FUNCTION SPACES VIII
BANACH CENTER PUBLICATIONS, VOLUME 79
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

ON THE APPROXIMATION BY EULER, BOREL AND TAYLOR MEANS IN ENLARGED HÖLDER CLASSES

BOGDAN SZAL

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Szafrana 4a, 65-516 Zielona Góra, Poland
E-mail: B.Szal@wmie.uz.zgora.pl

Abstract. We generalize and improve in some cases the results of Mahapatra and Chandra [7]. As a measure of Hölder norm approximation, generalized modulus-type functions are used.

1. Introduction. Let f be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be its Fourier series. Denote by $S_n(x) = S_n(f,x)$ the n-th partial sum of (1.1).

The usual supremum norm will be denoted by $\|\cdot\|_C$.

Let ω be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties

$$\omega(0) = 0, \quad \omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2).$$

Such functions will be called moduli of continuity. If

$$\omega(f,\delta) := \sup_{|x-y|\le\delta} |f(x) - f(y)|$$

denotes the modulus of continuity of $f \in C_{2\pi}$, then the class of functions $f \in C_{2\pi}$ for which

$$\omega(f,\delta) \le A\omega(\delta) \quad \text{if } 0 \le \delta \le 2\pi,$$

will be denoted by H^ω, and equipped with the norm

$$\|f\|_\omega := \|f\|_C + \sup_{x,\ y} |\Delta^\omega f(x,y)|,$$

2000 *Mathematics Subject Classification*: 42A24, 40G05, 40G10.
Key words and phrases: trigonometric approximation, Borel, Euler and Taylor summability.
The paper is in final form and no version of it will be published elsewhere.

where

$$\Delta^{w} f(x,y)=\frac{|f(x)-f(y)|}{\omega(|x-y|)}, \qquad x\neq y.$$

In the case $\omega(\delta)=\delta^{\alpha}$ $(0<\alpha\le 1)$ we write, as usual, H^{α}, $\Delta^{\alpha} f(x,y)$ and $\|f\|_{\alpha}$ instead of $H^{\delta^{\alpha}}$, $\Delta^{\delta^{\alpha}} f(x,y)$ and $\|f\|_{\delta^{\alpha}}$, respectively.

Let $A=(a_{nk})$ be an infinite matrix. The A-transform of the Fourier series (1.1) is given by

$$A_n(f,x)=\sum_{k=0}^{\infty} a_{nk} S_k(x).$$

Series (1.1) is said to be A-summable to s if

$$\lim_{n\to\infty} A_n(f,x)=s.$$

If $a_{nk}=\binom{n}{k} q^{n-k}(1+q)^{-n}$ $(q\ge 0,\ k\le n)$ and $a_{nk}=0$, $(k>n)$, then the matrix $A=(a_{nk})$ is called the Euler matrix. If (a_{nk}) is given by the formula

$$\frac{(1-r)^{n+1}}{(1-r\theta)^{n+1}}=\sum_{k=0}^{\infty} a_{nk}\theta^{k} \ (0\le r<1,\ |r\theta|<1)$$

then the matrix A is called the Taylor matrix. We shall write $E_n^q(f,x)$ or $T_n^r(f,x)$ for $A_n(f,x)$ according as A is the Euler or the Taylor matrix, respectively.

We shall also consider the Borel transform of the series (1.1) defined by

$$B^{p}(f,x)=e^{-p}\sum_{n=0}^{\infty}\frac{p^{n}}{n!} S_n(x),$$

for $p>0$.

The series (1.1) is summable by the Borel method to s if

$$\lim_{p\to\infty} B^{p}(f,x)=s.$$

We shall use the additional notations:

$$\phi_x(t)=f(x+t)+f(x-t)-2f(x), \tag{1.2}$$

$$M(p,t)=e^{-p}\sum_{n=0}^{\infty}\frac{p^{n}}{n!}\sin\left(n+\frac{1}{2}\right)t, \tag{1.3}$$

$$E(n,t)=(1+q)^{-n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\sin\left(k+\frac{1}{2}\right)t, \tag{1.4}$$

$$L(n,r,t,\theta)=\left(\frac{1-r}{h}\right)^{n+1}\sin\left(\left(n+\frac{1}{2}\right)t+(n+1)\theta\right), \tag{1.5}$$

for $0\le r<1$, $|r\theta|<1$ and $1-re^{it}=he^{-i\theta}$.

