FUNCTION SPACES VIII
BANACH CENTER PUBLICATIONS, VOLUME 79
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

FUNCTION SPACES AND APPLICATION TO FUZZY ANALYSIS

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It is well known that Banach space theory, Orlicz space theory and theories of other function spaces are very important in mathematics and related sciences. Moreover, in 1965 Zadeh [58] introduced the concept of fuzzy subset (briefly, fuzzy set) of a classical set; this notion is an extension of ordinary subset. Then fuzzy set theory has been applied to many areas, such as control theory, signal processing, pattern recognition, neural networks, softcomputing, datamining, intelligent techniques, system theory, making decision, and so on (for example, see [16]). On the other hand, fuzzy set theory is also used in mathematics. In 1968 Chang [4] defined a fuzzy topological space, Pu and Liu [24] deeply studied this object by introducing a new neighbour relation "coincide with" between a fuzzy point and a fuzzy set, and Liu and Luo published a monograph [17] in 1998. In 1971 Rosenfeld [26] introduced fuzzy groups and started the research of fuzzy algebraic structures, in this direction, for example, see monograph [22]. For fuzzy analysis, in 1972 Chang and Zadeh [5] considered a fuzzy number, in 1982 Dubois and Prade [7] investigated the calculus of fuzzy-number-valued functions, about this subject see monographs [6, 44, 47] and monograph [15] summarizing the results on fuzzy differential equations; in 1974 Sugeno [30] discussed fuzzy measures and Sugeno fuzzy integrals, and monographs [11] and [34] described them in detail; in 1977 Katsaras and Liu [14] generalized topological linear spaces to the fuzzy case, naturally, they have a close connection with function spaces, Wu [35] also presented a different definition of fuzzy topological linear space, but these two definitions are not convenient to expand the theory.

The purpose of this paper is to show several applications in fuzzy analysis by means of function spaces and to explain some new function spaces in fuzzy analysis.

2000 *Mathematics Subject Classification*: Primary 54C35; Secondary 03E72.

This research is partially supported by the National Natural Science Foundation of China (10571035, 10271035).

The paper is in final form and no version of it will be published elsewhere.

1. Preliminaries. In this section, we recall some basic notions and notations of fuzzy mathematics which will be used in this paper and can be found in monographs [6, 15, 34, 47, 11].

DEFINITION 1.1. Let X be a nonempty set. Then we say that a fuzzy subset A is determined by a membership function $\mu_A : X \to [0,1]$.

When $\mu_A(x)$ is just the characteristic function $\chi_A(x) = 1$ $(x \in A)$, $= 0$ $(x \in X\backslash A)$, the membership function determines an ordinary subset, so a fuzzy subset is a generalization of the ordinary subset. Briefly, we use "fuzzy set" to replace "fuzzy subset".

Similar to the ordinary case, for any fuzzy sets A and B we define

$$A \cup B : \mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X$$

$$A \cap B : \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$$

$$A' : \mu_{A'}(x) = 1 - \mu_A(x), \forall x \in X$$

$$A \subset B : \mu_A(x) \le \mu_B(x), \forall x \in X.$$

DEFINITION 1.2. For any $x \in X$ and $\lambda \in (0,1]$, the fuzzy point x_λ is a special fuzzy set which is determined by $\mu_{\{x_\lambda\}}(x) = \lambda$, $\mu_{\{x_\lambda\}}(y) = 0$ $(y \ne x)$. We define two neighbor relations "belong to" and "coincide with" between a fuzzy point x_λ and a fuzzy set A as follows:

$$x_\lambda \in A : A(x) \ge \lambda \text{ ("belong to")}$$

$$x_\lambda \,\tilde{\in}\, A : x_\lambda \notin A' \Leftrightarrow \lambda > \mu_{A'}(x) \Leftrightarrow \lambda + \mu_A(x) > 1 \text{ ("coincide with").}$$

REMARK 1.1. The relation "coincide with" is important, because a fuzzy point belonging to the union of a class of fuzzy sets does not imply in general that the fuzzy point belongs to some fuzzy set of this class, but we can do for the relation "coincide with". See [17, 24, 44].

