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CHARACTERIZATION OF SURJECTIVE CONVOLUTION OPERATORS ON SATO'S HYPERFUNCTIONS

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Abstract. Let $\mu \in \mathcal{A}(\mathbb{R}^d)'$ be an analytic functional and let T_μ be the corresponding convolution operator on Sato's space $\mathcal{B}(\mathbb{R}^d)$ of hyperfunctions. We show that T_μ is surjective iff T_μ admits an elementary solution in $\mathcal{B}(\mathbb{R}^d)$ iff the Fourier transform $\widehat{\mu}$ satisfies Kawai's slowly decreasing condition (S). We also show that there are $0 \neq \mu \in \mathcal{A}(\mathbb{R}^d)'$ such that T_μ is not surjective on $\mathcal{B}(\mathbb{R}^d)$.

1. Introduction. Surjectivity of convolution operators has been characterized in many classical spaces of (generalized) functions including spaces of real analytic and holomorphic functions, spaces of (ultra)differentiable functions and spaces of (ultra)distributions and Fourier hyperfunctions, respectively. A selection of corresponding papers is contained in the references and it is intended only as a first hint towards the corresponding literature (see [1–7, 9, 11–16, 18, 20]).

For convolution operators on Sato's space $\mathcal{B}(\mathbb{R}^d)$ of hyperfunctions however, a characterization of surjectivity seems to be missing. One reason for this might be that $\mathcal{B}(\mathbb{R}^d)$ does not admit a suitable topology and hence topological methods are not directly applicable.

A sufficient condition for the surjectivity of T_μ on $\mathcal{B}(\mathbb{R}^d)$ is Kawai's slowly decreasing condition (S) (see [11] and (1) below). Using Kawai's condition, Okada [18] proved that T_μ is surjective on $\mathcal{B}(\mathbb{R}^d)$ for any ultradistribution $\mu \neq 0$ with compact support. Moreover, (S) can also be used to show that T_μ is surjective on $\mathcal{B}(\mathbb{R}^d)$ for any $0 \neq \mu \in \mathcal{A}(\{0\})'$ (see [18]), while the present paper shows that there are $0 \neq \mu \in \mathcal{A}(\mathbb{R}^d)'$ such that T_μ

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is not surjective on $\mathcal{B}(\mathbb{R}^d)$. In fact, we will show that (S) is also a necessary condition for surjectivity and we will clarify the connection of surjectivity and the existence of elementary solutions for T_μ. More precisely we will prove the following

Main Theorem. *For $\mu \in \mathcal{A}(\mathbb{R}^d)'$ the following are equivalent:*

a) $T_\mu : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ *is surjective.*
b) T_μ *admits an elementary solution* $E \in \mathcal{B}(\mathbb{R}^d)$.
c) $\widehat{\mu}$ *satisfies the following condition* (S)*: For any* $\delta > 0$ *there is* $C > 0$ *such that for any* $t \in \mathbb{R}^d$ *with* $|t| \geq C$ *there is* $\zeta \in \mathbb{C}^d$ *such that*

$$|\zeta - t| \leq \delta|t| \quad \text{and} \quad |\widehat{\mu}(\zeta)| \geq e^{-\delta|\zeta|}. \tag{1}$$

Recall that $E \in \mathcal{B}(\mathbb{R}^d)$ is an elementary solution for T_μ if $T_\mu(E) = \delta$, where δ denotes Dirac's δ-distribution.

Notice that we do not need a condition on the location of zeroes of $\widehat{\mu}$ for the characterization in our Main Theorem. This is different from the characterization of surjective convolution operators on Fourier hyperfunctions and on modified Fourier hyperfunctions (see [13] and [20]).

The implication "c) $\Rightarrow$ a)" was proved in [11] using convolution operators on holomorphic functions defined on tube domains. We will give a different proof here which is based on the space $\mathcal{R}(\mathbb{D}^d)$ of modified Fourier hyperfunctions (see 2.1 and 3.3). This space is also the main tool when proving that b) implies c) (see 3.1).

Using Baire's category theorem we then show that there are $0 \neq \mu \in \mathcal{A}(\mathbb{R}^d)'$ such that T_μ is not surjective on $\mathcal{B}(\mathbb{R}^d)$ (see 3.4).

2. Preliminaries. We will recall some basic notions and results concerning hyperfunctions and modified Fourier hyperfunctions in this section (see [11], [10] and [19] for a systematic study) and then show how topological methods can be used to characterize surjectivity of convolution operators on $\mathcal{B}(\mathbb{R}^d)$ (see 2.1).

