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ROUGHNESS OF TWO NORMS ON MUSIELAK-ORLICZ FUNCTION SPACES

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Abstract. Some criteria of strong roughness, roughness and pointwise roughness of Orlicz norm and Luxemburg norm on Musielak-Orlicz function spaces are obtained.

1. Introduction. Leach and Whitefield [1] introduced the concept of rough norm in 1973. Later John and Zizler [2] and Li [3] introduced the concept of strong rough norm and pointwise rough norm, respectively. X denotes a Banach space, X^* the dual of X. $S(X)$ is the unit sphere in X. $B(X)$ denotes the unit ball of X. The norm of X is said to be pointwise rough provided $\varepsilon(x) > 0$ for every point on $S(X)$, where

$$\varepsilon(x) = \sup\{\varepsilon > 0 : \exists f_n, g_n \in X^*, \|f_n\|, \|g_n\| \to 1, \\ f_n(x), g_n(x) \to 1, \overline{\lim_{n\to\infty}} \|f_n - g_n\| \geq \varepsilon\}.$$

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$\|\cdot\|$ is said to be rough provided $\inf\{\varepsilon(x) : x \in S(X)\} > 0$, $\|\cdot\|$ is said to be strongly rough provided $\inf\{\mathrm{diamA(x) : x \in S(x)}\} > 0$, where $A(x) = \{f \in S(X^*) : f(x) = \|x\| = 1\}$, that is, $A\,(x)$ is the gradient of x.

Criteria for roughness of the norm in Orlicz spaces are given in [10, 11]. This paper gives criteria for roughness of the norms in Musielak-Orlicz function spaces.

The triple (T, Σ, μ) stands for a finite nonatomic measure space. A mapping $T \times [0, \infty) \to [0, \infty]$ is said to be a Musielak-Orlicz function if it satisfies:

(i) $M(\cdot, u)$ is Σ-measurable for any $u \in [0, \infty)$.
(ii) for μ-a.e. $t \in T$, $M(t, u)$ is a left continuous and convex function of u.
(iii) for μ-a.e. $t \in T$, $M(t, 0) = 0$, $\lim_{u\to\infty} M(t, u) = \infty$, and there exists $u' \neq 0$ such that $M(t, u') < \infty$ for μ-a.e $t \in T$.

The complementary function of M is defined by

$$N(t, v) = \sup\{uv - M(t, u)\} \quad (\mu\text{-a.e. } t \in T; v \geq 0)$$

It is easy to see that $N(t, v)$ is also a Musielak-Orlicz function. We define $\hat{a}(t) = \sup\{v \geq 0 : N(t, v) < \infty\}$.

$p(t, u)$ and $p^-(t, u)$ (resp. $q(t, v)$ and $q^-(t, v)$) stand for the right and left derivative of $M(t, u)$ ($N(t, v)$, respectively). We know that for any $u, v \geq 0$

$$uv \leq M(t, u) + N(t, v) \quad (\mu\text{-a.e. } t \in T)$$

and $uv = M(t, u) + N(t, v)$ if and only if $p^-(t, u) \leq v \leq p(t, u)$ or $q^-(t, v) \leq u \leq q(t, v)$.

M is said to satisfy the Δ_2 condition (for short, $M \in \Delta_2$) if there exist $\lambda > 1$ and a measurable nonnegative function δ defined on T such that $\int_T \delta(t)d\mu < \infty$ and

$$M(t, 2u) \leq \lambda M(t, u) + \delta(t) \quad (\text{a.e. } t \in T, -\infty < u < +\infty).$$

By L^0 we denote the set of all (equivalence classes of) Σ-measurable real functions defined on T. The linear set

$$\left\{x(t) \in L^0 : \varrho_M(\lambda x) = \int_T M(t, \lambda x(t))d\mu \text{ for some } \lambda > 0\right\}$$

endowed with the Luxemburg norm

$$\|x\|_M = \inf\{\lambda > 0 : \rho_M(x/\lambda) \leq 1\}$$

or the Orlicz norm

$$\|x\|_M^o = \sup\left\{\int_T x(t)y(t)d\mu : \rho_N(y) \leq 1\right\} = \inf_{k>0}\left\{\frac{1}{k}(1 + \rho_M(kx))\right\}$$

is a Banach space. We call it the Musielak-Orlicz space and denote by L_M or L_M^o, respectively.