By K, K_1, $K_2,\dots$ we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same at each occurrence.

In [7] Mahapatra and Chandra proved the following theorems:

THEOREM 1. *Let* $0 \le \beta < \alpha$. *Then, for* $f \in H^a$,

$$\|E_n^q(f) - f\|_\beta = O(n^{-\frac{1}{2}(\alpha-\beta)}(\log n)^{\beta/\alpha}), \tag{1.6}$$

where $E_n^q(f,x)$ *is the Euler mean of the series (1.1).*

THEOREM 2. *Let* $0 \le \beta < \alpha$. *Then, for* $f \in H^a$,

$$\|B^p(f) - f\|_\beta = O(p^{-\frac{1}{2}(\alpha-\beta)}(\log p)^{\beta/\alpha}), \tag{1.7}$$

where $B^q(f,x)$ *is the Euler mean of the series (1.1).*

THEOREM 3. *Let* $0 \le \beta < \alpha$. *Then, for* $f \in H^a$,

$$\|T_n^r(f) - f\|_\beta = O(n^{-\frac{1}{2}(\alpha-\beta)}(\log n)^{\beta/\alpha}), \tag{1.8}$$

where $T_n^r(f,x)$ *is the Euler mean of the series (1.1).*

In the present paper we extend the validity of Theorems 1, 2 and 3 to the class H^ω. In some cases the present results are improvement of these theorems.

Let us define $\alpha = \alpha(\omega)$ as the infimum of those α' for which there exists a natural number $\mu = \mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega(2^{-n-\mu}) > 2\omega(2^{-n}) \tag{1.9}$$

for all n.

Let Ω_α denote the set of the moduli of continuity $\omega_\alpha(\delta)$ having the following additional property besides (1.9): For any natural number μ there exists a natural number $N(\mu)$ such that if $n > N(\mu)$ then

$$2^{\mu\alpha}\omega_a(2^{-n-\mu}) \le 2\omega_\alpha(2^{-n}).$$

2. Main results. Our main results are the following.

THEOREM 4. *If* $\omega_\alpha \in \Omega_\alpha$ *and* $\omega_\beta \in \Omega_\beta$ *where* $0 \le \beta < \alpha \le 1$, *then for* $f \in H^{\omega_\alpha}$ *and* $n \to \infty$,

$$\|E_n^q(f) - f\|_{\omega_\beta} = \begin{cases} O\left(\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}\right) & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ O\left(\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n})\right) & \text{for } \alpha = 1 \text{ and } \beta = 0. \end{cases} \tag{2.1}$$

THEOREM 5. *If* $\omega_\alpha \in \Omega_\alpha$ *and* $\omega_\beta \in \Omega_\beta$ *where* $0 \le \beta < \alpha \le 1$, *then for* $f \in H^{\omega_\alpha}$ *and* $p \to \infty$,

$$\|B^p(f) - f\|_{\omega_\beta} = \begin{cases} O\left(\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}\right) & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ O\left(\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}(1+\log\sqrt{p})\right) & \text{for } \alpha = 1 \text{ and } \beta = 0. \end{cases} \tag{2.2}$$

THEOREM 6. *If $\omega_\alpha \in \Omega_\alpha$ and $\omega_\beta \in \Omega_\beta$ where $0 \le \beta < \alpha \le 1$, then for $f \in H^{\omega_\alpha}$ and $n \to \infty$,*

$$\|T_n^r(f) - f\|_{\omega_\beta} = \begin{cases} O\left(\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}\right) & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ O\left(\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n})\right) & \text{for } \alpha = 1 \text{ and } \beta = 0. \end{cases} \tag{2.3}$$

REMARK 1. It is clear that Theorems 3, 4 and 5 include Theorems 1, 2 and 3, respectively and moreover if $\alpha < 1$ or $\beta > 0$ then (2.1), (2.2) and (2.3) give a better approximation order than (1.6), (1.7) and (1.8) do.

