LINEAR AND NON-LINEAR THEORY OF
GENERALIZED FUNCTIONS AND ITS APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 88
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

ON THE RELATION BETWEEN THE MASLOV–WHITHAM METHOD AND THE WEAK ASYMPTOTICS METHOD

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Abstract. We explain the relation between the weak asymptotics method introduced by the author and V. M. Shelkovich and the classical Maslov–Whitham method for constructing approximate solutions describing the propagation of nonlinear solitary waves.

It is well known that one of the most efficient methods for studying the special solutions of differential equations is the construction of asymptotic solutions, i.e., of functions satisfying the equation with a small discrepancy. Usually, the smallness is understood in the sense of the "maximum modulus" estimate, sometimes the remainder estimate can be refined by studying the behavior of its derivatives. For an important class of stabilizing solutions of nonlinear equations (such as solitons and kinks), a method for constructing such solutions was proposed by V. P. Maslov. Conceptually, this method originates from the Whitham–Kuzmak method, but has several principal distinctions related to the difference in the class of functions to which the solution belongs. Maslov also supplemented Whitham's approach with some ideas originating from the linear theory (the boundary-layer-type decompositions named as complex germs in Maslov's linear theory). The stabilizing asymptotic solutions which are the subject of Maslov's approach belong to the class of smooth functions of the variables $\tau \in \mathbb{R}^1$, $x \in \mathbb{R}^n$, $t \in [0,T]$, have the limits

$$\lim_{\tau\to\pm\infty} f(\tau,x,t) = f^{\pm}(x,t), \tag{1}$$

as $\tau \to \pm\infty$, and

$$\frac{\partial^{\alpha_1+|\alpha_2|}(f(\tau,x,t)-f^{\pm}(x,t))}{\partial t^{\alpha_1}\partial x_1^{\alpha_2^1}\dots\partial x_n^{\alpha_2^n}} = O(|\tau|^{-N}), \qquad |\tau|\to\infty,$$

2000 *Mathematics Subject Classification*: 35Q34, 76L05.
Key words and phrases: Whitham method, Maslov method, asymptotic solution, weak approximation.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc88-0-5

where $N > 0$ is a sufficiently large number (the same is true for the derivatives $f_\tau^{(\alpha)}$; moreover, $f^{\pm} = 0$ for $\alpha > 0$).

As an example, we use the well-known KdV equation ($n = 1$)

$$L_{KdV}u = \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = 0, \tag{2}$$

where $\varepsilon \to 0+0$ is a small parameter. We use this example, because it is the most popular equation with stabilizing solutions, and the explicit formulas for its famous solutions are widely known. In particular, the one-soliton solution of this equation has the form

$$u = \frac{6\beta^2}{\cosh^{-2}(\beta(x - Vt)/\varepsilon)} \stackrel{\text{def}}{=} 6\beta^2 \omega_0(\beta(x - Vt)/\varepsilon), \qquad V = 2\beta^2, \tag{3}$$

where $\beta > 0$ is an arbitrary constant.

The general one-soliton asymptotic solution of Eq. (2) has the form [1]

$$\begin{aligned} u = u(x, t, \varepsilon) &= u_0(x, t) + A(t)\omega_0\left(\beta(t)\frac{x - \varphi(t) - \varepsilon\varphi_1(t)}{\varepsilon}\right) \\ &+ \varepsilon\left(u_1(x, t)H\left(\frac{\varphi(t) + \varepsilon\varphi_1(t) - x}{\varepsilon}, t\right) + \omega_1\left(\beta(t)\frac{x - \varphi(t) - \varepsilon\varphi_1(t)}{\varepsilon}, t\right)\right) + O(\varepsilon^2), \end{aligned} \tag{4}$$

where $H(\tau, t)$ and $\omega_1(\tau, t)$ are stabilizing functions and $A(t)$, $\varphi(t)$, $\varphi_1(t)$, $\beta(t)$, $u_0(x, t)$, and $u_1(x, t)$ are smooth functions. The functions $A(t)$, $\varphi(t)$, and $u_0(x, t)$ satisfy the system of equations

$$\begin{cases} \dfrac{\partial u_0}{\partial t} + \dfrac{\partial u_0^2}{\partial x} = 0, \\ \dfrac{\partial \varphi}{\partial t} - 2u_0(\varphi, t) - \dfrac{2A(t)}{3} = 0, \\ \dfrac{d}{dt}(A^{3/2}) + \sqrt{6}A\dfrac{du_0(\varphi, t)}{dt} = 0, \\ \beta = (A/6)^{1/2}. \end{cases} \tag{5}$$

A solution of the form (4) satisfies the initial equation up to $O(\varepsilon)$, and the remainder can be refined by taking the terms with large powers of ε into account in (4). At this level of accuracy, the function $\varphi_1(t)$ cannot be determined, it also readily follows from the system (5) that the function $\varphi_1(t)$ is a correction of order ε to the solution $\varphi(t)$ and does not satisfy system (5) if we consider only first-order quantities.