DEFINITION 1.3 ([18]). Let X be a nonempty set and T be a family of fuzzy sets on X. If T satisfies:

(1) $U, V \in T$ imply $U \cap V \in T$;
(2) $U_\alpha \in T$ $(\alpha \in \Omega)$ imply $\bigcup_{\alpha\in\Omega} U_\alpha \in T$;
(3) $r \in [0,1]$ implies $r^* \in T$ (r^* means $\mu_{r^*}(x) = r$, $\forall x \in X$),

then we say that T is a fuzzy topology of X and (X,T) is a fuzzy topological space.

REMARK 1.2. Clearly, 0^* is empty set and 1^* is the whole set X, so the condition (3) is stronger than the usual case, but this definition of fuzzy topology is more convenient to discuss fuzzy topological linear spaces (cf. [36–40]).

DEFINITION 1.4. Let $u : R^n \to [0,1]$ be a fuzzy set on n-dimensional Euclidean space. If u satisfies the following four conditions:

(i) u is normal, i.e. there exists $t_0 \in R^n$ such that $u(t_0) = 1$;
(ii) $u(t)$ is upper semicontinuous;
(iii) u is fuzzy convex, that is, $u(\lambda t + (1-\lambda)s) \ge \min\{u(t), u(s)\}$ for any $t, s \in R^n$ and $\lambda \in [0,1]$;

(iv) $[u]^0 = cl(\bigcup_{0<r\leq 1}[u]^r)$ is bounded, where $[u]^r = \{t \in R^n : u(t) \geq r\}$ $\forall r \in (0,1]$,

then we say that u is an n-dimensional fuzzy number, denoted by $u \in E^n$.

In particular, for any $u \in E^1$, $[u]^r$ is a nonempty bounded closed interval $[u_-(r), u_+(r)]$ $(\forall r \in [0,1])$.

Proposition 1.1. *If $u \in E^1$, then u satisfies the following conditions:*

(1°) *$u_-(r)$ and $u_+(r)$ are left continuous on $(0,1]$ and right continuous at $r = 0$;*
(2°) *$u_-(1) \leq u_+(1)$;*
(3°) *$u_-(r)$ is nondecreasing and $u_+(r)$ is nonincreasing.*

Conversely, if two real functions $a(\lambda)$ and $b(\lambda)$ defined on $[0,1]$ satisfy conditions (1°)-(3°), then there exists a unique $u \in E^1$ such that $[u]^r = [a(\lambda), b(\lambda)]$ $(\forall r \in [0,1])$.

Algebraic operations in E^1: $\forall u, v \in E^1$, $c \in R^1$, $r \in [0,1]$

$$[u+v]^r = [u]^r + [v]^r$$

iff $(u+v)_-(r) = u_-(r) + v_-(r)$ and $(u+v)_+(r) = u_+(r) + v_+(r)$;

$$[cu]^r = c[u]^r$$

iff $(cu)_-(r) = cu_-(r)$ and $(cu)_+(r) = cu_+(r)$ for $c \geq 0$.

Ordering in E^1: $\forall u, v \in E^1$

$$u \leq v : [u]^r \leq [v]^r (\forall r \in [0,1])$$

iff $u_-(r) \leq v_-(r)$ and $u_+(r) \leq v_+(r)$ $(\forall r \in [0,1])$.

Metric in E^1: $\forall u, v \in E^1$

$$D(u,v) = \sup_{r\in[0,1]} d_H([u]^r, [v]^r) = \sup_{r\in[0,1]} \max\{|u_-(r) - v_-(r)|, |u_+(r) - v_+(r)|\}$$

Here d_H is the ordinary Hausdorff metric.

Proposition 1.2. *(E^1, D) is a complete, nonseparable metric space, and E^1 is a convex cone.*

The fuzzy-number-valued function is defined by

$$F : R^1 \to E^n.$$

Definition 1.5. Let X be a nonempty set, Σ be a σ-algebra of X and $\mu : \Sigma \to [0,\infty]$. μ is called a fuzzy measure and (X, Σ, μ) is a fuzzy measure space iff:

(FM1) $\mu(\phi) = 0$;
(FM2) $E \subset F$, $E, F \in \Sigma$ imply $\mu(E) \leq \mu(F)$ (monotonicity);
(FM3) $\{E_n\} \subset \Sigma$, $E_1 \subset E_2 \subset \cdots$ imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$;
(FM4) $\{E_n\} \subset \Sigma$, $E_1 \supset E_2 \supset \cdots$, $\mu(E_{n_0}) < \infty$ (for some positive integer n_0) imply $\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

Let $f : X \to [-\infty, +\infty]$. If for any $\alpha \in R^1$ we have $N_\alpha(f) = \{t \in X : f(t) \geq \alpha\} \in \Sigma$, then we say that f is a μ-measurable function.