3. Lemmas. To prove our theorems we need the following known lemmas.

LEMMA 1 ([4]). *If $0 \le \beta < \alpha \le 1$, $\omega_\beta \in \Omega_\beta$ and $\omega_\alpha \in \Omega_\alpha$, then for any n,*

$$\sum_{k=n}^{\infty} \frac{\omega_\alpha(2^{-k})}{\omega_\beta(2^{-k})} \le K \frac{\omega_\alpha(2^{-n})}{\omega_\beta(2^{-n})} \tag{3.1}$$

and

$$\sum_{k=1}^{n} \frac{\omega_\beta(2^{-k})}{\omega_\alpha(2^{-k})} \le K \frac{\omega_\beta(2^{-n})}{\omega_\alpha(2^{-n})}. \tag{3.2}$$

LEMMA 2 ([4]). *If $0 \le \beta < \alpha \le 1$, $\omega_\beta \in \Omega_\beta$, $\omega_\alpha \in \Omega_\alpha$, and $f \in H^{\omega_\alpha}$ then*

$$|\phi_x(t) - \phi_y(t)| \le K\omega_\beta(|x-y|)\frac{\omega_\alpha(t)}{\omega_\beta(t)} \tag{3.3}$$

for any x, y and positive t.

LEMMA 3 ([8]). *If $0 < \alpha < 1$ and $\omega_\alpha \in \Omega_\alpha$, then*

$$\sum_{n=0}^{m} 2^n \omega_\alpha(2^{-n}) \le K 2^m \omega_\alpha(2^{-m}) \tag{3.4}$$

and

$$\sum_{n=0}^{m} 2^{-n\frac{\alpha}{2}} (\omega_\alpha(2^{-n}))^{-1} \le K 2^{-m\frac{\alpha}{2}} (\omega_\alpha(2^{-m}))^{-1}. \tag{3.5}$$

LEMMA 4 ([7]). *If $q \ge 0$ and $0 < t < \pi$, then*

$$(1+q)^n E(n,t) = [P(q,t)]^{n/2} \sin\left(\frac{t}{2} + nQ(q,t)\right), \tag{3.6}$$

$$E(n,t) = O\left(\exp\left(\frac{-2nqt^2}{(1+q)^2\pi^2}\right)\right) \tag{3.7}$$

and

$$E(n,t) = O\left(\frac{1}{t\sqrt{n}}\right), \tag{3.8}$$

where

$$P(q,t) = 1 + q^2 + 2q\cos t, \qquad Q(q,t) = \tan^{-1}\left(\sin\frac{t}{q+\cos t}\right).$$

LEMMA 5 ([3]). *For* $0 \le r < 1$, $|r\theta| < 1$ *and* h *given by* $1 - re^{it} = he^{-i\theta}$ $(0 \le t \le \pi)$,

$$\left(\frac{1-r}{h}\right)^n \le \exp(-Knt^2)\ (0 \le t \le \pi) \tag{3.9}$$

and

$$\left(\frac{1-r}{h}\right)^n - \exp\left(\frac{-nr^2}{2(1-r)^2}\right) = O(nt^4)\ (0 \le t \le \pi). \tag{3.10}$$

LEMMA 6 ([5]). *If* $0 \le r < 1$, $|r\theta| < 1$, *then*

$$\theta - \frac{rt}{1-r} \le Kt^3 \quad \left(0 \le t \le \frac{\pi}{2}\right). \tag{3.11}$$

4. Proofs of Theorems

Proof of Theorem 4. Using the notations (1.2) and (1.4), a standard computation gives that

$$l_n(x) := E_n^q(f,x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}} E(n,t)dt.$$

Applying the inequality $\sin\frac{t}{2} \ge \frac{t}{\pi}$ $(0 \le t \le \pi)$ we obtain

$$|l_n(x) - l_n(y)| \le \frac{1}{\pi}\int_0^\pi \frac{|\phi_x(t) - \phi_y(t)|}{\sin\frac{t}{2}}|E(n,t)|dt$$

$$\le \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi/\sqrt{n}} + \int_{\pi/\sqrt{n}}^{\pi}\right)\frac{|\phi_x(t) - \phi_y(t)|}{t}|E(n,t)|dt = I_1 + I_2 + I_3.$$

Since $|E(n,t)| \le 3/2nt$ for $0 \le t \le \frac{\pi}{n}$ and $|E(n,t)| \le 1$, by (3.3), (3.1) and (3.2) the terms I_1 and I_2 can be estimated as follows:

$$I_1 \le \frac{3}{2}n\int_0^{\pi/n}|\phi_x(t) - \phi_y(t)|dt \le K\omega_\beta(|x-y|)\int_0^{\pi/n}\frac{\omega_\alpha(t)}{\omega_\beta(t)}dt \le K_1\omega_\beta(|x-y|)\frac{\omega_\alpha(\pi/n)}{\omega_\beta(\pi/n)}$$