All this is well known, but even in the case of the KdV equation, it is very difficult to construct an asymptotic solution describing the evolution of a multisoliton (two-soliton) solution with a smooth background (as the background we understand the smooth function $u_0(x, t) \not\equiv 0$) even at the level of formal asymptotic solutions.

But if we consider the KdV-type equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \varepsilon^2 \frac{\partial^3 u}{\partial x^3} = 0,$$

then the problem of constructing a formal asymptotic solution becomes much more complicated even in the case of polynomial functions $f(u)$, because, even in this case, the equation does not belong to the class of models integrable by the inverse scattering problem method.

The point is that, in the process of constructing the asymptotic one-soliton solution (4), one meets the standard ordinary differential equation, which, in the case of (2), takes the form

$$\dot{\omega}_0(-\varphi_t + 2u_0(\varphi, t)) + A\dot{\omega}_0^2 + \beta^2\ddot{\omega}_0 = 0. \tag{6}$$

Its solution is the well-known function from relation (3) under the condition that the second and the fourth equations in (5) are satisfied. But if we consider a two-soliton formal asymptotic solution, i.e., the case in which ω_0 is an unknown stabilizing function depending on the two arguments $(x - \varphi_1(t))/\varepsilon$ and $(x - \varphi_2(t))/\varepsilon$, then it is easy to see that this function is determined by a partial differential equation equivalent to the initial equation. In the case of (2), the solution of this equation (and correspondingly the solution to the initial equation) is given by an explicit formula describing the interaction of two KdV solitons.

In the nonintegrable case, there are very few results concerning the soliton interaction; some of them were obtained in [2, 3].

In the papers cited above, the formal asymptotic multisoliton solutions in the nonintegrable situation were analyzed by using the weak asymptotics method.

Definition 1. A *weak asymptotic solution* $\operatorname{mod} O_{\mathcal{D}'}(\varepsilon^2)$ of Eq. (3) is defined to be a sequence $u(x,t,\varepsilon) \in C^\infty\big((0,T); C^\infty(\mathbb{R}^1_x)\big)$ with $\varepsilon > 0$ such that the following relations hold for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}^1)$:

$$\frac{d}{dt}\int u\psi\,dx - \int u^2\frac{\partial\psi}{\partial x}\,dx = O(\varepsilon^2),$$

$$\frac{d}{dt}\int u^2\psi\,dx - \frac{4}{3}\int u^3\frac{\partial\psi}{\partial x}\,dx + 3\int\left(\varepsilon\frac{\partial u}{\partial x}\right)^2\frac{\partial\psi}{\partial x}\,dx = O(\varepsilon^2).$$

In fact, the weak asymptotic solution is constructed in the same class of stabilizing asymptotic solutions as in the Maslov–Whitham method, but this solution satisfies the equation whose remainder is small not in the sense of the uniform estimate but in the weak sense mentioned above.

Here the main notation is the notation of smallness in a weak sense. We say that the family of distributions $f(x,t,\varepsilon)$ depending on t and ε as parameters is of order $O_{D'}\varepsilon^\alpha$ if we have the following estimate for all the test functions $\psi(x)$:

$$\langle f(x,t,\varepsilon), \psi\rangle = O(\varepsilon^\alpha),$$

where the brackets $\langle , \rangle$ denote the action of a generalized function on the test function $\psi(x)$. Using this notation, we can rewrite the relations from Definition 1 as

$$L_{KdV}u = O_{D'}(\varepsilon^2), \qquad uL_{KdV}u = O_{D'}(\varepsilon^2).$$

Of course, this significantly simplifies the construction; namely, using Definition 1, we impose weaker conditions on the solution. In particular, in the framework of the weak asymptotics method for the one-soliton solution, we do not obtain Eq. (6) for determining the soliton profile, i.e., the function ω_0 in (3). Of course, we obtain system (5) but with arbitrary numerical coefficients and, only after the function from (3) is taken as ω_0, we obtain exactly system (5) as a consequence of Definition 1 [3, 4]. In other words, there is a gap: the Maslov–Whitham method does not permit constructing solutions describing the interaction between solitary nonlinear waves (it was not intended for those purposes), while the weak asymptotics method in the statement of Definition 1 leaves too much arbitrariness in the construction of the solution.