We define a closed subspace E_M^o of L_M^o by

$$E_M^o = \{x \in L^0 : \rho_M(\lambda x) < \infty \text{ for any } \lambda > 0\}.$$

The spaces E_M^o and L_M^o coincide if and only if $M \in \Delta_2$.

For any $x \in L_M^o$, write

$$d^o(x) = \inf\{\|x - y\|_M^0 : y \in E_M^o\}, d(x) = \inf\{\|x - y\|_M : y \in E_M\},$$

$$\theta_M(x) = \inf\{\lambda > 0 : \rho_M(x/\lambda) < \infty\} = \xi(x).$$

It is known that $\theta_M(x) = d^o(x) = d(x)$.

If $\int_T N(t,\hat{a}(t))d\mu > 1$, we have $[k_x^*, k_x^{**}] \neq \emptyset$ and we know (see [13], [14]) that $\|x\|_M^o = \frac{1}{k}(1+\rho_M(kx))$ if and only if $k \in [k_x^*, k_x^{**}]$, where

$$k_x^* = \inf\left\{k > 0 : \int_T N(t, p(t, k|x(t)|))d\mu \geq 1\right\},$$
$$k_x^{**} = \sup\left\{k > 0 : \int_T N(t, p(t, k|x(t)|))d\mu \leq 1\right\}.$$

The dual space of L_M is represented as $(L_M)^* = L_N^o \oplus \Phi$, i.e. every $f \in (L_M)^*$ is uniquely represented in the form $f = y + \phi$, where ϕ is a singular functional, i.e. $\phi(x) = 0$ for any $x \in E_M$, and y is a regular functional defined by the formula

$$\langle x, y\rangle = \int_T x(t)y(t)d\mu \qquad (\forall x \in L_M).$$

If any $f \in (L_M)^*$ is uniquely represented in the form $f = y + \phi$, then $\|f\|^o = \|y\|_N^o + \|\phi\|^o$ (see [4, Lemma 1.3]), where

$$\|\phi\| = \|\phi\|^o = \sup\{\phi(x) : \rho_M(x) < \infty\} = \sup\{\phi(x) : \rho_M(x) < \varepsilon\} = \sup_{\theta_M(x)\neq 0} \frac{\phi(x)}{\theta_M(x)}$$

if the norm $\|f\|^0$ is defined by $\|f\|^0 = \sup\{x^*(x) : \|x\|_M \leq 1\}$.

If $f \in (L_M^o)^*$, that is, $f = y + \phi$, where $y \in (L_N)$ is the regular functional defined by the formula

$$\langle x, y\rangle = \int_T x(t)y(t)d\mu \qquad (\forall x \in L_M^o)$$

and ϕ is a singular functional, then

$$\|f\| = \inf\{\xi > 0 : \rho_N(y/\xi) + \|\phi\|/\xi \leq 1\}$$

(see [14, Lemma 1.4]), if the norm $\|f\|$ is defined by the formula

$$\|f\| = \sup\{x^*(x) : \|x\|_M^0 \leq 1\}.$$

We start with auxiliary lemmas.

Lemma 1. *The spaces E_M^o and E_M are separable.*

Lemma 2. (i) *The spaces E_M^o and E_M are weakly Asplund.*

(ii) *The spaces E_M^o and E_M are Asplund if and only if $M \in \nabla_2$.*

Proof. By Lemma 1, the proof is similar to that for Orlicz spaces (see [13], Theorem 2.58). ∎

Lemma 3 ([6], Proposition 5.6). *For any sequence (u_n) in L_M we have that $\rho_M(u_n) \to 1$ if and only if $\|u_n\|_M \to 1$ if and only if $M \in \Delta_2$.*

Lemma 4 ([8], Lemma 1.7). *There is no nonzero singular functional $\phi \in (L_M^o)^*$ attaining its norm on $S(L_M^o)^*$.*

Lemma 5. *For any $f \in (L_M^o)^*, f = y + \phi$ (where $y \in (L_N), \phi$ is a singular functional). If $\|f\| = 1$ is attained on $S(L_M^o)$, we obtain*

$$\int_T N(t, y(t))d\mu + \|\phi\| = 1.$$

Proof. Since Lemma 4 holds, we can get the result in the same way as for Orlicz spaces (see [6], Theorem 1.42). ■

LEMMA 6. *The following conditions are equivalent:*

1) $N \in \Delta_2$, *i.e., there exist* $\lambda > 1$ *and* $0 \le \delta(t) \in L_1$ *satisfying*

$$N(t, 2v) \le \lambda N(t, v) + \delta(t) \quad (a.e.\ t \in T, v \in \mathbf{R}).$$

