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LOCAL MEANS AND WAVELETS IN FUNCTION SPACES

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Abstract. The paper deals with local means and wavelet bases in weighted and unweighted function spaces of type B^s_{pq} and F^s_{pq} on $\mathbb{R}^n$ and on $\mathbb{T}^n$.

1. Introduction. Compactly supported wavelets may serve in the theory of function spaces of type

$$B^s_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^s_{pq}(\mathbb{R}^n) \qquad \text{with } s \in \mathbb{R},\ 0 < p, q \le \infty, \tag{1.1}$$

($p < \infty$ for the F-scale) both as atoms and as kernels in local means,

$$k_{jm}(f) = \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{1.2}$$

having limited smoothness, supports in balls of radius $\sim 2^{-j}$, centred at $2^{-j}m$ and subject to some cancellations. Whereas one has for atoms satisfactory qualitative formulations in terms of natural conditions for smoothness and cancellations (which will be repeated below) the situation for kernels of local means is somewhat different. This comes mainly from the observation that it is totally sufficient for the theory of spaces of type (1.1) in $\mathbb{R}^n$, on smooth domains and even on Lipschitz domains to deal with local means as in (1.2) based on the special kernels

$$k_{jm}(y) = (\Delta^N k)(2^j y - m), \qquad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ y \in \mathbb{R}^n, \tag{1.3}$$

where k is a compactly supported C^∞ function and Δ^N is the Nth power of the Laplacian Δ ensuring the necessary cancellations. But (1.3) is too rigid when one constructs wavelet bases (para-bases, frames) for corresponding spaces in domains with fractal boundaries. Motivated by these observations we study in this paper local means of type (1.2) for the spaces in (1.1) at the same level of smoothness, location and cancellation as for atoms (Theorem 15 which might be considered as our main result). We shift the indicated

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application of these assertions to wavelet (para-)bases for function spaces of the above type in bounded domains with fractal boundaries (one may think about the snowflake curve as boundary) to a later occasion. But one gets almost as a by-product wavelet bases (isomorphisms) for the spaces in (1.1), some weighted generalisations and related periodic counterparts on the n-torus $\mathbb{T}^n$ under natural conditions for smoothness and cancellation. This will be outlined in Section 4 which might be considered as a continuation of [17].

2. Definitions and atoms

2.1. *Definitions.* We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \,|L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Big)^{1/p}$$

with the obvious modification if $p = \infty$. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with} \quad \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \sum_{j=1}^n \alpha_j.$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \qquad \text{(monomials)}.$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\, \varphi(x)\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n, \tag{2.1}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (2.1) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2,$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

Since $\sum_{j=0}^\infty \varphi_j(x) = 1$ for $x \in \mathbb{R}^n$, the φ_j form a dyadic resolution of unity. The entire analytic functions $(\varphi_j \widehat{f})^\vee(x)$ make sense pointwise for any $f \in S'(\mathbb{R}^n)$.

Definition 1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{2.2}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_j \widehat{f})^\vee\,|L_p(\mathbb{R}^n)\|^q\Big)^{1/q} < \infty \tag{2.3}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}. \tag{2.4}$$

Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}\,|(\varphi_j \widehat{f})^\vee(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| < \infty \tag{2.5}$$

(with the usual modification if $q = \infty$).

Remark 2. The theory of these spaces may be found in [14, 16, 18]. We only mention that these spaces are independent of φ (equivalent quasi-norms for admitted φ's). This justifies our omission of the subscript φ in (2.3), (2.5) in the sequel. We assume that the reader is familiar with basic assertions of the theory of the above spaces. But it seems to be in order to recall a few special cases. By the Paley-Littlewood theorem one has

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \qquad 1 < p < \infty,$$

and more generally,

$$F^s_{p,2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n), \qquad s \in \mathbb{N}_0, \quad 1 < p < \infty,$$

for the *classical Sobolev spaces*, usually normed by

$$\|f\,|W^s_p(\mathbb{R}^n)\| = \Big(\sum_{|\alpha|\le s} \|D^\alpha f\,|L_p(\mathbb{R}^n)\|^p\Big)^{1/p}.$$

Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \qquad (\Delta^{l+1}_h f)(x) = \Delta^1_h(\Delta^l_h f)(x),$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Then the *Hölder-Zygmund spaces* $\mathcal{C}^s(\mathbb{R}^n)$, $s > 0$, can be (equivalently) normed by

$$\|f\,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup |h|^{-s}\,|\Delta^m_h f(x)|$$

where $0 < s < m \in \mathbb{N}$. The second supremum is taken over all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$ with $0 < |h| \le 1$. One has

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \qquad s > 0.$$

If $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$ then $B^s_{pq}(\mathbb{R}^n)$ are the *classical Besov spaces* which again can be characterised in terms of the differences $\Delta^m_h f$ with $0 < s < m \in \mathbb{N}$. Otherwise we refer to [16, Chapter 1] and [18, Chapter 1] where one finds the history of these spaces, further special cases and classical characterisations.

2.2. *Atoms.* We give a detailed description of atomic representations of the spaces in (1.1) for two reasons. First we need these assertions later on for the proof of our main results about local means. Secondly, atoms and local means are dual to each other where the natural smoothness assumptions and the cancellation conditions change their roles.

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side-length of Q. Let χ_{jm} be the characteristic function of Q_{jm}.

Definition 3. Let $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq} is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\} \tag{2.6}$$

such that

$$\|\lambda \,|b_{pq}\| = \Big(\sum_{j=0}^{\infty} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{2.7}$$

and f_{pq} is the collection of all sequences λ as in (2.6) such that

$$\|\lambda \,|f_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jnq/p} \,|\lambda_{jm}\chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{2.8}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 4. Note that the factor $2^{jnq/p}$ in (2.8) can be omitted if one relies on the p-normalised characteristic function $\chi^{(p)}_{jm}(x) = 2^{\frac{n(j-1)}{p}} \chi_{jm}(x)$. This is the usual way to say what is meant by f_{pq}. But for our purposes the above version seems to be more appropriate. Of course, $b_{pp} = f_{pp}$.

Next we introduce atoms, which may be discontinuous.

Definition 5. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, and $d \ge 1$. Then L_∞-functions $a_{jm} : \mathbb{R}^n \to \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called (s,p)-*atoms* if

$$\operatorname{supp} a_{jm} \subset dQ_{jm}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n; \tag{2.9}$$

there exist all (classical) derivatives $D^\alpha a_{jm}$ with $|\alpha| \le K$ such that

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{2.10}$$

and

$$\int_{\mathbb{R}^n} x^\beta \, a_{jm}(x) \,\mathrm{d}x = 0, \quad |\beta| < L, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{2.11}$$

Remark 6. No cancellation (2.11) for $a_{0,m}$ is required. Furthermore, if $L = 0$ then (2.11) is empty (no condition). Of course, the above atoms depend on K, L, and d. But this will not be indicated. We put as usual

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+ \quad \text{and} \quad \sigma_{pq} = n\Big(\frac{1}{\min(p,q)} - 1\Big)_+ \tag{2.12}$$

where $b_+ = \max(b, 0)$ if $b \in \mathbb{R}$.