Let $f : X \to [0,\infty]$ be μ-measurable and $E \in \Sigma$. We define

$$(S)\int_E f(t)d\mu = \sup_{\alpha>0}\min\{\alpha, \mu(N_\alpha(f)\cap E)\}$$

as the Sugeno fuzzy integral of f on E. When $(s)\int_E f(t)d\mu < \infty$, we say that f is (S)-integrable on E.

2. Fuzzy topological vector spaces (ftvs). In this section by using fuzzy point, neighbor relation $\tilde{\in}$ "coincide with", definition 1.3 of fuzzy topological space (fts for short) and in term of introducing several kinds of "norms" we expanded the study of ftvs.

Definition 2.1 ([36]). Let X be a linear space and (X,T) be fuzzy topological space. If the addition and scalar multiplication are fuzzy continuous, then we say that (X,T) is a ftvs.

Definition 2.2 ([38]). Let (X,T) be a ftvs. If there is a family of fuzzy sets $\mathcal{U}= \{U\}$ such that for any $\lambda \in (0,1]$, $\mathcal{U}_\lambda = \{U \cap r^* : U \in \mathcal{U},\ r \in (1-\lambda,1]\}$ is a base of the quasineighborhood system of θ_λ (fuzzy set V is a quasineighborhood of x_λ iff there exists an open set $W \in T$ such that $W \subset V$ and $x_\lambda \tilde{\in} W$, see [34]). Then we say that (X,T) is a ftvs of (QL) type and $\mathcal{U} = \{U\}$ is called a prebase of (X,T).

Theorem 2.1 ([38]). *Let (X,T) be a ftvs of (QL) type. Then there exists a family of LaSalle's pseudo seminorms $\{\|x_\lambda\|_\alpha : \alpha \in D\}$ (D is a directed set with the order $\succ$) satisfying the following conditions:*

(i) *for any $x \in X$, $\lambda \in (0,1]$ and $\alpha \in D$, $\|x_\lambda\|_\alpha \geq 0$ and $\|\theta_\lambda\|_\alpha = 0$;*
(ii) *for any $x \in X$, $\lambda \in (0,1]$, $\alpha \in D$ and $k \in R^1$, $\|kx_\lambda\|_\alpha = |k|\|x_\lambda\|_\alpha$;*
(iii) *for each $\alpha \in D$ and $\mu \in (0,1]$, there exist $e \in D$ with $e \succ \alpha$ and $r \in (1-\mu,1]$ such that*

$$\|x_\lambda + y_\lambda\|_\alpha \leq \|x_\lambda\|_e + \|y_\lambda\|_e$$

for all $\lambda \in (1-r,1]$ and $x, y \in X$;
(iv) *for any $\alpha \in D$ and $x \in X$, $\|x_\mu\|_\alpha \geq \|x_\lambda\|_\alpha$ $(0 \leq \mu \leq \lambda \leq 1)$ and*

$$\lim_{\varepsilon\to 0^+} \|x_{\lambda-\varepsilon}\|_\alpha = \|x_\lambda\|_\alpha;$$

(v) *for any $\alpha \succ e$, $x \in X$ and $\lambda \in (0,1]$, $\|x_\lambda\|_\alpha \geq \|x_\lambda\|_e$.*

Conversely, if there exists a family of LaSalle's pseudo seminorms defined on the set of all fuzzy points of a linear space X and satisfying the conditions (i)-(v), then it determines a unique fuzzy topology T such that (X,T) is a ftvs of (QL) type.

Remark 2.1. We know that in ordinary case, tvs can be characterized by a family of LaSalle's pseudo seminorms. But in fuzzy case, it is quite different, we may construct a ftvs which is not of (QL) type (cf. [38]). Therefore, ftvs of (QL) type is remarkable in ftvs theory.