The goal of the present paper is to derive relations from which, as special cases, one can obtain the formulas arising in the Maslov–Whitham method and the formulas of the weak asymptotics method. We will write all formulas using the standard quadratic nonlinearity, but it is easy to rewrite them for arbitrary nonlinear $f(u)$, see also [2, 3, 4]. Those who do not believe in that can think only about the KdV equation.

We recall that the weak asymptotics method is based on the asymptotic expansions in the weak sense. Let $\omega(z) \in S(\mathbb{R}^1)$, $\beta = \beta(t)$, and $\varphi = \varphi(t)$ be continuous functions, and let $\psi(x) \in S(\mathbb{R}^1)$. We consider the expression $\langle \omega(\beta(x-\varphi)/\varepsilon), \psi\rangle$. We have

$$\left\langle \omega\left(\beta\frac{(x-\varphi)}{\varepsilon}\right), \psi\right\rangle = \int_{\mathbb{R}^1} \omega\left(\beta\frac{(x-\varphi)}{\varepsilon}\right)\psi(x)\,dx = \sum_{k\geq 0}\frac{\varepsilon^{k+1}}{\beta^{k+1}k!}(-1)^k\Omega_k\langle\delta^{(k)}(x-\varphi),\psi\rangle,$$

$$\Omega_k = \int_{\mathbb{R}^1} z^k\omega(z)\,dz. \tag{7}$$

The sum in this formula is understood as an asymptotic series in powers of ε, Ω_k are the kth moments of the function $\omega(z)$, and the entire expansion is called the *moment decomposition* in the theory of Colombeau algebras.

From this formula it is also easy to see that the possible correction $\varphi_1(t)$ adds a correction of order ε to all the summands, and this correction can be easily taken into account, but at this step, it is sufficient for us simply to write $\varphi(t)$.

In the weak asymptotics method (see Definition 1), we consider only several first terms from the above asymptotic series, i.e., we consider only several first moments of the asymptotic solution. In particular, some of these moments are coefficients in system (1). Since the function ω_0 in (4) is not determined in the framework of Definition 1, these coefficients also remain undetermined.

Moreover, it is easy to see that even if we know the entire asymptotic expansion in (6) (i.e., all the moments Ω_k of the function $\omega(z)$), we still cannot determine the function $\omega(z)$ itself. It is determined up to the summand all whose derivatives of the Fourier transform at zero are equal to zero. We can remove this ambiguity only by the assumption that the Fourier transform is analytic.

To find the relation between the formulas of the weak asymptotics method and the formulas of the Maslov–Whitham method, we note that the following relation holds for any test function $\psi(x)$ (from now on, we omit the numerical $(2\pi)^{\pm 1/2}$-type factors):

$$\begin{aligned}\left\langle \omega\left(\beta\frac{(x-\varphi)}{\varepsilon}\right), \psi\right\rangle &= \int \omega\left(\beta\frac{(x-\varphi)}{\varepsilon}\right)\psi(x)\,dx\\ &= \int \omega\left(\beta\frac{(x-\varphi)}{\varepsilon}\right)\int e^{ipx}\tilde{\psi}(p)\,dp dx,\end{aligned}$$

where $\tilde{\psi}(p) = \int e^{-ipx}\psi(x)\,dx$ is the Fourier transform.

This implies

$$\left\langle \omega\left(\beta\frac{(x-\varphi)}{\varepsilon}\right), \psi\right\rangle = \frac{\varepsilon}{\beta}\int \Omega\left(\frac{p\varepsilon}{\beta}\right)\tilde{\psi}(p)e^{-ip\varphi}\,dp, \tag{8}$$

where
$$\Omega\Big(\frac{p\varepsilon}{\beta}\Big) = \int \omega(z) e^{ip\varepsilon z/\beta}\, dz.$$

Obviously, (8) can be rewritten as
$$\Big\langle \omega\Big(\beta\frac{(x-\varphi)}{\varepsilon}\Big), \psi\Big\rangle = \frac{\varepsilon}{\beta}\Big[\Omega\Big(-\frac{i\varepsilon}{\beta}\frac{\partial}{\partial x}\Big)\psi\Big]\Big|_{x=\varphi(t)},$$
where $\Omega(\zeta) = F^{-1}_{z\to\zeta}\omega(z)$ is the inverse Fourier transform of the function $\omega(z)$. By assumption, we have $\Omega(\zeta) \in S(\mathbb{R}^1)$.