Theorem 7. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, *with*

$$K > s \quad \textit{and} \quad L > \sigma_p - s \tag{2.13}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{2.14}$$

where a_{jm} *are* (s,p)*-atoms according to Definition 5 with* (2.13) *and fixed* $d \geq 1$*, and* $\lambda \in b_{pq}$*. Furthermore,*

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda\,|b_{pq}\| \tag{2.15}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (2.14) *(for fixed* K, L, d*).*

(ii) *Let* $0 < p < \infty$*,* $0 < q \leq \infty$*,* $s \in \mathbb{R}$*. Let* $K \in \mathbb{N}_0$*,* $L \in \mathbb{N}_0$*, with*

$$K > s \quad \text{and} \quad L > \sigma_{pq} - s \tag{2.16}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (2.14) *where* a_{jm} *are* (s,p)*-atoms according to Definition 5 with* (2.16) *and fixed* $d \geq 1$*, and* $\lambda \in f_{pq}$*. Furthermore,*

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda\,|f_{pq}\| \tag{2.17}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (2.14) *(for fixed* K, L, d*).*

Remark 8. These formulations coincide essentially with [18, Section 1.5.1]. There one finds also some technical comments how the convergence in (2.14) must be understood. Atoms of the above type go back essentially to [4, 5]. But more details about the complex history of atoms may be found in [16, Section 1.9].

3. Local means

3.1. *Some definitions.* It is the main aim of this paper to prove estimates for local means which are dual to atomic representations according to Theorem 7 as far as smoothness assumptions and cancellation properties are concerned. First we collect some definitions. Let Q_{jm} be the same cubes in $\mathbb{R}^n$ as at the beginning of Section 2.2.

Definition 9. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Then L_∞-functions $k_{jm} : \mathbb{R}^n \to \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called *kernels* if

$$\operatorname{supp} k_{jm} \subset C Q_{jm}, \qquad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n; \tag{3.1}$$

there exist all (classical) derivatives $D^\alpha k_{jm}$ with $|\alpha| \leq A$ such that

$$|D^\alpha k_{jm}(x)| \leq 2^{jn + j|\alpha|}, \quad |\alpha| \leq A,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{3.2}$$

and

$$\int_{\mathbb{R}^n} x^\beta\, k_{jm}(x)\,\mathrm{d}x = 0, \quad |\beta| < B,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{3.3}$$

Remark 10. No cancellation (3.3) for $k_{0,m}$ is required. Furthermore, if $B = 0$ then (3.3) is empty (no condition). Compared with Definition 5 for atoms we have different normalisations in (2.10) and in (3.2) (also due to the history of atoms). We adapt the sequence spaces introduced in Definition 3 in connection with atoms to the above kernels.

Definition 11. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $\bar{b}^s_{pq}$ is the collection of all sequences λ as in (2.6) such that

$$\|\lambda \,|\bar{b}^s_{pq}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{3.4}$$

and $\bar{f}^s_{pq}$ is the collection of all sequences λ as in (2.6) such that

$$\|\lambda \,|\bar{f}^s_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jsq} |\lambda_{jm}\, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{3.5}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 12. The notation b^s_{pq} and f^s_{pq} (without bar) will be reserved for a slight modification of the above sequence spaces in connection with wavelet representations. One has $\bar{b}^s_{pp} = \bar{f}^s_{pp}$.

Definition 13. Let $f \in B^s_{pq}(\mathbb{R}^n)$ where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let k_{jm} be kernels according to Definition 9 with $A > \sigma_p - s$ where σ_p is given by (2.12) and $B \in \mathbb{N}_0$. Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y, \qquad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{3.6}$$

are local means, considered as a dual pairing for $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$. Furthermore,

$$k(f) = \{ k_{jm}(f) :\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n \}. \tag{3.7}$$

Remark 14. We justify the dual pairing (3.6). According to [14, Theorems 2.11.2, 2.11.3] one has for the dual space of $B^s_{pp}(\mathbb{R}^n)$ that

$$B^s_{pp}(\mathbb{R}^n)' = B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n), \quad s \in \mathbb{R}, \quad 0 < p < \infty, \tag{3.8}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ if } 1 \le p \le \infty \quad \text{and} \quad p' = \infty \text{ if } 0 < p < 1. \tag{3.9}$$

Since $k_{jm} \in C^A(\mathbb{R}^n)$ has a compact support one gets $k_{jm} \in B^{A-\varepsilon}_{uv}(\mathbb{R}^n)$ for any $\varepsilon > 0$ and $0 < u, v \le \infty$. By

$$B^s_{pq}(\mathbb{R}^n) \subset B^{s-\varepsilon}_{pp}(\mathbb{R}^n) \quad \text{and} \quad B^s_{\infty q}(\mathbb{R}^n) \subset B^{s,\mathrm{loc}}_{pq}(\mathbb{R}^n)$$

locally for any $\varepsilon > 0$, $0 < p < \infty$ and $0 < q \le \infty$ one gets by (3.8) and $A > \sigma_p - s$ that (3.6) makes always sense as a dual pairing. This applies also to $f \in F^s_{pq}(\mathbb{R}^n)$ since

$$F^s_{pq}(\mathbb{R}^n) \subset B^s_{p,\max(p,q)}(\mathbb{R}^n).$$

But one can also justify (3.6) for $f \in B^s_{pq}(\mathbb{R}^n)$ or $f \in F^s_{pq}(\mathbb{R}^n)$ by direct arguments, [18, Section 5.1.7].

3.2. *Main assertion.* At the end we ask for counterparts of atomic representations in Theorem 7 in terms of local means according to Definition 13 where k_{jm} are compactly supported wavelets. Since wavelets are special atoms one gets the desired estimates from above by Theorem 7. This brings us in the luxurious position that we only have to care for corresponding estimates from below. This will be done in the following theorem in terms of local means. The sequence spaces $\bar{b}^s_{pq}$ and $\bar{f}^s_{pq}$ have the same meaning as in Definition 11. Let σ_p and σ_{pq} be as in (2.12).

THEOREM 15. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *be kernels as in Definition 9 where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_p - s, \qquad B > s, \tag{3.10}$$

and $C > 0$ *are fixed. Let* $k(f)$ *be as in* (3.6), (3.7). *Then for some* $c > 0$ *and all* $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{b}^s_{pq}\| \le c\,\|f\,|B^s_{pq}(\mathbb{R}^n)\|. \tag{3.11}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *and* $k(f)$ *be the above kernels where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_{pq} - s, \qquad B > s, \tag{3.12}$$

and $C > 0$ *are fixed. Then for some* $c > 0$ *and all* $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c\,\|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{3.13}$$

Proof. Step 1. We prove (i). By Remark 14 the local means $k_{jm}(f)$ make sense. Let

$$f = \sum_{r=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{rl}\, a_{rl}(x), \qquad f \in B^s_{pq}(\mathbb{R}^n), \tag{3.14}$$

be an optimal atomic decomposition of Theorem 7 with a_{rl} as in (2.9)-(2.11) where

$$K = B > s \qquad \text{and} \qquad L = A > \sigma_p - s. \tag{3.15}$$

We call here and in the sequel an atomic decomposition *optimal* if there is some $c > 0$ such that for all $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|\lambda\,|b_{pq}\| \le c\,\|f\,|B^s_{pq}(\mathbb{R}^n)\|.$$