Remark 2.2. Wu and Li discussed locally convex ftvs in [40], Wu and Fang [39] considered locally bounded ftvs.

DEFINITION 2.3 ([37]). Let X be a linear space and $\bar{X}$ be the set of all fuzzy points x_λ ($x \in X, \lambda \in (0,1]$). If $\|\cdot\| : X \to [0,\infty)$ satisfies the following conditions:

1° for any $\lambda \in (0,1]$, $\|x_\lambda\| = 0$ iff $x = \theta$;
2° $\|kx_\lambda\| = |k|\|x_\lambda\|$;
3° $\|x_\lambda + y_\lambda\| \leq \|x_\lambda\| + \|y_\lambda\|$;
4° $\|x_\mu\| \geq \|x_\lambda\|$ $(0 \leq \mu \leq \lambda \leq 1)$ and $\lim_{\varepsilon \to 0^+ \|x_{\lambda-\varepsilon}\| = \|x_\lambda\|}$,

then we say that $\|\cdot\|$ is a fuzzy norm and $(X, \|\cdot\|)$ is a fuzzy normed space.

THEOREM 2.2 ([38]). *A fuzzy normed space is a separated ftvs of type (QL).*

Wu and Ma's book [44] gave the following two examples.

EXAMPLE 2.1. Let $L_P[0,1]$ $(P \geq 1)$ be the function space as usual. Denote $L^\omega[0,1] = \bigcap_{P\geq 1} L_P[0,1]$ and

$$\|f_\lambda\| = \|f\|_{\frac{1}{\lambda}} = \left(\int_0^1 |f(t)|^{\frac{1}{\lambda}} dt\right)^\lambda \qquad (f \in L^\omega[0,1], \lambda \in (0,1]).$$

Then $(L^\omega, \|\cdot\|)$ is a fuzzy normed space.

EXAMPLE 2.2. Let $H(U)$ be the set of all complex analytic functions on the unit disk U. For any $f \in H(U)$, $r \in [0,1)$, $P \geq 1$, denote

$$M_P(f,r) = (\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^P d\theta)^{\frac{1}{P}}, \qquad \|f\|_P = \lim_{r\to 1^-} M_P(f,r),$$

$$H^P = \{f \in H(U) : \|f\|_P < \infty\} \quad \text{(ordinary Banach space, called a Hardy space).}$$

Similar to the definition of the fuzzy norm in example 2.1, H^ω is a fuzzy normed space.

Moreover, for any $P \geq 1$, H^P is a fuzzy seminormed space with respect to the following fuzzy seminorm:

$$\|f_\lambda\|_P = M_P(f, 1-\lambda) \qquad (f \in H_P, \lambda \in (0,1])$$

(fuzzy seminorm means that the "norm" satisfies 2°−4° in definition 2.3 and $1' : \|\theta_\lambda\| = 0$, $\forall\lambda \in (0,1]$).

REMARK 2.3. Katsaras [12, 13] also considered ftvs and fuzzy normed spaces, but didn't use fuzzy points. In 1984 Wang [31] proved that definition 2.1 is equivalent to Katsaras's definition for ftvs, and Ma [19] showed that definition 2.3 is a little stronger than Katsaras's definition for fuzzy normed spaces, but it is not separated in Katsaras sense.

REMARK 2.4. Wu and Ma [43, 41] studied fuzzy topological algebras and fuzzy normed algebras.

In 1990, Wu, Ma and Bao [46] discussed the dual space of a fuzzy normed space and discovered that the dual space is lattice-valued. So, we must develop the theory of lattice-valued tvs; naturally, it is an extension of ftvs. In 1997 Fang and Yan proposed the notion of L-tvs and published a series of papers (see for example [8, 9]). In 2005, Yan and Wu [55, 56] also considered L-tvs.

3. Fuzzy measurable function spaces and the space of nonmonotonic fuzzy measures. In 2003 Wu and Mamadou [51] generalized the concept of Sugeno fuzzy integral from nonnegative measurable functions to the case of real measurable functions.