2) *for any* $\varepsilon > 0$, *there exist* $\lambda > 1$ *and* $0 \le \delta(t) \in L_1$, *such that*

$$N(t, v/\varepsilon) \le \lambda N(t, v) + \delta(t) \quad (a.e.\ t \in T, v \in \mathbf{R}).$$

3) *for any* $\varepsilon \in (0, 1)$, *there exist* $\theta \in (0, 1)$ *and* $0 \le \delta(t) \in L_1$ *satisfying*

$$M(t, \varepsilon u) \le \theta\varepsilon M(t, u) + \delta(t) \quad (a.e.\ t \in T, u \in \mathbf{R}).$$

4) *there exist* $\varepsilon, \theta \in (0, 1)$ *and* $0 \le \delta(t) \in L_1$ *satisfying:*

$$M(t, \varepsilon u) \le \theta\varepsilon M(t, u) + \delta(t) \quad (a.e.\ t \in T, u \in \mathbf{R}).$$

Proof. See [13]. ■

LEMMA 7 (see [19]). *If*

$$K_M := \sup_{\|x\|_M^o = 1} \left\{ k > 0 : \|x\|_M^o = \frac{1}{k}(1 + \rho_M(kx)) \right\},$$

then $K_M < \infty$ *if and only if* $N \in \Delta_2$.

Proof. Necessity. Since $N \notin \Delta_2$, taking any $\varepsilon > 0$, putting

$$\delta(t) = \sup\left\{ v \ge 0 : M(t, \varepsilon v) > \frac{\varepsilon}{\varepsilon + 1} M(t, v) \right\}$$

we have $\int_T M(t, \delta(t))dt = \infty$. Indeed otherwise

$$M(t, \varepsilon v(t)) \le \frac{\varepsilon}{\varepsilon + 1}(M(t, v) + M(t, \delta(t))) \quad \text{(a.e. } t \in T, v \in \mathbf{R}),$$

whence, by Lemma 6, we get $N \in \Delta_2$, a contradiction.

In this way, we get $u(t) \ge 0$ such that

$$M(t, \varepsilon v(t)) > \frac{\varepsilon}{\varepsilon + 1} M(t, v(t)) \text{ and } \int_T M(t, v(t))dt > \frac{\varepsilon + 1}{\varepsilon}.$$

Then $\int_T M(t, \varepsilon v(t))dt > 1$, whence $\|\varepsilon u\|^o > 1$. Therefore, there exists $\Omega \subset T$ such that $\|\varepsilon u|_\Omega\|^o = 1$. Take $k \in K(\varepsilon u|_\Omega)$, i.e.

$$1 = \|\varepsilon u\|^o = \frac{1}{k}(1 + \rho_M(k\varepsilon u)).$$

We have $k > 1$, so

$$\begin{aligned}\frac{1}{k} + \rho_M(k\varepsilon u|_\Omega) &\le \frac{1}{k}(1 + \rho_M(k\varepsilon u|_\Omega)) = \|\varepsilon u|_\Omega\|^o \\ &\le \varepsilon\left(1 + \rho_M\left(\frac{1}{\varepsilon} \cdot \varepsilon u|_\Omega\right)\right) = \varepsilon(1 + \rho_M(v|_\Omega)) \\ &\le \varepsilon\left(1 + \frac{\varepsilon + 1}{\varepsilon}\int_\Omega M(t, \varepsilon u(t))d\mu\right) = \varepsilon + (1 + \varepsilon)\rho_M(\varepsilon u|_\Omega).\end{aligned}$$

This shows that

$$\frac{1}{k} \le \varepsilon + \varepsilon \rho_M(\varepsilon u|_\Omega) \le 2\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we obtain that $K_M = \infty$, a contradiction.

Sufficiency. Because $N \in \Delta_2$, by Lemma 6, there exist $\eta > 0$ and $0 < \delta(t) \in L_1$ such that

$$M(t, 2u) \ge 2(1 + 2\eta)M(t, u) - \delta(t) \quad \text{(a.e. } t \in T, u \in \mathbb{R}).$$