and

$$I_2 \le \int_{\pi/n}^{\pi/\sqrt{n}}\frac{|\phi_x(t) - \phi_y(t)|}{t}dt \le K_2\omega_\beta(|x-y|)\int_{\pi/n}^{\pi/\sqrt{n}}\frac{\omega_\alpha(t)}{t\omega_\beta(t)}dt$$

$$\le K_2\omega_\beta(|x-y|)\sum_{k=\sqrt{n}}^{n-1}\int_{\pi/(k+1)}^{\pi/k}\frac{\omega_\alpha(t)}{t\omega_\beta(t)}dt \le K_3\omega_\beta(|x-y|)\sum_{m=\log(\sqrt{n}/\pi)}^{\log(n/\pi)}\frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}$$

$$\le K_4\omega_\beta(|x-y|)\frac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}.$$

By (3.8) and (3.3), we get

$$I_3 \le K_5\frac{1}{\sqrt{n}}\int_{\pi/\sqrt{n}}^{\pi}\frac{|\phi_x(t) - \phi_y(t)|}{t^2}dt \le K_6\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\int_{\pi/\sqrt{n}}^{\pi}\frac{\omega_\alpha(t)}{t^2\omega_\beta(t)}dt$$

$$\le K_6\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\sum_{k=1}^{\sqrt{n}-1}\int_{\pi/(k+1)}^{\pi/k}\frac{\omega_\alpha(t)}{t^2\omega_\beta(t)}dt$$

$$\le K_7\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\sum_{m=0}^{\log(\sqrt{n}/\pi)}2^m\frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}.$$

We estimate the last sum separately in different cases. Namely, if $\alpha < 1$, then by (3.4) we obtain

$$\sum_{m=0}^{\log(\sqrt{n}/\pi)} 2^m \frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})} \le \frac{K_8}{\omega_{\beta(\pi/\sqrt{n})}} \sum_{m=0}^{\log(\sqrt{n}/\pi)} 2^m \omega_\alpha(2^{-m}) \le K_9 \frac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}.$$

If $\alpha = 1$ and $\beta > 0$, using the monotonicity of the sequence $2^{m(1+\frac{\beta}{2})}\omega_\alpha(2^{-m})$, we get

$$\sum_{m=0}^{\log(\sqrt{n}/\pi)} 2^m \frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})} \le K_{10} \left(\frac{\sqrt{n}}{\pi}\right)^{1+\beta/2} \omega_\alpha\left(\frac{\pi}{\sqrt{n}}\right) \sum_{m=0}^{\log(\sqrt{n}/\pi)} 2^{-m\frac{\beta}{2}} (\omega_\beta(2^{-m}))^{-1}.$$

A standard calculation and (3.5) show that

$$\sum_{m=0}^{\log(\sqrt{n}/\pi)} 2^{-m\frac{\beta}{2}} (\omega_\beta(2^{-m}))^{-1} \le K_{11} \left(\frac{\sqrt{n}}{\pi}\right)^{-\beta/2} \frac{1}{\omega_\beta(\pi/\sqrt{n})},$$

whence

$$I_3 \le K_{12}\omega_\beta(|x-y|) \frac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}.$$

In the case $\alpha = 1$ and $\beta = 0$, we have

$$\sum_{m=0}^{\log(\sqrt{n}/\pi)} 2^m \frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})} \le K_{13} \frac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n}).$$

Consequently, collecting the partial results, we obtain

$$|l_n(x) - l_n(y)| \le \begin{cases} K_{14} \dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})} & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ K_{14} \dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n}) & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

Next, we have

$$|l_n(x)| = |E_n^q(f,x) - f(x)| \le \frac{1}{\pi}\int_0^\pi \frac{|\phi_x(t)|}{\sin\frac{t}{2}}|E(n,t)|dt$$

$$\le \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi/\sqrt{n}} + \int_{\pi/\sqrt{n}}^{\pi}\right) \frac{|\phi_x(t)|}{t}|E(n,t)|dt = J_1 + J_2 + J_3.$$