Example 1. We consider the KdV equation (2). Following the above remark about the phase shift, we seek its solution in the form
$$u = A\omega_0\Big(\beta\frac{x-\varphi}{\varepsilon}\Big), \tag{9}$$
where $\beta = \text{const}$ and $\varphi = Vt$.

It is well known that such a solution has the form (3), where $A = 6\beta^2$, $V = 2\beta^2$, and $\beta = \text{const} > 0$, and all this follows from Eq. (6), which, in this case, has the form
$$-V\dot{\omega}_0(z) + A(\dot{\omega}_0^2) + \beta^2\dddot{\omega}_0 = 0. \tag{10}$$
We substitute the solution of the form (9) into Eq. (2) and apply the result of this substitution to the test function $\psi(x)$. This means that we understand the equation in the sense of generalized functions, not approximately as in Definition 1, but exactly. We obtain
$$\Big\langle \frac{\partial}{\partial t}A\omega_0\Big(\beta\frac{(x-\varphi)}{\varepsilon}\Big), \psi\Big\rangle = iA\varphi_t\Big(\frac{i\beta}{\varepsilon}A\varphi_t(\Omega(\hat{p})\hat{p})\Big)\Big|_{x=\varphi},$$
where $\hat{p} = -\frac{i\varepsilon}{\beta}\frac{\partial}{\partial x}$.

Similarly, we obtain
$$\Big\langle \frac{\partial}{\partial x}A\omega_0\Big(\beta\frac{(x-\varphi)}{\varepsilon}\Big), \psi\Big\rangle = -\Big\langle A^2\omega_0^2\Big(\beta\frac{(x-\varphi)}{\varepsilon}\Big), \psi'_x\Big\rangle = -\frac{i\beta}{\varepsilon}A^2\Omega_{(2)}(\hat{p})\hat{p}\psi\Big|_{x=\varphi},$$
where $\Omega_{(2)}(\xi) = F^{-1}_{z\to\xi}\omega_0^2(\xi)$.

The last term in (2) can be rewritten as
$$\varepsilon^2\Big\langle \frac{\partial^3}{\partial x^3}A\omega_0, \psi\Big\rangle = -\varepsilon^2 A\Big\langle \omega_0\Big(\beta\frac{(x-\varphi)}{\varepsilon}\Big), \psi'''_{xxx}\Big\rangle = \varepsilon^{-1}i\beta^3 A(\Omega(\hat{p})\hat{p}^3)\psi\Big|_{x=\varphi},$$
Finally, Eq. (2) in the weak sense can be represented as
$$\Omega_L(-i\hat{p})\psi|_{x=\varphi} = (\varphi_t\Omega(\hat{p})\hat{p} - A\Omega_{(2)}(\hat{p})\hat{p} + \beta^2\Omega(\hat{p})\hat{p}^3)\psi(x)|_{x=\varphi} = 0, \tag{11}$$
where $\hat{p} = -\frac{i\varepsilon}{\beta}\frac{\partial}{\partial x}$ and the symbol of the operator Ω_L is the symbol of the operator in parentheses in the right-hand side of relation (11).

Since the function φ is *a priori* unknown, we can set (in general, this is only a sufficient condition for (11) to be satisfied)
$$[\varphi_t\omega(\hat{p})\hat{p} - A\Omega_{(2)}(\hat{p})\hat{p} + \beta^2\Omega(\hat{p})\hat{p}^3]\psi = 0. \tag{12}$$

As is easy to see, applying the Fourier transformation to the symbol of the operator in the left-hand side of (12)), we obtain exactly the standard equation (10), which describes the profile of the soliton (3).

It is of interest to note that, to ensure the uniqueness of the KdV equation solution in the sense mentioned above, one can replace the condition that the Fourier transform is analytic by the condition that there is no localization in Eq. (12) (i.e., the substitution $x = \varphi$ is used).

Now we note that the test function $\psi(x)$ is smooth. Therefore, the left-hand side of (12) can be expanded into an asymptotic series in powers of ε, and this expansion can be obtained by expanding the operators $\Omega(\hat{p})$ and $\Omega_{(2)}(\hat{p})$ by the Taylor formula at zero.

For example,

$$\Omega(\hat{p})\hat{p} = \sum_{k\ge 0} \frac{\Omega^{(k)}(0)}{k!}(\hat{p})^{k+1}.$$

Hence we have

$$(\Omega(\hat{p})\hat{p})\psi = \sum_{k\ge 0} (i)^k \Omega_k (-i)^{k+1} \beta^{k+1} \frac{\varepsilon^{k+1}}{k!} \psi^{(k+1)}(x),$$

where Ω_k are moments of the function $\omega_0(z)$ introduced in (7).