DEFINITION 3.1. Let (X, Σ, μ) be a fuzzy measure space, $f : X \to [-\infty, \infty]$ be μ-measurable and

$$f^+(t) = \begin{cases} f(t) & (f(t) \geq 0) \\ 0 & (f(t) < 0) \end{cases}, \qquad f^-(t) = \begin{cases} 0 & (f(t) \geq 0) \\ -f(t) & (f(t) < 0) \end{cases}.$$

If f^+ and f^- are (S)-integrable on E ($E \in \sum$), then we say that f is (S)-integrable on E and define

$$(S)\int_E f(t)d\mu = (S)\int_E f^+(t)d\mu - (S)\int_E f^-(t)d\mu$$

Let $S(\mu) = \{f : X \to [-\infty, \infty] : f$ is μ-measurable and finite μ-a.e. on $X\}$.

THEOREM 3.1 ([51]). *Let (X, Σ, μ) be a fuzzy measure space and $\mu(X) < 1$. If we denote*

$$\rho(f, g) = (S)\int_X \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} d\mu \qquad (f, g \in S(\mu)),$$

then $(S(\mu), \rho(\cdot, \cdot))$ is a pseudo-metric space iff μ is subadditive.

REMARK 3.1. For the case of $\mu(X) < \infty$, the sufficiency of Theorem 3.1 is also true. But we do not know whether the necessity holds in this case.

REMARK 3.2. For the further discussions of the Sugeno integral for real functions, see Wu and Zhao's [54] in 2006.

REMARK 3.3. Wu and Ma [42] discussed the convergence in the space of all (S)-integrable nonnegative functions.

In 2003 Narukawa, Murofushi and Sugeno [23] considered the convergence in the Banach space

$$FM(X, \Sigma) = \{\mu : \mu \text{ is the nonmonotonic fuzzy measure on } (X, \Sigma) \text{ and such that } \|\mu\| = |\mu|(X) < \infty\},$$

where (X, Σ) is a measurable space and nonmonotonic fuzzy measure μ means that $\mu : X \to (-\infty, \infty)$ and $\mu(\emptyset) = 0$, and

$$|\mu|(X) = \sup\Big\{\sum_{k=1}^{n} |\mu(A_k) - \mu(A_{k-1})|\Big\}$$

(the "sup" is taken over all finite sequences

$$\phi = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = X,\ A_k \in \Sigma \quad (k = 1, 2, \cdots, n-1))$$

(or see [1]).

Wu and Ren [48] obtained a characterization of the separability of the space of nonmonotonic fuzzy measures.

THEOREM 3.2 ([48]). *$(FM(X, \Sigma), \|\cdot\|)$ is separable if and only if the σ-algebra Σ is a finite set.*

4. Function spaces and their application to fuzzy numbers, calculus of fuzzy-number-valued functions and fuzzy differential equations. In 1983 Puri and Ralescu [25] proved that the fuzzy number space E^n can be embedded into a Banach space isometrically and isomorphically. But this Banach space is not concrete. Wu and Ma [45], Wu and Zhang [53], Ma [20] and Wang and Wu [32] constructed several concrete Banach function spaces as the embedding spaces of E^n and some kinds of its subspaces. Here we only present the results in [45] for the case of E^1.

By using Proposition 1.1, concerning the representation theorem of E^1 we know that the embedding problem can be expressed as how to construct two Banach function spaces Y, Z such that the embedding $j : u \in E^1 \to (u_-, u_+) \in Y \times Z$ is isometric and isomorphic from E^1 into $Y \times Z$. Naturally, we take

$$Y = Z = \bar{C}[0,1] = \{f : f \text{ is left continuous on } [0,1] \text{ and has a right limit for } t \in (0,1], \text{ in particular it is right continuous at } t = 0\},$$

$$\|f\|_{\bar{C}[0,1]} = \sup_{t\in[0,1]} |f(t)|$$

and as usual

$$\bar{C}[0,1] \times \bar{C}[0,1] = \{(f,g) : f, g \in \bar{C}[0,1]\},$$

$$\|(f,g)\|_{\bar{C}[0,1]\times\bar{C}[0,1]} = \max\{\|f\|_{\bar{C}[0,1]}, \|g\|_{\bar{C}[0,1]}\}.$$