An elementary calculation shows that if $f \in H^{\omega_\alpha}$ then

$$J_1 \le \frac{3}{2}n\int_0^{\pi/n} |\phi_x(t)|dt \le K_{15}n\int_0^{\pi/n} \omega_\alpha(t)dt \le K_{16}\omega_\alpha\left(\frac{\pi}{n}\right)$$

and

$$J_2 \le K_{17}\int_{\pi/n}^{\pi/\sqrt{n}} \frac{|\phi_x(t)|}{t}dt \le K_{18} \sum_{m=\log(\sqrt{n}/\pi)}^{\log(n/\pi)} \omega_\alpha(2^{-m})$$

$$\le K_{19}\omega_\beta\left(\frac{\pi}{\sqrt{n}}\right) \sum_{m=\log(\sqrt{n}/\pi)}^{\log(n/\pi)} \frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}.$$

Using (3.1) we obtain

$$J_2 \le K_{20}\omega_\alpha(\pi/\sqrt{n}).$$

Applying the same considerations as in the estimate of I_3 we can show that

$$J_3 \le \begin{cases} K_{21}\omega_\alpha(\pi/\sqrt{n}) & \text{if } \alpha < 1, \\ K_{21}\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})} & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ K_{21}\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n}) & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

Now, collecting our partial results we obtain that (3.8) holds, and this completes the proof. ■

Proof of Theorem 5. Using the notations (1.2) and (1.5), we can write

$$l_p(x) := B^p(f,x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}} M(p,t)dt.$$

Since

$$M(p,t) = e^{-2p\sin^2\frac{t}{2}}\sin\left(\frac{t}{2} + p\sin t\right)$$

we have

$$|l_p(x) - l_p(y)| \le \frac{1}{\pi}\int_0^\pi \frac{|\phi_x(t)-\phi_y(t)|}{\sin\frac{t}{2}}\left|e^{-2p\sin^2\frac{t}{2}}\sin\left(\frac{t}{2}+p\sin t\right)\right|dt$$
$$= I_1 + I_2 + I_3,$$

where for $p \ge 1$

$$I_1 = \frac{1}{\pi}\int_0^{\pi/p} \frac{|\phi_x(t)-\phi_y(t)|}{\sin\frac{t}{2}}\left|e^{-2p\sin^2\frac{t}{2}}\sin\left(\frac{t}{2}+p\sin t\right)\right|dt,$$

$$I_2 = \frac{1}{\pi}\int_{\pi/p}^{\pi/\sqrt{p}} \frac{|\phi_x(t)-\phi_y(t)|}{\sin\frac{t}{2}}\left|e^{-2p\sin^2\frac{t}{2}}\sin\left(\frac{t}{2}+p\sin t\right)\right|dt$$

and

$$I_3 = \frac{1}{\pi}\int_{\pi/\sqrt{p}}^{\pi} \frac{|\phi_x(t)-\phi_y(t)|}{\sin\frac{t}{2}}\left|e^{-2p\sin^2\frac{t}{2}}\sin\left(\frac{t}{2}+p\sin t\right)\right|dt.$$

Since $\sin\frac{t}{2} \ge \frac{t}{\pi}$ $(0 \le t \le \pi)$, $|\sin(\frac{t}{2}+p\sin t)| \le pt + \frac{t}{2}$ $(0 \le t \le \frac{\pi}{p})$ and $|M(p,t)| \le 1$, by (3.3), (3.1) and (3.2) the quantities I_1 and I_2 can be estimated as follows:

$$I_1 \le \int_0^{\pi/p} \frac{|\phi_x(t)-\phi_y(t)|}{t}\left|\sin\left(\frac{t}{2}+p\sin t\right)\right|dt$$

$$\le K_1\left(p+\frac{1}{2}\right)\omega_\beta(|x-y|)\int_0^{\pi/p}\frac{\omega_\alpha(t)}{\omega_\beta(t)}dt \le K_2\omega_\beta(|x-y|)\frac{\omega_\alpha(\pi/p)}{\omega_\beta(\pi/p)}$$

and

$$I_2 \le \int_{\pi/p}^{\pi/\sqrt{p}} \frac{|\phi_x(t)-\phi_y(t)|}{t}dt \le K_3\omega_\beta(|x-y|)\int_{\pi/p}^{\pi/\sqrt{p}}\frac{\omega_\alpha(t)}{\omega_\beta(t)}dt$$

$$\leq K_3\omega_\beta(|x-y|)\sum_{k=\sqrt{p}}^{p}\int_{\pi/(k+1)}^{\pi/k}\frac{\omega_\alpha(t)}{t\omega_\beta(t)}dt \leq K_4\omega_\beta(|x-y|)\sum_{m=\log(\sqrt{p}/\pi)}^{\log(p/\pi)}\frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}$$

$$\leq K_5\omega_\beta(|x-y|)\frac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}.$$