Thus, up to $O(\varepsilon^2)$, equating the coefficients of ε^1, we obtain from (12):

$$\varepsilon^1 : \quad \varphi_t \Omega_0 - A\Omega_{(2)(0)} = 0, \tag{13}$$

where

$$\Omega_{(2)(0)} = \int \omega_0^2(z)\,dz, \qquad \Omega_0 = \int \omega_0(z)\,dz,$$

Calculating the integrals ($\Omega_0 = 2$, $\Omega_{(2)(0)} = \frac{4}{3}$), we see that (13) exactly coincides with the second equation in system (5) for $u_0 \equiv 0$.

We continue the process of expansion. In the next order in ε, we obtain the relation

$$\varepsilon^2 : \quad \varphi_t \Omega'(0) - A\Omega'_{(2)}(0) = 0,$$

which holds automatically since $\omega_0(z)$ is even.

The next relation is given by the formula

$$\varepsilon^3 : \quad \varphi_t \Omega''(0) - A\Omega''_{(2)}(0) + 2\beta^2\Omega(0) = 0,$$

which implies the relation between β and A, $\beta = (A/6)^{1/2}$, already presented as a consequence of Eq. (10). The other relations obtained by expanding in ε can be considered similarly.

Until now, we considered the case in which the asymptotic solution constructed by the Maslov–Whitham method coincides with the exact solution. In this case, the term $u_0(x,t)$ and all the other terms in (4) admitting the estimate $O(\varepsilon)$ are absent. Hence the relations which follow from the second relation in Definition 1 (written in the form similar to (12)) either give the same result as ones derived from (12) or are fulfilled identically.

But if (4) contains terms of order ε, (namely $\varepsilon\omega_1$), then the above-proposed procedure of expanding in the Taylor formula results in an infinite chain of equations. This chain is arranged so that each first m equations contain more than m unknown functions, and hence such an approach cannot be used.

In general, this fact and its correction, i.e., the process of obtaining a closed system of equations in this case can be explained as follows.

The first relation in Definition 1 considered up to terms of the order of $\varepsilon\omega_2$ does not contain dispersion, and hence there is no soliton spirit in it.

Let the function $u_0(x,t)$ in (4) satisfy the first equation in system (5). Then the result of substitution of (4) into Eq. (2) can be represented up to $O(\varepsilon^2)$ (in the usual sense) as

$$L_{\mathrm{KdV}}u = \left(L_0(\hat\omega_0) + \varepsilon\frac{\delta L_0}{\delta\hat\omega_0}(\hat\omega_0)\omega_1\right)\Big|_{z=p(x-\varphi)/\varepsilon} + \varepsilon F + O(\varepsilon^2) = 0, \tag{14}$$

where $\hat\omega_0 = A(t)\omega_0(z)$,

$$L_0(\hat\omega_0) = \beta\left[-\varphi_t\frac{\partial\hat\omega_0}{\partial z} + \frac{\partial}{\partial z}(u_0(\varphi,t)+\hat\omega_0)^2 + \beta^2\frac{\partial^3\hat\omega_0}{\partial z^3}\right],$$

$\frac{\delta L_0}{\delta\hat\omega_0}(\hat\omega_0)$ is the linearization of the operator $L_0(\hat\omega_0)$ on $\hat\omega_0$, and F is a function containing β_t, φ_t, A_t, the derivatives of $u_0(x,t)$ for $x=\varphi$, and the derivatives of the function ω_0, see [1].

In the framework of Maslov–Whitham approach we have $L_0(\hat\omega_0) = 0$ (this implies not only expressions for ω_0 but also the second equation in system (5) and the relation $\beta = (A/6)^{1/2}$) and, as a consequence, the following equation for ω_1:

$$\frac{\delta L_0}{\delta\hat\omega_0}(\hat\omega_0)\omega_1 + F = 0.$$

This equation is solvable under the condition that F is orthogonal to the kernel of the operator $\frac{\delta L_0}{\delta\hat\omega_0}(\hat\omega_0)^*$, where $\left(\frac{\delta L_0}{\delta\hat\omega_0}(\hat\omega_0)\right)^*$ is a formally adjoint operator, see [1].