For $j \in \mathbb{N}_0$ we split (3.14) into

$$f = f_j + f^j = \sum_{r=0}^{j} \cdots + \sum_{r=j+1}^{\infty} \cdots \tag{3.16}$$

and get

$$\int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\, \mathrm{d}y + \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\, \mathrm{d}y. \tag{3.17}$$

Let $r \le j$ and let $l \in l^j_r(m)$ where C and d in

$$l^j_r(m) = \{l : \ CQ_{jm} \cap dQ_{rl} \ne \emptyset\} \tag{3.18}$$

have the same meaning as in (3.1), (2.9). Then card $l^j_r(m) \sim 1$ and

$$\begin{aligned} &2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, a_{rl}(y)\, \mathrm{d}y \right| \\ &\le c\, 2^{j(s-\frac{n}{p})} \sum_{|\gamma|=B} \sup_x |D^\gamma a_{rl}(x)| \int_{\mathbb{R}^n} |k_{jm}(y)| \cdot |y - 2^{-j}m|^B\, \mathrm{d}y \\ &\le c\, 2^{(j-r)(s-\frac{n}{p})} 2^{rB} 2^{-jB} 2^{jn} 2^{-jn} \\ &= c\, 2^{(j-r)(s-\frac{n}{p}-B)}. \end{aligned} \tag{3.19}$$

Hence for any $\varepsilon > 0$ (modification if $p = \infty$),

$$2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y) f_j(y)\,\mathrm{d}y \right|^p \le c \sum_{r=0}^{j} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p \, 2^{(j-r)(s-\frac{n}{p}-B+\varepsilon)p}. \tag{3.20}$$

For fixed r, j with $r \le j$ and $l \in \mathbb{Z}^n$ one has

$$\mathrm{card}\, \{m \in \mathbb{Z}^n : \ l \in l_r^j(m)\} \sim 2^{(j-r)n}. \tag{3.21}$$

Then it follows from (3.20), (3.21) that

$$\begin{aligned} 2^{j(s-\frac{n}{p})p} \sum_{m \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f_j(y)\,\mathrm{d}y \right|^p &\le c \sum_{r=0}^{j} 2^{(j-r)n}\, 2^{p(j-r)(s-\frac{n}{p}-B+\varepsilon)} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p \\ &\le c \sum_{r=0}^{j} 2^{p(j-r)(s-B+\varepsilon)} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p. \end{aligned} \tag{3.22}$$

Let $r > j$. Then one gets by (3.15) for $l \in l_r^j(m)$ that

$$\begin{aligned} &2^{j(s-\frac{n}{p})} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, a_{rl}(y)\,\mathrm{d}y \right| \\ &\le 2^{j(s-\frac{n}{p})} \sum_{|\gamma|=A} \sup_x |D^\gamma k_{jm}(x)| \int_{\mathbb{R}^n} |a_{rl}(y)| \cdot |y - 2^{-r}l|^A \,\mathrm{d}y \\ &\le c\, 2^{(j-r)(s-\frac{n}{p})}\, 2^{jn+Aj}\, 2^{-rA-rn} \le c\, 2^{(j-r)(s-\frac{n}{p}+n+A)}. \end{aligned} \tag{3.23}$$

By (3.16), (3.17) it follows that for any $\varepsilon > 0$,

$$2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\frac{n}{p}+n+A-\varepsilon)p} \Big(\sum_{l \in l_r^j(m)} |\lambda_{rl}| \Big)^p. \tag{3.24}$$

If $p \le 1$ then one gets

$$2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\sigma_p+A-\varepsilon)p} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p. \tag{3.25}$$

If $1 < p < \infty$ then it follows from (3.24), Hölder's inequality and card $l_r^j(m) \sim 2^{n(r-j)}$ that

$$\begin{aligned} 2^{j(s-\frac{n}{p})p} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y \right|^p &\le c \sum_{r>j} 2^{(j-r)(s-\frac{n}{p}+n+A-\varepsilon)p} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p 2^{(r-j)(p-1)n} \\ &\le c \sum_{r>j} 2^{(j-r)(s+A-\varepsilon)p} \sum_{l \in l_r^j(m)} |\lambda_{rl}|^p. \end{aligned}$$

Summation over $m \in \mathbb{Z}^n$ results in

$$2^{j(s-\frac{n}{p})p} \sum_{m \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y \right|^p \le c \sum_{r>j} 2^{(j-r)(s-\sigma_p+A-\varepsilon)p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p. \tag{3.26}$$

By (3.22), (3.26) and (3.10) it follows that for some $\varkappa > 0$,

$$2^{j(s-\frac{n}{p})p} \sum_{m \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\,\mathrm{d}y \right|^p \le c \sum_{r=0}^{\infty} 2^{-|j-r|\varkappa p} \sum_{l \in \mathbb{Z}^n} |\lambda_{rl}|^p.$$

Using (3.7), (3.4) and (2.15) based on (2.7) one gets by standard arguments that

$$\|k(f)\,|\bar{b}^s_{pq}\| \le c\,\|\lambda\,|b_{pq}\| \sim \|f\,|B^s_{pq}(\mathbb{R}^n)\|.$$

This proves (3.11).

Step 2. The proof of (ii) will be based on the vector-valued maximal inequality

$$\Big\|\Big(\sum_{k=0}^{\infty}(M|g_k|^w)\,(\cdot)^{q/w}\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| \le c\,\Big\|\Big(\sum_{k=0}^{\infty}|g_k(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| \tag{3.27}$$

due to [3] where

$$0<p<\infty, \quad 0<q\le\infty, \quad 0<w<\min(p,q).$$

Here M is the Hardy-Littlewood maximal function,

$$(Mg)(x) = \sup|Q|^{-1}\int_Q |g(y)|\,\mathrm{d}y, \qquad x\in\mathbb{R}^n,$$

where the supremum is taken over all cubes Q centred at x. A short proof may also be found in [13, pp. 303-305]. By Remark 14 the local means $k_{jm}(f)$ make sense. Let again

$$f=\sum_{r=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\lambda_{rl}\,a_{rl}(x), \qquad f\in F^s_{pq}(\mathbb{R}^n),$$

be an optimal atomic decomposition of Theorem 7 with a_{rl} as in (2.9)-(2.11) where now

$$K=B>s \quad \text{and} \quad L=A>\sigma_{pq}-s.$$

We rely again on the splitting (3.16)-(3.18). Let $r\le j$. We assume $q<\infty$ (if $q=\infty$ one has to modify what follows appropriately). Then the counterpart of (3.19), (3.20) is given by

$$2^{jsq}\chi_{jm}(x)\,\Big|\int_{\mathbb{R}^n}k_{jm}(y)\,f_j(y)\,\mathrm{d}y\Big|^q \le c\sum_{r=0}^{j}\sum_{l\in l^j_r(m)}|\lambda_{rl}|^q\,2^{\frac{rn}{p}q}\chi_{jm}(x)\,2^{(j-r)(s-B+\varepsilon)q}$$

where $x\in\mathbb{R}^n$. For fixed j,r, and l the summation $\sum\chi_{jm}(x)$ over those $m\in\mathbb{Z}^n$ with $l\in l^j_r(m)$ is comparable with $\chi_{rl}(x)$ and can be estimated from above by its maximal function. Hence, one gets for any $w>0$,

$$\begin{aligned}&2^{jsq}\sum_{m\in\mathbb{Z}^n}\chi_{jm}(x)\,\Big|\int_{\mathbb{R}^n}k_{jm}(y)\,f_j(y)\,\mathrm{d}y\Big|^q\\ &\le c\sum_{r=0}^{j}2^{(j-r)(s-B+\varepsilon)q}\sum_{l\in\mathbb{Z}^n}M(|\lambda_{rl}|^w\,2^{\frac{rn}{p}w}\,\chi_{rl}(\cdot))(x)^{q/w}.\end{aligned} \tag{3.28}$$