Hence, for $u \ge 0$ satisfying $M(t, u) > \delta(t)/2\eta$, we obtain

$$M(t, 2u) \ge 2(1 + \eta)M(t, u). \tag{1}$$

Pick D big enough such that

$$D - 1 - \frac{1}{2\eta}\int_T \delta(t) \ge 1.$$

For any $x \in S(L_M^o)$, denote

$$H_x = \{t \in T : M(t, D|x(t)|) > \delta(t)/2\eta\}.$$

In view of $1 = ||x||_M^o \le \frac{1}{D}(1 + \rho_M(Dx))$, we get $\rho_M(Dx) \ge D - 1$. Moreover

$$\begin{aligned}\int_{H_x} M(t, Dx(t))d\mu &= \rho_M(Dx) - \int_{T\setminus H_x} M(t, Dx(t))d\mu \\ &\ge D - 1 - \frac{1}{2\eta}\int_{T\setminus H_x}\delta(t)d\mu \ge D - 1 - \frac{1}{2\eta}\int_T \delta(t)d\mu \ge 1. \end{aligned} \tag{2}$$

Since $N \in \Delta_2$, we have $\hat{a}(t) = \infty$ a.e. For any $x \in S(L_M^o)$, we obtain that $K(x) \ne \emptyset$. If $k \in K(x)$, then $k \le D$ or $k > D$. If $k > D$, there exists $j \ge 1$ such that $2^{j-1}D < k \le 2^j D$. From (1) and (2), we have

$$\begin{aligned}2^j D \ge k = 1 + \rho_M(kx) &> \int_{H_x} M(t, kx(t))d\mu \ge \int_{H_x} M(t, 2^{j-1}D|x(t)|)d\mu \\ &\ge 2^{j-1}(1 + \eta)^{j-1}\int_{H_x} M(t, D|x(t)|)d\mu \ge (1 + \eta)^{j-1}2^{j-1}.\end{aligned}$$

This shows that $j - 1 \le \log_{1+\eta}^2 D$. Therefore $k \le D \cdot 2^{\log_{1+\eta}^2 D + 1}$, so $K_M < \infty$. ■

Lemma 8. *The space E_M is smooth if and only if $p(t, u)$ is continuous with respect to u, $t \in T$. The support functional of $u \in E_M$ is the function*

$$v(t) = \frac{p(t, u(t)/\|u\|_M)\operatorname{sign} u(t)}{\|p(t, u(t)/\|u\|_M)\|^o} \quad \mu\text{-a.e.}$$

Proof. It is analogous to the proof of Theorem 2.15 in [6]. ■

Lemma 9 ([8], Lemma 1.9). *If $x \in (L_M^o)$ and $\theta_M(x) > 0$, there exist two distinct singular functionals ϕ_i, $\|\phi_i\| = 1$, satisfying $\phi_i(u) = \xi_0(u), i = 1, 2$.*

Lemma 10. *If $p(t, u)$ is continuous with respect to u, $t \in T$, then μ-a.e. $u \in S(L_M)$ is a smooth point of $B(L_M)$ if and only if the support functional of u belongs to $S(L_N^o)$, and $u \ne \theta$.*

Proof. By Lemmas 8, 9, the proof is similar to the proof for Orlicz spaces (see [6], Theorem 2.17). ■

LEMMA 11. *$u \in S(L_M)$ is a smooth point of $B(L_M)$ if and only if*

(1) $|u(t)| < B(t)$ (a.e. $t \in T$);
(2) $\rho_M(u) = 1, p^-(|u|) \in E_N^o$;
(3) $p^-(t, |u(t)|) = p(t, |u(t)|)$, *where* $B(T) = \sup\{u \geq 0 : M(t, u) < \infty\}$.

Proof. Since Lemma 10 holds, we can repeat the proof from [16]. ∎

LEMMA 12 ([17]). *$x \in S(L_M^o)$ is an extreme point of $B(L_M^o)$ if and only if*

(*a*) *the set $K(x)$ consists of one element from $(0, +\infty)$,*
(*b*) *$kx(t) \in S_M$ for μ-a.e. $t \in T$, where $\{k\} = K(x)$.*

LEMMA 13. *$x_0 \in S(X)$ is a smooth point if and only if its support functional is an extreme point of $B(X^*)$.*

LEMMA 14 (see [18]). *If $M \in \Delta_2$ and $x_0 \in S(L_M)$ is an extreme point of $B(L_M^o)$, then x_0 is an H-point.*

2. The Orlicz norm

THEOREM 2.1. *The following conditions are equivalent:*

(i) *L_M^o is rough.*
(ii) *L_M^o is pointwise rough.*
(iii) *$M \notin \nabla_2$.*

Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii) If $M \in \nabla_2$, by Lemma 2(ii), we obtain that E_M^o is an Asplund space. By Corollary 2 in [12], p. 177, there is at least one F-differential point x_0 of E_M^o on $S(E_M^o)$. Now, we need only prove that x_0 is also an F-differential point of L_M^o.