It is easy to verify that (see [1])

$$\left(\frac{\delta L_0}{\delta\hat\omega_0}(\hat\omega_0)\right)^*(\hat\omega_0+\hat u_0) = 0, \tag{15}$$

where $\hat u_0 = u_0(\varphi,t)$. Then, (14) implies the relation

$$\int_{\mathbb{R}^1}(\hat u_0+\hat\omega_0)F\,dz = 0,$$

which, in turn, gives the last equation in system (5).

All of that is well known, and now we will try to understand what is going on if we consider (14) in our weak sense. The first difference is that the orders w.r.t ε are changed. Namely, it is easy to see that all terms in the expression $L_0(\hat\omega_0)$ have the same properties as the function ω in (7), and thus $L_0(\hat\omega_0) = O_{\mathcal{D}'}(\varepsilon)$. Moreover, all other terms in (14) except the summand of order $O(\varepsilon^2)$ have the same order. To explain this, we consider the function from (7) and restrict our consideration only to the time-derivative. For all functions ω_1 with the same properties as ω, we have

$$\left\langle\frac{dA\omega\left(\frac{\beta(t)(x-\varphi(t))}{\varepsilon}\right)}{dt}, \psi(x)\right\rangle = \varepsilon^{-1}\langle\beta\varphi_t' + \beta_t'(x-\varphi(t))\omega', \psi\rangle + \frac{dA}{dt}\langle\omega,\psi\rangle = O_{\mathcal{D}'}(\varepsilon),$$

and

$$\left\langle \varepsilon\omega\Big(\frac{\beta(t)(x-\varphi(t))}{\varepsilon}\Big)\frac{dA\omega_1\big(\frac{\beta(t)(x-\varphi(t))}{\varepsilon}\big)}{dt}, \psi(x)\right\rangle = O_{D'}(\varepsilon),$$
$$\left\langle \varepsilon\omega_1\Big(\frac{\beta(t)(x-\varphi(t))}{\varepsilon}\Big)\frac{dA\omega\big(\frac{\beta(t)(x-\varphi(t))}{\varepsilon}\big)}{dt}, \psi(x)\right\rangle = O_{D'}(\varepsilon),$$

because the product $\varepsilon\omega\frac{dA\omega_1}{dt}$ has the same properties as the function ω from (7). The summands coming to the expression for F and for $\frac{\delta L_0}{\delta\hat{\omega}_0}(\hat{\omega}_0)\omega_1$ are made just in the same way.

So if we consider (14) in the weak sense, all the terms in the right-hand side except $O(\varepsilon^2)$ will be of the same order, and the moment decomposition can contain terms coming from ω_1. But they will be equal to zero because of Eq. (15). In other words, all the terms of order $O(\varepsilon)$ in the moment decomposition containing ω_1 have the form

$$\int_{\mathbb{R}^1}\left(\frac{\delta L_0}{\delta\hat{\omega}_0}(\hat{\omega}_0)\right)\omega_1 dz,$$

and they are equal to zero because of (15). This follows from the fact that the unity belongs to the kernel of the operator $\big(\frac{\delta L_0}{\delta\hat{\omega}_0}(\hat{\omega}_0)\big)^*$.

The same is true if we consider the second relation in Definition 1. In this case, all the terms containing ω_1 have the form

$$\int_{\mathbb{R}^1}\left((\hat{\omega}_0+\hat{u}_0)\frac{\delta L_0}{\delta\hat{\omega}_0}(\hat{\omega}_0)\right)\hat{\omega}_1 dz,$$

and they also are equal to zero because of (15).

As was already said above, we cannot restrict our consideration only to the first relation from the Definition 1. Even if we consider this relation up to higher degrees of ε, we will not derive a closed system of equations as a result of the moment decomposition, because, in contrast to the Maslov–Whitham method (see above), here it does not have the triangle form. Adding the second relation to the definition of the weak asymptotic solution, we obtain this system, and it coincides with system (5) (under the assumption that $\hat{\omega}_0$ is the exact soliton-type solution of the equation $L(\hat{\omega}_0)=0$).

Clearly, the right-hand side of (14) can be used to write the relations from Definition 1 in a weak sense, i.e., in the form (12).

For example, the second relation can be written in the form

$$\left\langle (u_0+\hat{\omega}_0+\varepsilon\omega_1)\Big(\frac{1}{\varepsilon}L_0(\hat{\omega}_0)+\frac{\delta L_0}{\delta\hat{\omega}_0}(\hat{\omega}_0)\omega_1+F\Big), \psi\right\rangle = O(\varepsilon^2). \tag{16}$$

Just as above, the left-hand side of this relation can be represented as some ε-pseudodifferential operator applied to the function $\psi(x)$.