This is the counterpart of (3.22). Let now $r>j$. Then it follows as in (3.23), (3.24) that

$$2^{js}\,\chi_{jm}(x)\,\Big|\int_{\mathbb{R}^n}k_{jm}(y)\,f^j(y)\,\mathrm{d}y\Big| \le c\,\chi_{jm}(x)\sum_{r>j}2^{(j-r)(s+A+n)}\sum_{l\in l^j_r(m)}2^{r\frac{n}{p}}|\lambda_{rl}|.$$

For $x \in \mathbb{R}^n$ with $\chi_{jm}(x) = 1$ and $0 < w < 1$ the last factor can be estimated by

$$\Big(\sum_{l \in l^j_r(m)} 2^{r\frac{n}{p}} |\lambda_{rl}| \Big)^w \le \sum_{l \in l^j_r(m)} 2^{r\frac{nw}{p}} |\lambda_{rl}|^w \le c 2^{rn} 2^{-jn} 2^{jn} \int_{\mathbb{R}^n} \sum_{l \in l^j_r(m)} 2^{r\frac{nw}{p}} |\lambda_{rl}|^w \chi_{rl}(y)\,\mathrm{d}y$$
$$\le c\, 2^{(r-j)n} M\Big(\sum_{l \in l^j_r(m)} 2^{r\frac{nw}{p}} |\lambda_{rl}|^w \,\chi_{rl}(\cdot)\Big)(x).$$

Assuming again $q < \infty$ one gets

$$2^{jsq}\chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y \right|^q$$
$$\le c\,\chi_{jm}(x) \sum_{r>j} 2^{(j-r)(s+A+n-\frac{n}{w}-\varepsilon)q}\, M\Big(\sum_{l \in l^j_r(m)} 2^{r\frac{nw}{p}} \,|\lambda_{rl}|^w\, \chi_{rl}(\cdot)\Big)(x)^{q/w}. \tag{3.29}$$

Since

$$A + s > \sigma_{pq} = \frac{n}{\min(1,p,q)} - n$$

one may choose w with $w < \min(1,p,q)$ and $\varepsilon > 0$ in (3.29) such that

$$A + s + n - \frac{n}{w} - \varepsilon > 0.$$

In what follows we decompose $2^{-r}\mathbb{Z}^n$ with $r > j$ into clusters (controlled overlapping) around $2^{-j}m$ with $m \in \mathbb{Z}^n$. Let $l \in l^j_r(m)$ as in (3.18). With $r = j + t$ one gets for some $\varkappa > 0$ that

$$2^{jsq} \sum_{m \in \mathbb{Z}^n} \chi_{jm}(x) \left| \int_{\mathbb{R}^n} k_{jm}(y)\, f^j(y)\,\mathrm{d}y \right|^q$$
$$\le c \sum_{t=1}^{\infty} 2^{-t\varkappa q} \sum_{m \in \mathbb{Z}^n} M\Big(\sum_{l \in l^j_{j+t}(m)} 2^{(j+t)\frac{nw}{p}}\, |\lambda_{j+t,l}|^w\, \chi_{j+t,l}(\cdot)\Big)(x)^{q/w}. \tag{3.30}$$

Summation over j in (3.28) with $s - B + \varepsilon < 0$ and in (3.30) gives by (3.5),

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c \Big\| \Big(\sum_{r=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \Big[M\Big(|\lambda_{rl}|^w\, 2^{\frac{rn}{p}w}\, \chi_{rl}(\cdot)\Big)\Big]^{q/w} \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\|$$
$$+\, c \sum_{t=1}^{\infty} 2^{-t\varkappa'} \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \Big[M\Big(\sum_{l \in l^j_{j+t}(m)} \cdots \Big)\Big]^{q/w} \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| \tag{3.31}$$

for some $0 < \varkappa' < \varkappa$. By construction, the lattice $2^{-j}\mathbb{Z}^n$ refined by $l \in l^j_{j+t}(m)$ is related to $2^{-j-t}\mathbb{Z}^n$. Since $w < \min(1,p,q)$ one can apply (3.27). Then one gets by (2.8) and (2.17) that

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c\,\|\lambda\,|f_{pq}\| \sim \|f\,|F^s_{pq}(\mathbb{R}^n)\|.$$

This proves (3.13). ■

Remark 16. As far as technicalities are concerned we refer in this context to [9]. In particular we took over from this paper the idea to decompose the lattice $2^{-j-t}\mathbb{Z}^n$ in connection with (3.30), (3.31) into clusters around $2^{-j}\mathbb{Z}^n$. These clusters disappear after the vector-valued maximal inequality is applied. According to Definition 9 the kernels

k_{jm} have compact supports. This was of some use. But there is little doubt that one can replace the compactness assumption (3.1) by a sufficiently strong decay. We refer again to [9].

4. Wavelet bases

4.1. *Wavelets.* As said in the Introduction the interplay between the sharp and natural Theorems 7 and 15 can be taken as a starting point for a comprehensive theory for wavelet (para-)bases in function spaces on (bounded) fractal domains in $\mathbb{R}^n$. But this task will be shifted to a later occasion. A first step in this direction has been done in [20] where we formulated Theorem 15 without proof referring to a later occasion (just this one now). Using a weaker version of Theorem 15 (seen from the point of view of applications to wavelets) we dealt in [17] and [18, Section 3.1] with wavelet bases (isomorphisms) in the spaces

$$B^s_{pq}(\mathbb{R}^n), \quad F^s_{pq}(\mathbb{R}^n) \qquad \text{where } s \in \mathbb{R},\ 0 < p, q \le \infty, \tag{4.1}$$

(with $p < \infty$ for the F-spaces). This has been extended in [19] and [18, Section 4.2] to corresponding spaces in bounded Lipschitz domains. Our aim in the present paper is more modest. We wish simply to substitute the predecessor of Theorem 15 by its sharper version obtained now and describe the consequences for wavelet bases (isomorphisms) in the spaces in (4.1). However we extend these assertions to some weighted spaces and some periodic spaces. But first we recall the needed basic notation of wavelet theory.