Let $f_n \in (L_M^o)^*, f_n = y_n + \phi_n, \|f_n\| \to 1$ and $f_n(x_0) \to 1$, where $y_n \in L_N$ and ϕ_n is a singular functional,

$$f_n(x) = \int_T x(t) y_n(t) d\mu + \phi_n(x) \quad (x \in L_M^o).$$

Since $\int_T x_0(t) y_n(t) d\mu = \int_T x_0(t) y_n(t) d\mu + \phi_n(x_0) = f_n(x_0)$, $\underline{\lim}_{n\to\infty} \|y_n\|_N \geq 1$. If $\overline{\lim}_{n\to\infty} \|y_n\|_N > 1$, there exists $\lambda > 0$ such that $\|y_n\|_N > 1$ for an infinite number of n. Therefore, $\|\frac{y_n}{1+\lambda}\| > 1$ and $\rho_N(\frac{y_n}{1+\lambda}) \geq 1$. We have

$$\|f_n\| = \inf\{\xi > 0 : \rho_N(y_n/\xi) + \phi_n/\xi \leq 1\}. \tag{2.1}$$

It is easy to prove that $\|f_n\| \geq 1 + \lambda$. This contradiction proves that

$$\overline{\lim_{n\to\infty}} \|y_n\|_N \leq 1,$$

whence $\|y_n\|_N \to 1$. By $\lim_{n\to\infty} \|f_n\| = 1$ and (2.1), there exists $\xi_n \to 1$ satisfying

$$\rho_N(y_n/\xi_n) + \|\phi_n\|/\xi_n \leq 1.$$

Since $N \in \Delta_2$ and $\|y_n\|_N \to 1$, by Lemma 3, we get $\rho_N(y_n/\xi_n) \to 1$, so $\|\phi_n\|/\xi_n \to 0$, i.e, $\|\phi_n\| \to 0$. Since x_0 is an F-differential point of E_M^o, and $\|y_n\|_N \to 1$, so $\{y_n\}$ is a Cauchy sequence. Combining this fact with $\|\phi_n\| \to 0$, we get that $\{f_n\}$ is a Cauchy sequence, too. Hence x_0 is an F-differential point of L_M^o.

(iii)⇒(i) Suppose that $x_0 \in S(L_M^o), f = y + \phi \in (L_M^o)^*, \|f\| = 1$ and $f(x_0) = \|x\|_M^o = 1$. By Lemma 5,

$$\int_T N(t, y(t))d\mu + \|\phi\| = 1 \quad \text{for a.e. } t \in T, \tag{2.2}$$

Since $N \notin \Delta_2$, there exists $v \in S(L_N)$ and $\varepsilon_0 > 0$ satisfying

$$\|v\chi_{T_i}\| \geq \varepsilon_0, \qquad i = 1, 2, \ldots,$$

where $T_i = \{t \in T : |v(t)| \geq i\}$.

Defining $v_i = v\chi_{T\setminus T_i}/\|v\chi_{T\setminus T_i}\|$, we have $\|v_i\| = 1$. Define $f_i, g_i \in (L_M^o)^*$ by

$$f_i(x) = \int_{T\setminus T_i} x(t)y(t)d\mu + \int_{T_i} v_i x(t)d\mu + \phi(x),$$

$$g_i(x) = \int_{T\setminus T_i} x(t)y(t)d\mu - \int_{T_i} v_i x(t)d\mu + \phi(x) \quad (x \in L_M^o).$$

In view of

$$\int_{T\setminus T_i} x_0(t)y(t)d\mu \to \int_T x_0(t)y(t)d\mu$$

and

$$\int_{T_i} v_i x(t)d\mu \leq \int_{T_i} M(t, x_0(t))d\mu + \int_{T_i} N(t, v_i)d\mu \to 0,$$

we get $\underline{\lim}_{i\to\infty} f_i(x_0) = 1$, whence $\underline{\lim}_{i\to\infty}\|f_i\| \geq 1$. On the other hand, as $i \to \infty$, for any $\varepsilon > 0$, there exists T_{i_o} such that $\int_{T_{i_o}} N(t, v_{i_o})dt < \varepsilon$, a.e. $t \in T$. By (2.2), we know that

$$\int_{T\setminus T_i} N(t, y(t))dt + \int_{T_i} N(t, v_i(t))dt + \|\phi\| \leq \rho_N(y) + \|\phi\| + \varepsilon = 1 + \varepsilon, \qquad i \geq i_0.$$

Taking into account (2.1), $\|f_i\| \leq 1 + \varepsilon$. Therefore $\lim_{i\to\infty}\|f_i\| = 1$.