It was shown that the expansion of this operator by the Taylor formula is equivalent to the moment decomposition of the left-hand side in the angular brackets in (16). The same can be performed for the first relation in Definition 1. If we omit the localization (see above) and consider the expansion of this operator in powers of ε, then we obtain the usual chain of equation from the Maslov–Whitham method. The same procedure applied to the second relation gives an equivalent chain.

Thus, Definition 1 gives (in the above weak sense) a closed system of equations for determining the soliton parameters, and this system does not contain the contribution of the correction ω_1. The last follows from the construction of the relations in Definition 1: they are modeling the solvability conditions for the equation for the first correction in the Maslov–Whitham method.

EXAMPLE 2. In the case of description of the soliton interaction, we construct the formal multisoliton (multiphase) asymptotic solution of Eq. (2) in the form [2, 3]

$$u = \omega\biggl(\beta_1\frac{(x-\varphi_1)}{\varepsilon}, \beta_2\frac{(x-\varphi_2)}{\varepsilon}, t, \varepsilon\biggr),$$

under the assumption that

$$\omega(z_1, z_2, t, \varepsilon) = +A_1\omega_0(z_1) + A_2\omega_0(z_2) + O(\varepsilon^N) \tag{17}$$

for $|z_1 - z_2| \geq c\varepsilon^{-\delta}$, where $c = \mathrm{const} > 0$, $\delta > 0$ is any positive number, and $N \gg 1$. In other words, we assume that, outside the domains of interaction ($\varphi_1 \sim \varphi_2$), the desired solution is close to the sum of solitary solitons.

Our goal is to demonstrate that the technique developed above can also be generalized to the case of the multisoliton ansatz.

Just as above, in this case, we have

$$\begin{aligned}\langle u, \psi\rangle &= \int \omega\biggl(\beta_1\frac{(x-\varphi_1)}{\varepsilon}, \beta_2\frac{(x-\varphi_2)}{\varepsilon}, t, \varepsilon\biggr)\psi(x)\,dx \\ &= \int \omega\biggl(\beta_1\frac{(x-\varphi_1)}{\varepsilon}, \beta_2\frac{(x-\varphi_2)}{\varepsilon}, t, \varepsilon\biggr)\tilde{\psi}(p)e^{ipx}\,dp\,dx.\end{aligned}$$

In contrast to the preceding cases, here it is possible to reduce the last integral to the form "ε-pseudodifferential operator applied to ψ" in two different ways by using the changes

$$\beta_1\frac{(x-\varphi_1)}{\varepsilon} = z \qquad \text{or} \qquad \beta_2\frac{(x-\varphi_2)}{\varepsilon} = z.$$

In both cases, we obtain

$$\langle u, \psi\rangle = \frac{\varepsilon}{\beta_1}\Omega_1\biggl(-\frac{i\varepsilon}{\beta_1}\frac{\partial}{\partial x}, t, \varepsilon\biggr)\psi\bigg|_{x=\varphi_1} = \frac{\varepsilon}{\beta_2}\Omega_2\biggl(-\frac{i\varepsilon}{\beta_2}\frac{\partial}{\partial x}, t, \varepsilon\biggr)\psi\bigg|_{x=\varphi_2}, \tag{18}$$

where

$$\begin{aligned}\Omega_1\biggl(-\frac{i\varepsilon}{\beta}\frac{\partial}{\partial x}, t, \varepsilon\biggr) &= \frac{\varepsilon}{\beta_1}\int e^{\frac{ip\varepsilon}{\beta_1}z}\omega\biggl(z, z\frac{\beta_2}{\beta_1} + \beta_2\rho, t, \varepsilon\biggr)\,dz \\ &= \frac{\varepsilon}{\beta_2}e^{-ip\varepsilon\rho}\int e^{\frac{ip\varepsilon}{\beta_2}z}\omega\biggl(z\frac{\beta_1}{\beta_2} - \beta_1\rho, z, t, \varepsilon\biggr)\,dz \stackrel{\mathrm{def}}{=} \frac{\varepsilon}{\beta_2}\Omega_2\end{aligned} \tag{19}$$

and $\rho = (\varphi_2 - \varphi_2)/\varepsilon$.