We suppose that the reader is familiar with wavelets in $\mathbb{R}^n$ of Daubechies type and the related muli-resolution analysis. The standard references are [1, 10, 11, 21]. A short summary of what is needed may also be found in [18, Section 1.7]. As usual, $C^u(\mathbb{R})$ collects all (complex-valued) continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order $u \in \mathbb{N}$ (inclusively). Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \qquad u \in \mathbb{N}, \tag{4.2}$$

be real compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x)\, x^v \,\mathrm{d}x = 0 \quad \text{for all } v \in \mathbb{N}_0 \text{ with } v < u. \tag{4.3}$$

Recall that ψ_F is called the *scaling function* (father wavelet) and ψ_M is the associated (mother) *wavelet.* We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor procedure. Let

$$G = (G_1, \ldots, G_n) \in G^0 = \{F, M\}^n \tag{4.4}$$

if G_r is either F or M. Let

$$G = (G_1, \ldots, G_m) \in G^j = \{F, M\}^{n*}, \qquad j \in \mathbb{N}, \tag{4.5}$$

if G_r is either F or M where * indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^n \psi_{G_r}(2^j x_r - m_r), \qquad G \in G^j, \quad m \in \mathbb{Z}^n, \tag{4.6}$$

where $j \in \mathbb{N}_0$. We always assume tha ψ_F and ψ_M in (4.2) have L_2-norm 1. Then

$$\{\Psi^j_{G,m} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n\} \tag{4.7}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ (for any $u \in \mathbb{N}$) and

$$f = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m} \tag{4.8}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \, \Psi^j_{G,m}(x) \, \mathrm{d}x = 2^{jn/2} \, (f, \Psi^j_{G,m}) \tag{4.9}$$

is the corresponding expansion, adapted to our later needs, where $2^{-jn/2} \, \Psi^j_{G,m}$ are uniformly bounded functions.

4.2. *Spaces on* $\mathbb{R}^n$*.* One may ask whether the orthonormal basis (4.7) in $L_2(\mathbb{R}^n)$ remains to be an (unconditional) basis in other spaces on $\mathbb{R}^n$. First candidates are $L_p(\mathbb{R}^n)$ with $1 < p < \infty$ but also related (fractional) Sobolev spaces and classical Besov spaces. Something may be found in the above-mentioned books [1, 10, 11, 21]. One may also consult [18, Remarks 1.63, 1.65, pp. 32,34] for more details and further references. An extension of this theory to all spaces in (4.1) has been given in [9, 17] and [18, Section 3.1.3, Theorem 3.5, p. 154]. Basically one identifies the wavelets in (4.6) both with atoms and with kernels of local means and applies (now) Theorems 7 and 15 for related estimates from below and from above. In [17, 18] we used Theorem 7 but we relied on the heavy machinery of rather general equivalent quasi-norms in the spaces (4.1) instead of the tailored (for our purposes) Theorem 15. Then one gets somewhat unnatural conditions for $u \in \mathbb{N}$ in (4.2). Although this outcome is not elegant it does not matter very much as long as one deals with spaces on $\mathbb{R}^n$. But the situation is different if one wishes to extend this theory to spaces of type (4.1) on (bounded) domains in $\mathbb{R}^n$ with fractal boundary. However this must be shifted to a later occasion, where [20] might be considered as a first step in this direction. Here we restrict ourselves to (unweighted and weighted) spaces on $\mathbb{R}^n$ and periodic spaces.

First we adapt the sequence spaces in Definition 11 to wavelets having the additional parameter $G \in G^j$ as in (4.4), (4.5).

Definition 17. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then b^s_{pq} is the collection of all sequences

$$\lambda = \big\{ \lambda^{j,G}_m \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ m \in \mathbb{Z}^n \big\} \tag{4.10}$$

such that

$$\|\lambda \, | b^s_{pq}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{4.11}$$

and f^s_{pq} is the collection of all sequences (4.10) such that

$$\|\lambda \, | f^s_{pq}\| = \Big\| \Big(\sum_{j,G,m} 2^{jsq} \, |\lambda^{j,G}_m \, \chi_{jm}(\cdot)|^q \Big)^{1/q} | L_p(\mathbb{R}^n) \Big\| < \infty$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

REMARK 18. We refer to the comments in Remark 12. As explained at the beginning of [18, Section 3.1.3] we may abbreviate the right-hand side of (4.8) by

$$\sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_{G,m}^j$$

since the conditions on the sequences λ always ensure that the corresponding series converge unconditionally (which means that any rearrangement converges to the same limit) at least in $S'(\mathbb{R}^n)$. Local convergence in $B^\sigma_{pq}(\mathbb{R}^n)$ means convergence in $B^\sigma_{pq}(K)$ for any ball K in $\mathbb{R}^n$. Similarly for $F^\sigma_{pq}(\mathbb{R}^n)$. Recall that σ_p and σ_{pq} are given by (2.12).

THEOREM 19. (i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\Psi^j_{G,m}$ be the wavelets in* (4.6) *based on* (4.2) *with*

$$u > \max(s, \sigma_p - s). \tag{4.12}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_{G,m}^j, \qquad \lambda \in b^s_{pq}, \tag{4.13}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any space $B^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. The representation (4.13) *is unique,*

$$\lambda_m^{j,G} = 2^{jn/2}\, (f, \Psi^j_{G,m}), \tag{4.14}$$

and

$$I : \quad f \mapsto \{2^{jn/2}\, (f, \Psi^j_{G,m})\} \tag{4.15}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ onto b^s_{pq}. If, in addition, $p < \infty$, $q < \infty$, then $\{\Psi^j_{G,m}\}$ is an unconditional basis in $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and*

$$u > \max(s, \sigma_{pq} - s). \tag{4.16}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda_m^{j,G}\, 2^{-jn/2}\, \Psi_{G,m}^j, \qquad \lambda \in f^s_{pq}, \tag{4.17}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any space $F^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. The representation (4.17) *is unique with* (4.14) *and I in* (4.15) *is an isomorphic map of $F^s_{pq}(\mathbb{R}^n)$ onto f^s_{pq}. If, in addition, $q < \infty$, then $\{\Psi^j_{G,m}\}$ is an unconditional basis in $F^s_{pq}(\mathbb{R}^n)$.*

Proof. Step 1. This theorem coincides with corresponding assertions in [17] and [18, Section 3.1.3, Theorem 3.5, pp. 153-156] under the more restrictive smoothness and cancellation assumptions (4.2), (4.3) with

$$u > \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right) \quad \text{and} \quad u > \max\left(s, \frac{2n}{\min(p,q)} + \frac{n}{2} - s\right) \tag{4.18}$$

for $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, respectively. But all technicalities such as the unconditional convergence of (4.13), (4.17), the uniqueness (4.14), the isomorphic maps (4.15), the use of duality, (3.8), (3.9), the unconditional bases, can be taken over verbatim. Hence it remains to make clear that (4.12) and (4.16) are natural and sufficient.