Hyperfunctions may be defined as boundary values of holomorphic functions as follows: let $\mathbb{C}^d_\sharp := \{z \in \mathbb{C}^d \mid \Im(z_l) \neq 0 \text{ for any } l\}$ and $\mathbb{C}^d_{\sharp,j} := \{z \in \mathbb{C}^d \mid \Im(z_l) \neq 0 \text{ for any } l \neq j\}$. Then the space of hyperfunctions is given by

$$\mathcal{B}(\mathbb{R}^d) := \mathcal{H}(\mathbb{C}^d_\sharp)/\Big(\sum_{j \leq d} \mathcal{H}(\mathbb{C}^d_{\sharp,j})\Big). \tag{2}$$

We always assume in this paper that $\mu \in \mathcal{A}(\mathbb{R}^d)'$ is fixed. $\widehat{\mu}$ then is an entire function and there is K such that for any $\varepsilon > 0$ there is C_ε such that

$$|\widehat{\mu}(z)| \leq C_\varepsilon e^{\varepsilon|z| + K|\Im(z)|}. \tag{3}$$

For $u \in \mathcal{H}(\mathbb{C}^d_\sharp)$ let

$$\mu * u(z) := \langle {}_\xi\mu, u(z - \xi)\rangle \quad \text{ if } z \in \mathbb{C}^d_\sharp. \tag{4}$$

The convolution operator T_μ on $\mathcal{B}(\mathbb{R}^d)$ is then defined by

$$T_\mu([u]) := [\mu * u] \quad \text{ if } [u] \in \mathcal{B}(\mathbb{R}^d). \tag{5}$$

In the following, we will usually write $u \in \mathcal{B}(\mathbb{R}^d)$ and we will thus not distinguish in notation between the defining function and its equivalence class in $\mathcal{B}(\mathbb{R}^d)$.

Similarly, the space $\mathcal{R}(\mathbb{D}^d)$ of modified Fourier hyperfunctions on the radial compactification $\mathbb{D}^d$ of $\mathbb{R}^d$ may be defined as a space of boundary values using in (2) spaces of exponentially increasing functions defined as follows (see [19, p. 257 ff]): let $\mathbb{Y}^d$ be the radial compactification of $\mathbb{C}^d = \mathbb{R}^{2d}$. For W open in $\mathbb{Y}^d$ let

$$\mathcal{O}_{inc}(W) := \{f \in \mathcal{H}(\mathbb{C}^d \cap W) \mid \sup_Z |f(z)|e^{-|z|/j} < \infty \text{ for any } Z \Subset W \text{ and any } j\}$$

where $Z \Subset W$ means that Z is relatively compact in W (and open). Let $U := \{z \in \mathbb{C}^d \mid |\Im(z)| < (1 + |\Re(z)|^2)^{1/2}/2\}$, $V := (\overline{U})^\circ \subset \mathbb{Y}^d$, $V_\sharp := \{z \in V \mid \Im(z_l) \neq 0 \text{ for any } l\}$ and $V_{\sharp,j} := \{z \in V \mid \Im(z_l) \neq 0 \text{ for any } l \neq j\}$. The modified Fourier hyperfunctions on $\mathbb{R}^d$ are defined by

$$\mathcal{R}(\mathbb{D}^d) := \mathcal{O}_{inc}(V_\sharp)/\Big(\sum_{j \leq d} \mathcal{O}_{inc}(V_{\sharp,j})\Big). \tag{6}$$

Alternatively, $\mathcal{R}(\mathbb{D}^d)$ may be defined by duality (see [19, Thm. 4.25]), namely as the dual space $\widetilde{P}_*(\mathbb{D}^d)'$ of $\widetilde{P}_*(\mathbb{D}^d) := \liminf_{j\to\infty} \widetilde{P}_{*,j}$, where

$$\widetilde{P}_{*,j} := \{f \in \mathcal{H}(W_j) \mid \|f\|_j := \sup_{z \in W_j} |f(z)| \exp(|z|/j) < \infty\}$$

for $W_j := \{z \in \mathbb{C}^d \mid |\Im(z)| < (1 + |\Re(z)|^2)^{1/2}/j\}$. The duality of $\mathcal{R}(\mathbb{D}^d)$ and $\widetilde{P}_*(\mathbb{D}^d)$ is defined for $u \in \mathcal{R}(\mathbb{D}^d)$ and $f \in \widetilde{P}_*(\mathbb{D}^d)$ by

$$(u, f) = \sum_{\sigma \in \{1,-1\}^d} \operatorname{sign}(\sigma) \int_{\gamma_{j,\sigma}} u(z) f(z) dz \tag{7}$$

where $\gamma_{j,\sigma}$ is the path defined by $x + ie_\sigma(1 + |x|^2)^{1/2}/j$ for a unit vector e_σ in the corresponding orthant and large j. By (6) and (2), the restriction

$$|_{\mathbb{R}^d} : \mathcal{R}(\mathbb{D}^d) \to \mathcal{B}(\mathbb{R}^d) \text{ is well defined and } \mathcal{R}(\mathbb{D}^d)|_{\mathbb{R}^d} = \mathcal{B}(\mathbb{R}^d) \tag{8}$$