Similarly, $\lim_{i\to\infty} g_i(x_0)=1$, $\lim_{i\to\infty}\|g_i\|=1$. But we have proved that $\lim_{i\to\infty}\|v_i\chi_{T_i}\| = 1$, hence $\overline{\lim}_{i\to\infty}\|f_i - g_i\| = 2$. This shows that $\varepsilon(x_0) \geq 2$. In view of the arbitrariness of x_0, we get that $\inf\{\varepsilon(x) : x \in S(L_M^o)\} = 2$, and L_M^o is rough. ∎

THEOREM 2.2. *The space L_M^o is not strongly rough.*

Proof. By Lemma 1, Lemma 2(i) and Lemma 5, the proof is the same as for Orlicz spaces, see [10]. ∎

3. The Luxemburg norm

THEOREM 3.1. *The space L_M is pointwise rough if and only if $M \notin \nabla_2$.*

Proof. Sufficiency. A non-smooth point must be a rough point. Next we prove that every smooth point is a rough point.

Suppose that $x_0 \in S(L_M)$ is a smooth point. By Lemma 10, there exists a support functional of $x_0 - y_0$, which belongs to $S(L_N^o)$. By the proof of Lemma 11, we know that $K(y_0) \neq \emptyset$. Choose $k_0 > 0$ satisfying $\frac{1}{k_0}(1+\rho_N(k_0y_0)) = \|y_0\|_N^o = 1$. Since $N \notin \Delta_2$, we get from the proof of Theorem 2.1 that $v_i = v\chi_{T\setminus T_i}/\|v\chi_{T\setminus T_i}\|$, where $T_i = \{t \in T : |v(t)| \geq i\}$

satisfies $\|v_i\| = 1$. Therefore, as $i \to \infty$, for any $\varepsilon > 0$, there exists T_{i_o} such that $\int_{T_{i_o}} N(t, v_{i_o})dt < \varepsilon$ for a.e. $t \in T$ and $i_o > i$. Define $y_i, z_i \in L_N^o$ by

$$y_i(t) = \begin{cases} y_0(t), & t \in T \setminus T_i, \\ v_i/k_0, & t \in T_i, \end{cases} \qquad z_i(t) = \begin{cases} y_0(t), & t \in T \setminus T_i, \\ -v_i/k_0, & t \in T_i. \end{cases}$$

Then

$$\int_{T\setminus T_i} x_0(t)y_0(t)d\mu + \frac{v_i}{k_0}\int_{T_i} x_0(t)dt \to \int_T x_0(t)y_0(t)d\mu = 1.$$

Hence $\underline{\lim}_{i\to\infty}\|y_i\|_N^o \geq 1$. Moreover,

$$\begin{aligned}\|y_i\|_N^o &\leq \frac{1}{k_0}(1 + \rho_N(k_0 y_i)) = \frac{1}{k_0}(1 + \int_{T\setminus T_i} N(t, k_0 y_0(t))dt + \int_{T_i} N(t, v_i)dt) \\ &\leq \frac{1}{k_0}(1 + \rho_N(k_0 y_i)) + \varepsilon\frac{1}{k_0} = 1 + \varepsilon\frac{1}{k_0}.\end{aligned}$$

Therefore, $\overline{\lim}_{n\to\infty}\|y_i\|_N^o \leq 1$, whence $\lim_{i\to\infty}\|y_i\|_N^o = 1$.

In the same way one can prove that $\int_T x_0(t)z_i(t)d\mu \to 1, \|z_i\|_N^o \to 1$. But

$$\|y_i - z_i\|_N^o \geq \|y_i - z_i\|_N \geq \frac{2}{k_0}\|v_i\|_N.$$

Hence $\overline{\lim}_{n\to\infty}\|y_i - z_i\|_N^o \geq 2/k_0$, which shows that x_0 is a rough point of L_M.