Step 2. Let f be given by (4.13). Then

$$a^G_{jm} = 2^{-j(s-\frac{n}{p})}\, 2^{-jn/2}\, \Psi^j_{G,m}, \qquad G \in G^j,$$

are atoms in $B^s_{pq}(\mathbb{R}^n)$ according to Definition 5 and (2.13) with $K = L = u$ (up to unimportant constants). Having the different normalisations for b^s_{pq} in (4.11) and for b_{pq} in (2.7) in mind one gets by Theorem 7(i) that $f \in B^s_{pq}(\mathbb{R}^n)$ and

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\| \le c\, \|\lambda \,|b^s_{pq}\| \tag{4.19}$$

where the extra summation over G does not influence this argument. Conversely, if $f \in B^s_{pq}(\mathbb{R}^n)$ then

$$k^G_{jm} = 2^{jn/2}\, \Psi^j_{G,m}, \qquad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n,$$

are kernels of corresponding local means in $B^s_{pq}(\mathbb{R}^n)$ according to Definitions 9, 13, (4.6) and (3.10) with $A = B = u$ (neglecting unimportant constants). The modification

$$k(f) = \big\{k^G_{jm}(f) :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\}$$

of (3.7) is immaterial for the application of Theorem 15 resulting in

$$\|k(f) \,|b^s_{pq}\| \le c\, \|f \,|B^s_{pq}(\mathbb{R}^n)\|. \tag{4.20}$$

Similarly for $F^s_{pq}(\mathbb{R}^n)$. Armed with (4.19), (4.20) and the F-counterparts the rest is the same as in [17, 18]. ■

4.3. *Weighted spaces.* Based on [8] we extended in [18, Section 6.2] Theorem 19 with u as in (4.18) to some weighted spaces. There is no difficulty to apply the corresponding arguments to the above improved version with u as in (4.12), (4.16). The main reason for giving an explicit formulation comes from the somewhat surprising (for the author) application to get wavelet bases for periodic spaces subject of the next subsection.

DEFINITION 20. Let w be a real C^∞ function in $\mathbb{R}^n$ such that for all $\gamma \in \mathbb{N}^n_0$ and suitable positive numbers c_γ,

$$|D^\gamma w(x)| \le c_\gamma\, w(x) \qquad \text{for all } x \in \mathbb{R}^n,$$

and

$$0 < w(x) \le c\, w(y)\, (1 + |x-y|^2)^{\alpha/2} \quad \text{for all } x \in \mathbb{R}^n,\ y \in \mathbb{R}^n,$$

and some constants $c > 0$ and $\alpha \ge 0$.

(i) Let $0 < p \le \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Then $B^s_{pq}(\mathbb{R}^n, w)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{pq}(\mathbb{R}^n, w)\| = \|wf \,|B^s_{pq}(\mathbb{R}^n)\| < \infty.$$

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Then $F^s_{pq}(\mathbb{R}^n, w)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^s_{pq}(\mathbb{R}^n, w)\| = \|wf \,|F^s_{pq}(\mathbb{R}^n)\| < \infty.$$

REMARK 21. Usually these spaces are introduced in the same way as in Definition 1 with $L_p(\mathbb{R}^n, w)$ in place of $L_p(\mathbb{R}^n)$. Then the above version is one of the basic observations of the theory of these spaces. These weighted spaces (avoiding local singularities) played some role in the theory of function spaces and had been studied in detail in [2, Chapter 4] (mainly based on [6, 7]) and [18, Chapter 6] where one finds also references to the literature. Here we are only interested in an extension of Theorem 19 to these spaces. For this purpose one has first to modify the sequence spaces in Definition 17.

DEFINITION 22. Let w be a weight according to Definition 20 and let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $b^s_{pq}(w)$ is the collection of all sequences (4.10) such that

$$\|\lambda\,|b^s_{pq}(w)\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \sum_{G\in G^j} \Big(\sum_{m\in\mathbb{Z}^n} w(2^{-j}m)^p\,|\lambda^{j,G}_m|^p\Big)^{q/p}\Big)^{1/q} < \infty$$

and $f^s_{pq}(w)$ is the collection of all sequences (4.10) such that

$$\|\lambda\,|f^s_{pq}(w)\| = \Big\|\Big(\sum_{j,G,m} 2^{jsq}\,w(2^{-j}m)^q\,|\lambda^{j,G}_m\,\chi_{jm}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| < \infty$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

REMARK 23. This extends Definition 17 to the weighted case and coincides with [18, Definition 6.11, p. 269]. We extend now Theorem 19 to the weighted case again in the sense of Remark 18, now with respect to the spaces $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ as in Definition 20.

THEOREM 24. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and let* $\Psi^j_{G,m}$ *be the wavelets in* (4.6) *based on* (4.2) *with*

$$u > \max(s, \sigma_p - s). \tag{4.21}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m\,2^{-jn/2}\,\Psi^j_{G,m}, \qquad \lambda \in b^s_{pq}(w), \tag{4.22}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and in any space* $B^\sigma_{pq}(\mathbb{R}^n, \widetilde{w})$ *with* $\sigma < s$ *and* $\widetilde{w}(x)w^{-1}(x) \to 0$ *if* $|x| \to \infty$. *Furthermore, the representation* (4.22) *is unique,*

$$\lambda^{j,G}_m = 2^{jn/2}\,(f, \Psi^j_{G,m}) \tag{4.23}$$

and

$$I: \quad f \mapsto \{2^{jn/2}\,(f, \Psi^j_{G,m})\} \tag{4.24}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n, w)$ *onto* $b^s_{pq}(w)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\{\Psi^j_{G,m}\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{R}^n, w)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and*

$$u > \max(s, \sigma_{pq} - s). \tag{4.25}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda^{j,G}_m\,2^{-jn/2}\,\Psi^j_{G,m}, \qquad \lambda \in f^s_{pq}(w), \tag{4.26}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and in any space* $F^\sigma_{pq}(\mathbb{R}^n, \widetilde{w})$ *with* $\sigma < s$ *and* $\widetilde{w}(x)w^{-1}(x) \to 0$ *if* $|x| \to \infty$. *Furthermore, the representation* (4.26) *is unique with* (4.23) *and I in (4.24) is an isomorphic map of $F^s_{pq}(\mathbb{R}^n,w)$ onto $f^s_{pq}(w)$. If, in addition, $q<\infty$, then $\{\Psi^j_{G,m}\}$ is an unconditional basis in $F^s_{pq}(\mathbb{R}^n,w)$.*

Proof. This theorem with u as in (4.18) coincides with [18, Theorem 6.15, pp. 270/271]. The corresponding proof reduces the weighted case to the unweighted one using localisation and interpolation. But this works also for u as in (4.21), (4.25) relying now on Theorem 19. ∎

4.4. *Periodic spaces.* The main reason for inserting Theorem 24 is the somewhat surprising application to wavelet bases in periodic spaces. However we will be very brief shifting more comprehensive considerations to a later occasion. As far as spaces on the n-torus $\mathbb{T}^n$ are concerned we refer to [12, Chapter 3]. We rely here on the close connection of these spaces to periodic spaces on $\mathbb{R}^n$ as considered in [14, Chapter 9] with a reference to [15]. There one finds further details and explanations of what follows. We specify the weight w in Definition 20 now by

$$w_\alpha(x)=(1+|x|^2)^{\alpha/2},\qquad \alpha\in\mathbb{R}.$$

Definition 25. Let $s\in\mathbb{R}$, $0<p\le\infty$ ($p<\infty$ for the F-spaces), $0<q\le\infty$ and $\alpha<-\frac{n}{p}$. Let either $A=B$ or $A=F$. Then

$$A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)=\left\{f\in A^s_{pq}(\mathbb{R}^n,w_\alpha):\ f(\cdot)=f(\cdot+k),\ k\in\mathbb{Z}^n\right\}. \tag{4.27}$$