Using the inequality $\sin\frac{t}{2} \geq \frac{t}{\pi}$ $(0 \leq t \leq \pi)$ we obtain

$$e^{-2p\sin^2\frac{t}{2}} \leq e^{-2p\frac{t^2}{\pi^2}} \leq \frac{K_6}{t\sqrt{p}} \ (0 < t \leq \pi).$$

From this and (3.3) we have

$$I_3 \leq K_6\frac{1}{\sqrt{p}}\int_{\pi/\sqrt{p}}^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{t^2}dt \leq K_7\frac{1}{\sqrt{p}}\omega_\beta(|x-y|)\int_{\pi/\sqrt{p}}^{\pi}\frac{\omega_\alpha(t)}{t^2\omega_\beta(t)}dt$$

$$\leq K_7\frac{1}{\sqrt{p}}\omega_\beta(|x-y|)\sum_{k=1}^{\sqrt{p}-1}\int_{\pi/(k+1)}^{\pi/k}\frac{\omega_\alpha(t)}{t^2\omega_\beta(t)}dt$$

$$\leq K_8\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\sum_{m=0}^{\log(\sqrt{n}/\pi)}2^m\frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}.$$

The last sum we can estimate similarly like in the proof of Theorem 4. Thus

$$I_3 \leq \begin{cases} K_9\omega_\beta(|x-y|)\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})} & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ K_9\omega_\beta(|x-y|)\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}(1+\log\sqrt{p}) & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases}$$

Consequently, collecting the partial results, we obtain

$$|l_p(x)-l_p(y)| \leq \begin{cases} K_{10}\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})} & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ K_{10}\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}(1+\log\sqrt{p}) & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases} \tag{4.1}$$

Applying the same considerations as in the proof of Theorem 4 we can show that if $f \in H^{\omega_\alpha}$, then

$$\|l_p\|_C = \begin{cases} O\left(\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}\right) & \text{if } \alpha < 1 \text{ or } \beta > 0, \\ O\left(\dfrac{\omega_\alpha(\pi/\sqrt{p})}{\omega_\beta(\pi/\sqrt{p})}(1+\log\sqrt{p})\right) & \text{if } \alpha = 1 \text{ and } \beta = 0. \end{cases} \tag{4.2}$$

From (4.1) and (4.2) we obtain (2.2). This completes the proof of Theorem 5. ■

Proof of Theorem 6. Denoting

$$l_n(x) := T_n^r(f,x) - f(x)$$

and using the definition of the Taylor transform of the Fourier series of f and the notations (1.2) and (1.5), we get

$$l_n(x) = \frac{1}{\pi}\sum_{k=0}^{\infty} a_{nk}\int_0^{\infty}\frac{\phi_x(t)}{\sin\frac{t}{2}}\sin\left(k+\frac{1}{2}\right)tdt = \frac{1}{\pi}\int_0^{\infty}\frac{\phi_x(t)}{\sin\frac{t}{2}}L(n,r,t,\theta)dt$$

with $1 - re^{it} = he^{-i\theta}$.

Using the inequality $\sin\frac{t}{2} \geq \frac{t}{\pi}$ $(0 \leq t \leq \pi)$ we have

$$|l_n(x) - l_n(y)| \leq \int_0^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{t}\,|L(n,r,t,\theta)|\,dt = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^{\pi/n}\frac{|\phi_x(t)-\phi_y(t)|}{t}\,|L(n,r,t,\theta)|\,dt,$$

$$I_2 = \int_{\pi/n}^{\pi/\sqrt{n}}\frac{|\phi_x(t)-\phi_y(t)|}{t}\,|L(n,r,t,\theta)|\,dt$$

and

$$I_3 = \int_{\pi/\sqrt{n}}^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{t}\,|L(n,r,t,\theta)|\,dt.$$