THEOREM 4.1 ([45, I]). *For $u \in E^1$, denote $j(u) = (u_-, u_+)$. Then $j(E^1)$ is a closed convex cone with vertex 0 in $\bar{C}[0,1] \times \bar{C}[0,1]$ and satisfies*

(a) $\|j(u) - j(v)\|_{\bar{C}[0,1]\times\bar{C}[0,1]} = D(u,v),\ \forall u, v \in E^1$;
(b) $j(su + tv) = sj(u) + tj(v),\ \forall u, v \in E^1,\ s, t \geq 0$;
(c) $cl(j(E^1) - j(E^1)) = \bar{C}[0,1] \times \bar{C}[0,1]$.

If $u : R^1 \to [0,1]$ only satisfies the conditions (i)-(iii) in definition 1.4, namely, $u_-(0)$ and $u_+(0)$ may not be defined, then we say that u is a noncompact fuzzy number, denoted by $u \in E^1_\infty$. We can construct a concrete Fréchet function space as the embedding space of E^1_∞.

In fact, comparing E^1 and E^1_∞, we construct the space

$$\bar{C}(0,1] = \{f : f \text{ is a left continuous function on } (0,1] \text{ and has a right limit for } t \in (0,1]\}.$$

Similarly to Theorem 4.1, applying the method from [21], we can get the following embedding theorem.

THEOREM 4.2 ([52, I]). *For $u \in E^1_\infty$, denote $j(u) = (u_-, u_+)$. Then $j(E^1_\infty)$ is a closed convex cone with vertex 0 in $\bar{C}(0,1] \times \bar{C}(0,1]$ and $j : E^1_\infty \to \bar{C}(0,1] \times \bar{C}(0,1]$ satisfies the statements* (a)–(c) *in Theorem* 4.1*, where the quasi norm of Fréchet space $\bar{C}(0,1]$ is:*

$$\|f\|_{\bar{C}(0,1]} = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f\|_k}{1 + \|f\|_k}, \qquad \|f\|_k = \sup_{t\in[\frac{1}{k},1]} |f(t)| \ (k = 1, 2, \cdots).$$

An application of the above embedding theorem to the calculus of fuzzy-number-valued functions we can find in [32], [45, II] and [52, II]. Here we only present a result for one dimensional case as follows.

Theorem 4.3 ([45, II]). *If $F : [a,b] \to E^1$, then the following statements are equivalent:*

(1) *$F(t)$ is (K) integrable on $[a,b]$ (namely, there exists a Lebesgue integrable function $h : [a,b] \to R^1$ such that for any $r \in [0,1]$ and $x \in [F(t)]^r$ we have $|x| \le h(t)$ $(\forall t \in [a,b])$ and there exists a fuzzy number $u \in E^1$ such that for any $r \in [0,1]$ we have*

$$[u]^r = \left\{ (L) \int_a^b f(t)dt : f(t) \in [F(t)]^r \ (\forall t \in [a,b]) \text{ is measurable selection} \right\},$$

see [6, 44]);
(2) *$j \circ F(t)$ is Pettis integrable for $\bar{C}[0,1] \times \bar{C}[0,1]$ on $[a,b]$;*
(3) *$F_-(t), F_+(t)$ are Pettis integrable for $\bar{C}[0,1]$ on $[a,b]$;*
(4) *$F_-(t)(r), F_+(t)(r)$ are Lebesgue integrable on $[0,1]$ $(\forall t \in [a,b])$.*

Remark 4.1. For the proof of Theorem 4.2 we need to calculate the conjugate space of the embedding function space $\bar{C}[0,1]$.

The references in Lakshmikantham and Mohapatras' monograph [15] give the list of papers on fuzzy differential equations published before 2003, including some papers of Wu and Song. One of the basic problems in this area is to consider the initial value problem (IVP for short) for the fuzzy differential equation (FDE)

$$u' = f(t,u), \qquad u(t_0) = u_0,$$

where $' = \frac{d}{dt}$, $f \in C[J \times E^n, E^n]$, $J = [t_0, t_0 + a]$, $t_0 \ge 0$, $a > 0$, $u_0 \in E^n$ (fuzzy-number-valued function $F : [a,b] \to E^n$ being differentiable at $t_1 \in [a,b]$ means that there exists $F'(t_1) \in E^n$ such that

$$\lim_{h \to 0^+} D\left(\frac{F(t_1 + h) - F(t_1)}{h}, F'(t_1)\right) = 0 \quad \text{and} \quad \lim_{h \to 0^+} D\left(\frac{F(t_1) - F(t_1 - h))}{h}, F'(t_1)\right) = 0$$

and for any $u, v \in E^n$, H-difference $u - v$ means that there exists $w \in E^n$ such that $u = v + w$, see [6] or [47]).