Remark 26. We justify this definition. First we remark that $f\in S'(\mathbb{R}^n)$ is periodic,

$$f(\cdot)=f(\cdot+k)\qquad\text{for all } k\in\mathbb{Z}^n,$$

if, and only if, it can be represented as

$$f=\sum_{m\in\mathbb{Z}^n} a_m\, e^{i2\pi mx},\qquad x\in\mathbb{R}^n, \tag{4.28}$$

with $\{a_m\}\subset\mathbb{C}$ of at most polynomial growth, hence

$$|a_m|\le c\,(1+|m|)^{\varkappa}\quad\text{for some } c>0,\ \varkappa>0 \text{ and all } m\in\mathbb{Z}^n. \tag{4.29}$$

One may consult the above references. With φ_j as in Definition 1 one checks easily that

$$(\varphi_j\widehat{f})^\vee(x)=\sum_{m\in\mathbb{Z}^n} a_m\,\varphi_j(2\pi m)\,e^{i2\pi mx},\qquad x\in\mathbb{R}^n.$$

These are trigonometrical polynomials. Then one gets by Definition 20 for these periodic distributions that

$$\|f\,|B^s_{pq}(\mathbb{R}^n,w_\alpha)\|=\Big(\sum_{j=0}^\infty 2^{jsq}\Big\|w_\alpha(x)\sum_{m\in\mathbb{Z}^n} a_m\varphi_j(2\pi m)\,e^{i2\pi mx}\,|L_p(\mathbb{R}^n)\Big\|^q\Big)^{1/q} \tag{4.30}$$

and

$$\|f\,|F^s_{pq}(\mathbb{R}^n,w_\alpha)\|=\Big\|w_\alpha(x)\Big(\sum_{j=0}^\infty 2^{jsq}\Big|\sum_{m\in\mathbb{Z}^n} a_m\,\varphi_j(2\pi m)\,e^{i2\pi mx}\Big|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\|. \tag{4.31}$$

Now it is quite clear that (4.27) makes sense and that these periodic spaces are independent of α with $\alpha<-n/p$ (equivalent quasi-norms).

Let

$$\mathbb{T}^n=\{x\in\mathbb{R}^n:\ 0\le x_j\le 1\}$$

be the n-torus, where opposite points are identified in the usual way. Then the related periodic distributions $f \in D'(\mathbb{T}^n)$ can be represented by

$$f = \sum_{m \in \mathbb{Z}^n} a_m \, e^{i2\pi mx}, \qquad x \in \mathbb{T}^n, \tag{4.32}$$

where $\{a_m\} \subset \mathbb{C}$ satisfies (4.29). Next we introduce the periodic counterpart on $\mathbb{T}^n$ of Definition 1.

DEFINITION 27. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the same dyadic resolution of unity as in Definition 1.

(i) Let p, q, s as in (2.2). Then $B^s_{pq}(\mathbb{T}^n)$ is the collection of all $f \in D'(\mathbb{T}^n)$, represented by (4.32), such that

$$\|f \,|B^s_{pq}(\mathbb{T}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big\| \sum_{m \in \mathbb{Z}^n} a_m \, \varphi_j(2\pi m) \, e^{i2\pi mx} \,|L_p(\mathbb{T}^n) \Big\|^q \Big)^{1/q} < \infty. \tag{4.33}$$

(ii) Let p, q, s as in (2.4). Then $F^s_{pq}(\mathbb{T}^n)$ is the collection of all $f \in D'(\mathbb{T}^n)$, represented by (4.32), such that

$$\|f \,|F^s_{pq}(\mathbb{T}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \sum_{m \in \mathbb{Z}^n} a_m \, \varphi_j(2\pi m) \, e^{i2\pi mx} \Big|^q \Big)^{1/q} |L_p(\mathbb{T}^n) \Big\| < \infty. \tag{4.34}$$

REMARK 28. We refer again to [12, Chapter 3] for details, explanations, properties and special cases. In particular, these spaces are independent of φ.

PROPOSITION 29. *Let $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$. Let $A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\mathbb{T}^n)$ be the spaces introduced in the Definitions 25, 27, where either $A = B$ or $A = F$. Let $f \in D'(\mathbb{T}^n)$ be represented by* (4.32) *with* (4.29). *Then the extension operator* $\mathrm{ext}^{\mathrm{per}}$,

$$\mathrm{ext}^{\mathrm{per}} f = \sum_{m \in \mathbb{Z}^n} a_m \, e^{i2\pi mx}, \qquad x \in \mathbb{R}^n,$$

(*appropriately interpreted*) *maps $A^s_{pq}(\mathbb{T}^n)$ isomorphically onto $A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$,*

$$\mathrm{ext}^{\mathrm{per}} A^s_{pq}(\mathbb{T}^n) = A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n).$$

Proof. This follows immediately from the above preparations, in particular by (4.32) compared with (4.28) and (4.30), (4.31) compared with (4.33), (4.34) having in mind that $\alpha < -n/p$, hence $w_\alpha \in L_p(\mathbb{R}^n)$. ∎

The periodic spaces $A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ are closed subspaces of the weighted spaces $A^s_{pq}(\mathbb{R}^n, w_\alpha)$ with $\alpha < -n/p$ for which we have the wavelet representations according to Theorem 24. The question arises whether these assertions can be adapted to the periodic spaces and transferred by Proposition 29 to corresponding spaces on $\mathbb{T}^n$. The idea to periodise wavelet bases on $\mathbb{R}$ according to (4.6) (with $n = 1$) to get (periodic) wavelet bases on the 1-torus $\mathbb{T}$, preferably for $L_2(\mathbb{T})$, appears several times in literature. We refer to [1, pp. 304/305], [10, Section 7.5.1, pp. 282/283] and [21, Section 2.5, pp. 38-42]. We extend these constructions to the periodic spaces on $\mathbb{R}^n$ and the spaces on $\mathbb{T}^n$. We rely on the notation introduced in (4.2)-(4.6) now with 2^{j+L} in place of 2^j for some $L \in \mathbb{N}$ such that

$$\mathrm{supp}\ \Psi^0_{G,0} \subset \{x \in \mathbb{R}^n : \ |x| < 1/2\}, \qquad G \in G^0 = \{F, M\}^n, \tag{4.35}$$

for the starting terms and, hence, for all $\Psi^j_{G,0}$ with $j \in \mathbb{N}_0$. Let

$$\mathbb{P}^n_j = \{k \in \mathbb{Z}^n : \ 0 \le k_r < 2^{j+L}\}, \qquad j \in \mathbb{N}_0,$$

be the $2^{(j+L)n}$ lattice points in $2^{j+L}\mathbb{T}^n$. We put

$$\Psi^j_{G,k,\mathrm{per}}(x) = \sum_{l \in \mathbb{Z}^n} \Psi^j_{G,k}(x-l) = \sum_{l \in \mathbb{Z}^n} \Psi^j_{G,k+2^{j+L}l}(x), \qquad x \in \mathbb{R}^n, \tag{4.36}$$

where $j \in \mathbb{N}_0$ and $k \in \mathbb{P}^n_j$. In other words, one extends the distinguished wavelets with off-points in $\mathbb{T}^n$ periodically to $\mathbb{R}^n$ and with $u \in \mathbb{N}$ as in (4.2)-(4.6) one gets by Definition 25,

$$\Psi^j_{G,k,\mathrm{per}} \in C^u(\mathbb{R}^n) \cap A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n), \qquad s < u,$$

for all admitted p, q. Let

$$\Psi^{j,\mathrm{per}}_{G,k}(x) = \Psi^j_{G,k,\mathrm{per}}(x), \qquad x \in \mathbb{T}^n, \quad k \in \mathbb{P}^n_j, \tag{4.37}$$

be the restriction or $\Psi^j_{G,k,\mathrm{per}}$ to the n-torus $\mathbb{T}^n$. Then one gets the periodic wavelet bases on $\mathbb{R}^n$ and $\mathbb{T}^n$ we are looking for with

$$\Psi^{j,\mathrm{per}}_{G,k} \in C^u(\mathbb{T}^n) \cap A^s_{pq}(\mathbb{T}^n), \qquad s < u, \quad k \in \mathbb{P}^n_j.$$