Necessity. If $M \in \nabla_2$, by Lemma 2(ii), E_M is an Asplund space. Moreover, $\|f_n\|_M^o = \|y_n\|_M^o + \|\phi_n\|$, whence the proof is similar to the proof for Orlicz spaces. ■

THEOREM 3.2. *The space L_M is not strongly rough.*

Proof. Since Lemma 1 and Lemma 2 (1) hold, the proof is as in the case of Orlicz spaces, see [10]. ■

THEOREM 3.3. *The space L_M is rough if and only if $M \notin \nabla_2$ and $M \in \Delta_2$.*

Proof. Sufficiency. Since $M \in \Delta_2$, the support functional at any point x_0 on $S(L_M)$ belongs to $S(L_N)$. Since $M \notin \nabla_2$, by the proof of the sufficiency in Theorem 2.1, $\varepsilon(x_0) \geq 2/k_y$, where $\|y\|_N^o = 1, \int_T x_0(t)y(t)d\mu = 1$. Again by Lemma 7, we have $\sup\{k_y : \|y\|_N^o = 1\} = k < \infty$, whence $\inf\{\varepsilon(x) : x \in S(L_M)\} \geq 2/k$, which shows that L_M is rough.

Necessity. Roughness implies pointwise roughness, so it follows immediately from Theorem 2.1 that $M \notin \nabla_2$.

In order to prove that $M \in \Delta_2$, we consider the following four steps.

(I) If $M \notin \Delta_2$, for any positive integer n, there exists a smooth point x_0 on $S(L_M)$. From the proof of Lemma 11, we get y_0 in the support of x_0 satisfying $K(y_o) \neq \emptyset$, moreover we prove that $k_{y_o} > n$.

Assume that $M \notin \Delta_2$, take any $\varepsilon > 0$ ($\varepsilon = 1/2n$) and put

$$\delta(t) = \sup\left\{v \geq 0 : N(t, \varepsilon v) > \frac{\varepsilon}{\varepsilon + 1}N(t, v)\right\}.$$

Then $\int_T N(t, \delta(t))dt = \infty$. Indeed, otherwise

$$N(t, \varepsilon v(t)) \leq \frac{\varepsilon}{\varepsilon + 1}(N(t, v) + N(t, \delta(t))) \quad (\text{a.e. } t \in T, v \in \mathbf{R}),$$

so by Lemma 6, we get that $M \in \Delta_2$, a contradiction.

In this way, we get $v(t) \geq 0$, satisfying: $v(t)$ is a strict increase point of $q(t,v)$ with respect to v for each $t \in T$ and

$$N(t, \varepsilon v(t)) > \frac{\varepsilon}{\varepsilon+1} N(t, v(t)), \qquad \int_T N(t, v(t))dt > \frac{\varepsilon+1}{\varepsilon}.$$

Therefore $\int_T N(t, \varepsilon v(t))dt > 1$, whence $\|\varepsilon v\| > 1$. Moreover, $\|\varepsilon v\|^o \geq \|\varepsilon v\|$, so we obtain $\|\varepsilon v\|^o > 1$. Consequently there exists $\Omega \subset T$ such that $\|\varepsilon v|_\Omega\|^o = 1$. Take $k \in K(\varepsilon v|_\Omega)$, i.e.

$$1 = \|\varepsilon v\|^o = \frac{1}{k}(1 + \rho_N(k)).$$

Then $k > 1$ and

$$\begin{aligned} \frac{1}{k} + \rho_N(k\varepsilon v|_\Omega) &\leq \frac{1}{k}(1 + \rho_N(k\varepsilon v|_\Omega)) = \|\varepsilon v|_\Omega\|^o \\ &\leq \varepsilon\left(1 + \rho_N\left(\frac{1}{\varepsilon} \cdot \varepsilon v|_\Omega\right)\right) = \varepsilon(1 + \rho_N(v|_\Omega)) \\ &\leq \varepsilon\left(1 + \frac{\varepsilon+1}{\varepsilon} \int_\Omega N(t, \varepsilon v(t))d\mu\right) = \varepsilon + (1+\varepsilon)\rho_N(\varepsilon v|_\Omega), \end{aligned}$$

which shows that $\frac{1}{k} \leq \varepsilon + \varepsilon\rho_N(\varepsilon v|_\Omega) \leq 2\varepsilon$, i.e. $k\varepsilon v|_\Omega > n$.