Note that $C[J \times E^n, E^n]$ is the set of all continuous fuzzy mappings from $J \times E^n$ to E^n and $(f(t,u(t)) \in) C[J, E^n]$ is the set of all continuous fuzzy-number-valued functions from J to E^n, and the metric in $C[J, E^n]$ as usual is defined by

$$\rho(u,v) = \sup_{t \in J} D(u(t), v(t)) = \sup_{t \in J} \sup_{r \in [0,1]} d_H([u(t)]^r, [v(t)]^r).$$

Similarly, $C^1[J, E^n]$ is the set of all continuously differentiable fuzzy-number-valued functions from J to E^n.

Remark 4.2. In the monograph [15], the metric

$$\rho'(u,v) = \sup_{t \in J} D(u(t), v(t)) e^{-\lambda t},$$

replaces the metric $\rho(u,v)$ in $C[J, E^n]$ to consider FDEs for some cases.

Using the embedding theorem of E^n, Wu and Song [50] discussed approximate solutions, existence and uniqueness for IVP of FDE. Song et al. [27] got a global existence theorem. Wu and Song [49] gave an existence theorem under a compactness-type condition. Song et al. [28] considered asymptotic equilibrium and stability, and so on (see Wu Congxin, Some notes of fuzzy and nonfuzzy differential equations, Conference on

Differential and Difference Equations, and Applications, Melbourne, Florida, USA, 2005, August 1-5 (invite lecture)). Now we only mention that Song and Wu [29] investigated the IVP of FDE in the sense of the level differential which is weaker than the concept of the differential (this notion is so strong, because for some $u, v \in E^n$, H-difference does not exist).

THEOREM 4.4 ([29]). *Assume that*

(a) $f \in C_w[J \times B(u_0, b), E^n]$ *and* $D(f(t,u), 0) \leq M$ *for all* $(t,u) \in J \times B(u_0, b)$ *where* $B(u_0, b) = \{u \in E^n : D(u, u_0) < b\}, u_0 \in E^n$*;*
(b) $g \in C^1[J \times [0,b], R^1]$, $g(t,0) \equiv 0$ *and* $0 \leq g(t,x) \leq M_1$ *for all* $t \in J$ *and* $x \in [0,b]$ *such that* $g(t,x)$ *is nondecreasing on* x*, and the ordinary IVP*

$$x'(t) = g(t, x(t)), \; x(t_0) = 0$$

has only the solution $x(t) = 0$ *on* J*;*
(c) $D(f(t,u), f(t,v)) \leq g(t, D(u,v))$ *for all* $t \in J$, $u, v \in B(u_o, b)$ *and* $D(u,v) \leq b$.

Then in the sense of level differential the IVP for FDE has a unique solution $u \in C^1_w[[t_0, t_0 + r], B(u_0, b)]$ *on* $[t_0, t_0 + r]$ *where* $r = \min\{a, \frac{b}{M}, \frac{b}{M_1}\}$.

Note that $u \in C_w$ means that $u(t)$ is level-continuous and $u \in C^1_w$ means that $u(t)$ is level-continuously level-differentiable. Here a fuzzy-number-valued function $F : T \subset R^1 \to E^n$ is level-differentiable at $t_1 \in T$ iff for any $\alpha \in [0,1]$ there exists a compact convex set of R^n, denoted by $DF_\alpha(t_1)$ such that

$$\lim_{h \to 0^+} d_H\left(\frac{[F(t_1 + h)]^\alpha - [F(t_1)]^\alpha}{h}, DF_\alpha(t_1)\right) = 0,$$