But in this case the equation symbol already contains the two unknown functions

$$\frac{\partial\omega}{\partial z_1} \qquad \text{and} \qquad \frac{\partial\omega}{\partial z_2}.$$

Indeed, for example, if $\beta_i = \mathrm{const}$, then

$$\frac{\partial u}{\partial t} = \frac{\partial\omega}{\partial t} + (-\varphi_{1t}\beta_1)\frac{\partial\omega}{\partial z_1} + (-\varphi_{2t}\beta_2)\frac{\partial\omega}{\partial z_2},$$

and the derivatives $\frac{\partial\omega}{\partial z_i}$ are associated with different symbols:

$$\left\langle \frac{\partial\omega}{\partial z_i}, \psi \right\rangle = \frac{\varepsilon}{\beta_1}\Omega_{11}\Big(-\frac{i\varepsilon}{\beta_1}\frac{\partial}{\partial x}, t, \varepsilon\Big)\psi\Big|_{x=\varphi_1} = \frac{\varepsilon}{\beta_2}\Omega_{21}\Big(-\frac{i\varepsilon}{\beta_2}\frac{\partial}{\partial x}, t, \varepsilon\Big), \tag{20}$$

where, for example,

$$\Omega_{11} = \frac{\varepsilon}{\beta_1}\int e^{\frac{ip\varepsilon}{\beta_1}z}\frac{\partial\omega}{\partial z_1}\Big(z, z\frac{\beta_2}{\beta_1} + \beta_2\rho, t, \varepsilon\Big)\,dz,$$

and Ω_{21} is defined similarly.

We use (19), (20) to represent the result of substitution of the function $u = u(x,t,\varepsilon)$ given by formula (17) into the KdV equation (understood in the weak sense) in the form

$$\Omega_{1L}\Big(-\frac{i\varepsilon}{\beta_1}\frac{\partial}{\partial x}, t, \varepsilon\Big)\psi\Big|_{x=\varphi_1} = 0 \tag{21}$$

or

$$\Omega_{2L}\Big(-\frac{i\varepsilon}{\beta_2}\frac{\partial}{\partial x}, t, \varepsilon\Big)\psi\Big|_{x=\varphi_2} = 0. \tag{22}$$

These representations are equivalent, their choice is determined by the choice of one of the changes in (18).

Just as in the case of a single soliton ansatz, we omit the substitution $x = \varphi_i$ and see that (21) or (22) is equivalent to the *partial differential* equation for determining the function $\omega(z_1, z_2, t, \varepsilon)$. In the case of the KdV equation itself, this function is obtained by the inverse scattering problem method. In the nonintegrable generalizations of the KdV equation, the formulas for the exact solution are unknown.

But we can try to construct an approximate (asymptotic in ε) solution of Eqs. (21) (or (22)), by expanding the operator in powers of ε (by the Taylor formula, see above). In this case, in [2, 3], the function ω was constructed in the form

$$\omega = g_1(z)\omega_0\Big(\beta_1\frac{x-\phi(t,\tau,\varepsilon)}{\varepsilon}\Big) + g_2\omega_0\Big(\beta_1\frac{x-\phi(t,\tau,\varepsilon)}{\varepsilon}\Big), \tag{23}$$

where $g_i = A_i + S_i(\tau)$, $A_i = \text{const}$, $S_i(\tau) \in S(\mathbb{R}^1)$, $\phi_i = \varphi_{i0}(t) + \varepsilon\varphi_{i1}(\tau)$, and $\tau = \beta_1(\varphi_{20}(t) - \varphi_{10}(t))/\varepsilon$. Here $(\varphi_{i0}(t))'_t$ is the velocity of the solitary soliton with amplitude A_i, $\varphi_{20}(0) < \varphi_{10}(0)$, $A_1 > A_2 > 0$; $\beta_i = (A_i/6)^{1/2} = \text{const}$.

We do not prove the existence of a solution of the form (23) but only construct a weak asymptotic solution by using Definition 1. One can say that this definition was constructed on the basis of the one-soliton solution, but, in this case the numbers of equations is also equal to the number of unknown functions arising in the two-soliton case. Of course, the existence of two-soliton solution for nonintegrable KdV type equations is unknown, but there is an example of a nonintegrable equation having an explicit exact solution, which describes the interaction of two nonlinear waves, see [5].

To derive the equations describing the weak asymptotic solution (23), we used the integral identities from Definition 1.

Obviously, the same results can be obtained by representing the relations from Definition 1 in the form of ε-pseudodifferential operators and then expanding the symbols of these operators in powers of ε. This means that, using the same relation which is treated in different ways, we obtain different necessary conditions for its solution to exist.

Acknowledgments. This work was supported by the Russian Foundation for Basic Research under grant no. 05-01-00912 and DFG project 436 RUS 13/895/0-1.

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