Since $\int_T N(t, \delta(t)) = \infty, \varepsilon = 1/2n$ and $v(t) < \delta(t)$, we get $\int_T N(t, \varepsilon v(t)) < \infty$. Therefore $N(t, \varepsilon v(t))$ is bounded for μ-a.e. $t \in T$. Hence $\varepsilon v(t)$ is bounded for μ-a.e. $t \in T$. Therefore $\varepsilon v(t)|_\Omega \in E_N^0$. By the Hahn–Banach Theorem, there is an $x_0 \in S(L_M)$ such that $x_0(\varepsilon v|_\Omega) = \|\varepsilon v|_\Omega\|_N^0 = 1 = \|x_0\|_M$. Therefore $\varepsilon v|_\Omega$ is a support functional at x_0.

Setting $y_0 = \varepsilon v|_\Omega$, from the above proof we obtain that $k_{y_0} > n$.

(II) Since εv is a strict increase point of $q(t,v)$ with respect to v for each $t \in T$, we know that $N(t,v)$ is strictly convex with respect to v for each $t \in T$, ky_0 is a strictly convex point of $N(t,v)$. By Lemma 12, we obtain that y_0 is an extreme point of L_N^0. So by Lemma 13, x_0 is a smooth point of L_M.

(III) Put $y_n \in L_N^0, \|y_n\|_N^0 = 1$ and $\int_T x_0(t)y_n(t)d\mu \to 1$. Then $y_n(t) - y_0(t) \xrightarrow{\mu} 0$. By the same method of Lemma 14, we get $y_n(t) - y_0(t) \xrightarrow{\mu} 0$ on Ω.

Next we will prove that $y_n(t) - y_0(t) \xrightarrow{\mu} 0$ on $t \in T \setminus \Omega$. Since $\int_T y_n(t)\chi_\Omega x_0(t)d\mu \to 1$, we have $\|y_n\|_N^0 \to 1$. Notice that

$$1 \leftarrow \|y_n\|_N^0 = \frac{1}{k_n}(1 + \rho_N(k_n y_n \chi_{T\setminus\Omega})) \geq \|y_n \chi_{T\setminus\Omega}\|_N^0 + \rho_N(y_n \chi_{T\setminus\Omega}).$$

Therefore $\rho_N(y_n \chi_{T\setminus\Omega}) \to 0$, whence $y_n \to 0$ on $t \in T \setminus \Omega$.

(IV) We have $\overline{\lim}_{n\to\infty}\|y_n - y_0\|_N^0 \leq 4/k$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $e \subset t, \mu(e) < \delta$ implies $\rho_N(ky_0\chi_e) < \varepsilon$ and $\|y_0\chi_e\|_N^0 < \varepsilon$. Pick $e_0 \subset t$ such that $\mu(e_0) < \delta$, and $y_n(t)$ converges to $y_0(t)$ uniformly on $T \setminus e_0$. Then for n large enough, we have

$$\begin{aligned} 1 + \varepsilon \geq \|y_n\|_N^0 &= \frac{1}{k_n}(1 + \rho_N(k_n y_n \chi_{T\setminus e_0}) + \rho_N(k_n y_n \chi_{e_0})) \\ &\geq \|y_n \chi_{T\setminus e_0}\|_N^0 + \frac{\rho_N(k_n y_n \chi_{e_0})}{2k} \geq \|y_n \chi_{T\setminus e_0}\|_N^0 - \varepsilon \\ &\quad + \frac{\rho_N(k_n y_n \chi_{e_0})}{2k} \geq 1 - 2\varepsilon + \frac{\rho_N(k_n y_n \chi_{e_0})}{2k}, \end{aligned}$$

whence $\rho_N(k_n y_n \chi_{e_0}) < \varepsilon$. Therefore, for n large enough

$$\rho_N\left(\frac{k}{4}(y_n - y_0)\right) \le \rho_N\left(\frac{k}{4}(y_n - y_0)\chi_{T\setminus e_0}\right) + \frac{1}{2}\rho_N\left(\frac{k-k_n}{2} y_n \chi_e\right) + \frac{1}{4}\rho_N(k_n y_n \chi_{e_0}) + \frac{1}{4}\rho_N(k y_0 \chi_{e_0}) = o(\varepsilon).$$

Hence

$$\|y_n - y_0\|_N^0 \le \frac{4}{k}\left(1 + \rho_N\left(\frac{k}{4}(y_n - y_0)\right)\right) \le \frac{4}{k} + o(\varepsilon).$$

So $\lim_{n\to\infty} \|y_n - y_0\|_N^0 \le \frac{4}{k}$. This shows that $\varepsilon(x_0) \le 8/k \le 8/n$, whence $\inf\{\varepsilon(x) : x \in S(L_M)\} = 0$, i.e. L_M is not rough. ∎

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