Now one can construct wavelet bases both for the spaces $A^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ and the spaces on $\mathbb{T}^n$. We restrict ourselves to the spaces on $\mathbb{T}^n$. Furthermore we will be very brief shifting details to later occasions.

PROPOSITION 30. *Let $u \in \mathbb{N}$. Then*

$$\{\Psi^{j,\mathrm{per}}_{G,k} : \ j \in \mathbb{N}_0, \ G \in G^j, \ k \in \mathbb{P}^n_j\} \tag{4.38}$$

is an orthonormal basis in $L_2(\mathbb{T}^n)$.

Proof. One obtains from the corresponding assertion for $\Psi^j_{G,k}$ and (4.35) that the functions in (4.38) are orthonormal. The completeness follows from Proposition 29 and a related assertion for $L_2(\mathbb{R}^n, w_\alpha)$. ■

Next we introduce the periodic counterpart of the sequence spaces in Definition 17. Let χ_{jk} below be the characteristic function of a cube with the left corner $2^{-j-L}k$ and of side-length 2^{-j-L}, where $j \in \mathbb{N}_0$ and $k \in \mathbb{P}^n_j$ which is a subcube of $\mathbb{T}^n$.

DEFINITION 31. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $b^{s,\mathrm{per}}_{pq}$ is the collection of all sequences

$$\mu = \{\mu^{j,G}_k \in \mathbb{C} : \ j \in \mathbb{N}_0, \ G \in G^j, \ k \in \mathbb{P}^n_j\} \tag{4.39}$$

such that

$$\|\mu \,| b^{s,\mathrm{per}}_{pq}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \Big(\sum_{k \in \mathbb{P}^n_j} |\mu^{j,G}_k|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{4.40}$$

and $f^{s,\mathrm{per}}_{pq}$ is the collection of all sequences (4.39) such that

$$\|\mu \,| f^{s,\mathrm{per}}_{pq}\| = \Big\| \Big(\sum_{j,G,k} 2^{jsq} |\mu^{j,G}_k \, \chi_{jk}(\cdot)|^q \Big)^{1/q} | L_p(\mathbb{T}^n) \Big\| < \infty \tag{4.41}$$

with the usual modifications if $p = \infty$ and /or $q = \infty$.

REMARK 32. Of course, the summation over j, G, k in (4.41) is the same as in (4.40). After these preparations one can now formulate the periodic counterpart of Theorem 19. While (f, g) stands for the dual pairing in $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$, we denote the dual pairing in $(D(\mathbb{T}^n), D'(\mathbb{T}^n))$ by $(f, g)_\pi$.

THEOREM 33. (i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\{\Psi^{j,\mathrm{per}}_{G,k}\}$ be the orthonormal wavelet basis in $L_2(\mathbb{T}^n)$ as in* (4.38) *with*

$$u > \max(s, \sigma_p - s).$$

Let $f \in D'(\mathbb{T}^n)$. Then $f \in B^s_{pq}(\mathbb{T}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,k} \mu^{j,G}_k \, 2^{-jn/2} \, \Psi^{j,\mathrm{per}}_{G,k}, \qquad \mu \in b^{s,\mathrm{per}}_{pq}, \tag{4.42}$$

unconditional convergence being in $D'(\mathbb{T}^n)$ and in any spaces $B^\sigma_{pq}(\mathbb{T}^n)$ with $\sigma < s$. The representation (4.42) *is unique,*

$$\mu^{j,G}_k = 2^{jn/2} \, (f, \Psi^{j,\mathrm{per}}_{G,k})_\pi, \quad j \in \mathbb{N}_0, \ G \in G^j, \ k \in \mathbb{P}^n_j, \tag{4.43}$$

and

$$I: \quad f \mapsto \{2^{jn/2} \, (f, \Psi^{j,\mathrm{per}}_{G,k})_\pi\} \tag{4.44}$$

is an isomorphic map of $B^s_{pq}(\mathbb{T}^n)$ onto $b^{s,\mathrm{per}}_{pq}$. If, in addition, $p < \infty$, $q < \infty$, then (4.38) *is an unconditional basis in $B^s_{pq}(\mathbb{T}^n)$.*

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and*

$$u > \max(s, \sigma_{pq} - s).$$

Let $f \in D'(\mathbb{T}^n)$. Then $f \in F^s_{pq}(\mathbb{T}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,k} \mu^{j,G}_k \, 2^{-jn/2} \, \Psi^{j,\mathrm{per}}_{G,k}, \qquad \mu \in f^{s,\mathrm{per}}_{pq}, \tag{4.45}$$

unconditional convergence being in $D'(\mathbb{T}^n)$ and in any space $F^\sigma_{pq}(\mathbb{T}^n)$ with $\sigma < s$. The representation (4.45) *is unique with* (4.43) *and I in* (4.44) *is an isomorphic map of $F^s_{pq}(\mathbb{T}^n)$ onto $f^{s,\mathrm{per}}_{pq}$. If, in addition, $q < \infty$, then* (4.38) *is an unconditional basis in $F^s_{pq}(\mathbb{T}^n)$.*

Proof. Without going into detail the key steps to prove the above assertion are quite clear by the above discussion. Let $A = B$. By Definition 25 any $f \in B^{s,\mathrm{per}}_{pq}(\mathbb{R}^n)$ can be considered as a periodic element of $B^s_{pq}(\mathbb{R}^n, w_\alpha)$ and expanded according to (4.22), (4.23). One checks that

$$(f, \Psi^j_{G,k+2^{j+L}l}) = (f, \Psi^{j,\mathrm{per}}_{G,k})_\pi, \qquad k \in \mathbb{P}^n_j, \quad l \in \mathbb{Z}^n. \tag{4.46}$$

Collecting the corresponding terms in (4.22) one gets by (4.36),

$$f = \sum_{j=0}^\infty \sum_{G \in G^j} \sum_{k \in \mathbb{P}^n_j} \mu^{j,G}_k \, 2^{-jn/2} \, \Psi^j_{G,k,\mathrm{per}}, \qquad \mu \in b^{s,\mathrm{per}}_{pq},$$

where we used (4.46) in (4.23), (4.43). Then (4.37) and Proposition 29 prove the theorem. ∎

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