Since $|1-r| \leq h$ and $|\sin((n+\frac{1}{2})t + (n+1)\theta)| \leq |(n+\frac{1}{2})t + (n+1)\theta|$ for $0 \leq t \leq \frac{\pi}{n}$, by (3.3), (3.11) and (3.1) the term I_1 can be estimated as follows:

$$\begin{aligned}
I_1 &= \int_0^{\pi/n}\frac{|\phi_x(t)-\phi_y(t)|}{t}\left|\left(\frac{1-r}{h}\right)^{n+1}\sin\left(\left(n+\frac{1}{2}\right)t + \left(n+1\right)\theta\right)\right|dt \\
&\leq \int_0^{\pi/n}\frac{|\phi_x(t)-\phi_y(t)|}{t}\left|\left(n+\frac{1}{2}\right)t + (n+1)\theta\right|dt \\
&\leq K_1\omega_\beta(|x-y|)\int_0^{\pi/n}\frac{\omega_\alpha(t)}{t\omega_\beta(t)}\left(\left(n+\frac{1}{2}\right)t + (n+1)\left(K_2t^3 + \frac{rt}{1-r}\right)\right)dt \\
&\leq K_3(n+1)\omega_\beta(|x-y|)\int_0^{\pi/n}\frac{\omega_\alpha(t)}{\omega_\beta(t)}dt \leq K_4\omega_\beta(|x-y|)\frac{\omega_\alpha(\pi/n)}{\omega_\beta(\pi/n)}.
\end{aligned}$$

Further, we observe that

$$\begin{aligned}
|L(n,r,t,\theta)| &= \left|\left(\frac{1-r}{h}\right)^{n+1}\sin\left(\left(n+\frac{1}{2}\right)t + \left(n+1\right)\theta\right)\right| \\
&\leq \left|\sin\left(\left(n+\frac{1}{2}\right)t + (n+1)\theta\right)\right| \leq 1
\end{aligned}$$

since $|1-r| \leq h$. From this and by (3.3), (3.1) and (3.2) we get

$$I_2 \leq \int_{\pi/n}^{\pi/\sqrt{n}}\frac{|\phi_x(t)-\phi_y(t)|}{t}dt \leq K_5\omega_\beta(|x-y|)\int_{\pi/n}^{\pi/\sqrt{n}}\frac{\omega_\alpha(t)}{t\omega_\beta(t)}dt$$

$$\leq K_5\omega_\beta(|x-y|)\sum_{k=\sqrt{n}}^{n-1}\int_{\pi/(k+1)}^{\pi/k}\frac{\omega_\alpha(t)}{t\omega_\beta(t)}dt \leq K_6\omega_\beta(|x-y|)\sum_{m=\log(\sqrt{n}/\pi)}^{\log(n/\pi)}\frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}$$

$$\leq K_7\omega_\beta(|x-y|)\frac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}.$$

By (3.9) we obtain

$$I_3 \leq \int_{\pi/\sqrt{n}}^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{t}e^{-Knt^2}dt \leq K_8\frac{1}{\sqrt{n}}\int_{\pi/\sqrt{n}}^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{t^2}dt$$

since $e^{-nt^2} \leq \frac{1}{t\sqrt{n}}$ for $t > 0$. Using (3.3) we have

$$I_3 \leq K_{19}\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\int_{\pi/\sqrt{n}}^{\pi}\frac{\omega_\alpha(t)}{t^2\omega_\beta(t)}dt$$

$$\leq K_9\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\sum_{k=1}^{\sqrt{n}-1}\int_{\pi/(k+1)}^{\pi/k}\frac{\omega_\alpha(t)}{t^2\omega_\beta(t)}dt$$

$$\leq K_{10}\frac{1}{\sqrt{n}}\omega_\beta(|x-y|)\sum_{m=0}^{\log(\sqrt{n}/\pi)}2^m\frac{\omega_\alpha(2^{-m})}{\omega_\beta(2^{-m})}.$$

The last sum can be estimated as in the proof of Theorem 4. Thus

$$I_3 \leq \begin{cases} K_{11}\omega_\beta(|x-y|)\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})} & \text{if } \alpha<1 \text{ or } \beta>0, \\ K_{11}\omega_\beta(|x-y|)\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n}) & \text{if } \alpha=1 \text{ and } \beta=0. \end{cases}$$

Applying the same considerations as in the proof of Theorem 4 we can show that if $f \in H^{\omega_\alpha}$, then

$$\|l_n\|_C = \begin{cases} O\Big(\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}\Big) & \text{if } \alpha<1 \text{ or } \beta>0, \\ O\Big(\dfrac{\omega_\alpha(\pi/\sqrt{n})}{\omega_\beta(\pi/\sqrt{n})}(1+\log\sqrt{n})\Big) & \text{if } \alpha=1 \text{ and } \beta=0. \end{cases}$$

Now, collecting our partial results we obtain that (2.3) holds, and this completes the proof. ■

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