.** This follows immediately from Theorem 1 by G. V. Epifanov in [12]. \( \blacksquare \)

Since we are interested in situations for which the minimal projection onto \( X_{n-k} \) is unique, we may assume without loss generality that (see [24])

\[
X = \bigcap_{p=1}^k (f^{(p)})^{-1}(0)
\]

where the hyperplanes \( \{(f^{(p)})^{-1}(0)\}_{p=1}^k \) are given by the linearly independent functionals \( \{f^{(p)}\}_{p=1}^k \in (l_\infty^n)^* \) such that, for \( p = 1, \ldots, k \), we have

\[
\|f^{(p)}\|_1 = 1, \quad f^{(p)} = (f^{(p)}_1, \ldots, f^{(p)}_n)
\]

\[
0 < f^{(p)}_1 < f^{(p)}_2 < \cdots < f^{(p)}_{n-k} < \frac{1}{2},
\]

\[
f^{(k)}_{n-k+1} \geq \frac{1}{2}, \quad f^{(k)}_{n-k+2} \geq \frac{1}{2}, \ldots, f^{(k)}_n \geq \frac{1}{2},
\]

\[
f^{(p)}_i = 0 \text{ if } p + i \neq n, \ i = n - k + 1, \ldots, n.
\]
Moreover, if conditions (8)–(11) hold then the unique minimal projection from \(l_\infty^n\) onto \(X_{n-k}\) has norm one (see [3], Thm. 1; [26], Lemma 2.4.1 and [24], Chp. 2).

**Theorem 1.** Let \(Y = l_\infty^n\) \((n \geq 3)\) and \(X = X_{n-k} \subset Y\) be a subspace of dimension \(n - k\) given by

\[
X_{n-k} = \bigcap_{p=1}^{k} (f^{(p)})^{-1}(0)
\]

where \(\{ f^{(p)} \}_{p=1}^{k} \) satisfies (8)–(11). Let \(\pi_0\) be the minimal projection from \(Y\) onto \(X\). Then \(\pi_0\) is the SUM-projection with norm one and for the SUP-constant \(s_0 = s(\pi_0)\) we have the inequality

\[
\min \left\{ \frac{f^{(1)}_{n-k} - f^{(1)}_{n-k}}{f^{(1)}_{n-k} + f^{(1)}_{n-k}}, \frac{f^{(2)}_{n-k} - f^{(2)}_{n-k}}{f^{(2)}_{n-k} + f^{(2)}_{n-k}}, \ldots, \frac{f^{(k)}_{n-k} - f^{(k)}_{n-k}}{f^{(k)}_{n-k} + f^{(k)}_{n-k}} \right\} \leq s_0 < 1.
\]

**Remark 1.** This result extends the results of O. M. Martynov ([18] and [19]) regarding two and three dimensional subspaces of \(l_\infty^4\) and \(l_\infty^5\) respectively. (see Remark 2 below)

**Remark 2.** In general

\[
\hat{s} = \min \left\{ \frac{f^{(1)}_{n-k} - f^{(1)}_{n-k}}{f^{(1)}_{n-k} + f^{(1)}_{n-k}}, \frac{f^{(2)}_{n-k} - f^{(2)}_{n-k}}{f^{(2)}_{n-k} + f^{(2)}_{n-k}}, \ldots, \frac{f^{(k)}_{n-k} - f^{(k)}_{n-k}}{f^{(k)}_{n-k} + f^{(k)}_{n-k}} \right\}
\]

is not equal to the SUP-constant; indeed in the case \(k = 1, n \geq 3\) we have

\[
\hat{s} = \frac{f^{(1)}_{n} - f^{(1)}_{n-1}}{f^{(1)}_{n} + f^{(1)}_{n-1}} < \frac{f^{(1)}_{n} - f^{(1)}_{n-1} + (f^{(1)}_{n-2} + \cdots + f^{(1)}_{1})}{f^{(1)}_{n} + f^{(1)}_{n-1} + (f^{(1)}_{n-2} + \cdots + f^{(1)}_{1})} = 1 - 2f^{(1)}_{n-1}
\]

and, by [14] (Thm 2.3.1), \(1 - 2f^{(1)}_{n-1}\) is the SUP-constant.

**Corollary 1.** If \(k = n - 1, n \geq 3\), then under the hypotheses of Theorem 1 we have

\[
\hat{s} = \min \{ f^{(1)}_{2} - f^{(1)}_{1}, \ldots, f^{(n-1)}_{n} - f^{(n-1)}_{1} \}.
\]

**Conjecture 1.** Let \(f^{(0)}, \ldots, f^{(n-k)} \in (l_\infty^n)^*\), where

\[
\begin{align*}
&f^{(0)} = (f_1, \ldots, f_k, 0_{k+1}, \ldots, 0_n), \\
&f^{(1)} = (0_1, \ldots, 0_k, 1, 0_{k+2}, \ldots, 0_n), \\
&f^{(2)} = (0_1, \ldots, 0_{k+1}, 1, 0_{k+3}, \ldots, 0_n), \\
&\vdots \\
&f^{(n-k)} = (0_1, \ldots, 0_{n-1}, 1)
\end{align*}
\]

where \(\sum_{i=1}^{k} f_i = 1\) and \(0 < f_1 \leq f_2 \leq \cdots \leq f_{k-1} < f_k\). Let \(\hat{f} = (f_1, \ldots, f_k) \in (l_\infty^n)^*\) and let \(\pi_{\hat{f}}\) be the unique minimal projection onto \((\hat{f})^{-1}(0)\) from \(l_\infty^n\). Let

\[
H = \bigcap_{p=0}^{n-k} (f^{(p)})^{-1}(0)
\]

and \(\pi_H\) be the unique minimal projection onto \(H\) from \(l_\infty^n\). Then the SUP-constant \(s(\pi_H)\) is equal to the SUP-constant \(s(\pi_{\hat{f}})\).
References