$$\lim_{h \to 0^+} d_H\left(\frac{[F(t_1)]^\alpha - [F(t_1 - h)]^\alpha}{h}, DF_\alpha(t_1)\right) = 0,$$

where d_H is the ordinary Hausdorff metric and the family $\{DF_\alpha(t_1) : \alpha \in [0,1]\}$ determines a fuzzy number $F'(t_1) \in E^n$, i.e. $[F'(t_1)]^\alpha = DF_\alpha(t_1)$ for any $\alpha \in [0,1]$. Moreover, fuzzy mapping $f : J \times E^n \to E^n$ is level-continuous at $(t_1, u_1) \in T \times E^n$ iff for any fixed $\alpha \in [0,1]$ and arbitrary $\varepsilon > 0$ there exists $\delta(\alpha, \varepsilon) > 0$ such that

$$d_H([f(t,u)]^\alpha, [f(t_1, u_1)]^\alpha) < \varepsilon$$

whenever $|t - t_1| < \delta$ and $d_H([u]^\alpha, [u_1]^\alpha) < \delta$ for $t \in T$ and $u \in E^n$.

REMARK 4.3. Buckley [3] considered the space of continuous fuzzy-number-valued functions with different metrics. Gong and Wu [10] discussed the classes of bounded variation fuzzy-number-valued functions and of absolutely continuous fuzzy-number-valued functions, but we need to introduce the corresponding fuzzy or nonfuzzy metrics in future.

5. Fuzzy subdifferential and application in fuzzy mathematical programming. In 2003 Wang and Wu [33] introduced subdifferentials for fuzzy mappings and discussed various properties and relations, especially, for the case of convex fuzzy mappings and gave some applications to convex fuzzy programming.

DEFINITION 5.1 ([33]). Let $F : k \subset R^n \to E^1$, $x^0 = (x^0_1, x^0_2, \cdots, x^0_n) \in k$. $(u^0_1, u^0_2, \cdots, u^0_n)$ is a subgradient of F at x^0 iff $u^0_1, u^0_2, \cdots, u^0_n \in E^1$ and there exists $\delta_0 > 0$ such that

$$F(x) + \sum_{x_i < x_i^0} (x_i^0 - x_i)u_i^0 \geq F(x^0) + \sum_{x_i \geq x_i^0} (x_i - x_i^0)u_i^0$$

for any $x \in U(x^0, \delta_0) \cap k$ ($U(x^0, \delta_0) = \{x \in R^n : d(x, x^0) < \delta_0\}$).

And we say that $\partial F(x^0) = \{(u_1, u_2, \cdots, u_n) : (u_1, u_2, \cdots, u_n)$ *is a subgradient of* F at x^0, $u_i \in E^1 (i = 1, 2, \cdots, n)\}$ is the subdifferential of F at x^0.

Let $F : k \subset R^n \to E^1$ be a convex fuzzy mapping (i.e. $\forall x^{(1)}, x^{(2)} \in k,\ t \in [0, 1]$ we have $F(tx^{(1)} + (1-t)x^{(2)}) \leq tF(x^{(1)}) + (1-t)F(x^{(2)})$). Then convex fuzzy programming means the following problem

$$(FCP) \begin{cases} \min F(x) \\ x \in k \end{cases}$$

We can prove that the local minimum solutions and global solutions of (FCP) are equivalent (see [33]), so we call them the minimum solutions.

Theorem 5.1 ([33]). *Let* $F : k \subset R^n \to E^1$ *be a convex fuzzy mapping. Then* $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ *is a minimum solution of (FCP) if and only if there exists a subgradient* $(u_1, u_2, \cdots, u_n)$ *of* F *at* x^0 *such that*

$$\sum_{x_i \geq x_i^0} (x_i - x_i^0)u_i \geq \sum_{x_i < x_i^0} (x_i^0 - x_i)u_i, \quad \forall x \in k$$

Remark 5.1. In 2006 Bao and Wu [2] gave several sufficient conditions in order that the semicontinuity always implies convexity for fuzzy mappings.

Remark 5.2. In 2006 Yang and Wu [57] defined four kinds of subdifferentials for fuzzy mappings and applied them to fuzzy programming. They also obtained sufficient and necessary conditions for solution and some properties of a solution set.

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