(see [19, sect. 1.2 i)]). By [19, 3.2.3] we have

$$\ker(\ |_{\mathbb{R}^d}) = \mathcal{R}(\mathbb{D}^d \setminus \mathbb{R}^d) = \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)' \tag{9}$$

where $\widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d) := \liminf_{j\to\infty} \widetilde{P}_{*,\infty,j}$ for

$$\widetilde{P}_{*,\infty,j} := \{f \in \mathcal{H}(W_{\infty,j}) \mid \|f\|_{\infty,j} := \sup_{z \in W_{\infty,j}} |f(z)| \exp(|z|/j) < \infty\}$$

where $W_{\infty,j} := \{z \in W_j \mid |z| > j\}$.

We may define $T_\mu(u)$ for $u \in \mathcal{R}(\mathbb{D}^d)$ by using (4) and (5). Then $T_\mu(u) \in \mathcal{R}(\mathbb{D}^d)$ and we have

$$T_\mu(u)|_{\mathbb{R}^d} = T_\mu(u|_{\mathbb{R}^d}) \quad \text{ if } u \in \mathcal{R}(\mathbb{D}^d). \tag{10}$$

For $f \in \widetilde{P}_*(\mathbb{D}^d)$ and large j we get by (7)

$$
\begin{aligned}
(T_\mu(u), f) &= \sum_{\sigma\in\{1,-1\}^d} \operatorname{sign}(\sigma)\Big\langle {}_\xi\mu, \int_{\gamma_{j,\sigma}} u(z-\xi)f(z)dz\Big\rangle \\
&= \sum_{\sigma\in\{1,-1\}^d} \operatorname{sign}(\sigma)\Big\langle {}_\xi\mu, \int_{\widetilde{\gamma}_{j,\sigma}} u(z)f(z+\xi)dz\Big\rangle \\
&= \sum_{\sigma\in\{1,-1\}^d} \operatorname{sign}(\sigma)\Big\langle {}_\xi\mu, \int_{\gamma_{j,\sigma}} u(z)f(z+\xi)dz\Big\rangle = (u, \check{\mu} * f) \quad \text{if } u \in \mathcal{R}(\mathbb{D}^d)
\end{aligned} \tag{11}
$$

where $\widetilde{\gamma}_{j,\sigma}$ is defined by $x + ie_\sigma(1+|x|^2)^{1/2}/j - \xi$ and j is large.

We finally notice that the Fourier transform

$$\widehat{f}(z) := \int f(x)e^{-i\langle x,z\rangle}dx, \qquad f \in \widetilde{P}_*(\mathbb{D}^d),$$

defines a topological isomorphism in $\widetilde{P}_*(\mathbb{D}^d)$ (and hence in $\mathcal{R}(\mathbb{D}^d)$ by duality, see [19]). Moreover,

$$\widehat{\check{\mu} * f} = \widehat{\check{\mu}}\widehat{f} \quad \text{if } f \in \widetilde{P}_*(\mathbb{D}^d). \tag{12}$$

Lemma 2.1. *Let $\mu \in \mathcal{A}(\mathbb{R}^d)'$. Then the following are equivalent:*

a) *T_μ is surjective on $\mathcal{B}(\mathbb{R}^d)$.*

b) *The mapping*

$$S : \mathcal{R}(\mathbb{D}^d) \times \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)' \to \mathcal{R}(\mathbb{D}^d), \qquad S(\nu,\eta) := T_\mu(\nu) - \eta,$$

is surjective.

c) *B is bounded in $\widetilde{P}_*(\mathbb{D}^d)$ if B is bounded in $\widetilde{P}_*(\mathbb{D}^d\setminus\mathbb{R}^d)$ and $\widehat{\mu}\widehat{B}$ is bounded in $\widetilde{P}_*(\mathbb{D}^d)$.*

Proof. "a) $\Rightarrow$ b)" For $\kappa \in \mathcal{R}(\mathbb{D}^d)$ there is $\nu \in \mathcal{B}(\mathbb{R}^d)$ by a) such that $T_\mu(\nu) = \kappa|_{\mathbb{R}^d}$. By (8) there is $\overline{\nu} \in \mathcal{R}(\mathbb{D}^d)$ such that $\overline{\nu}|_{\mathbb{R}^d} = \nu$. Then

$$T_\mu(\overline{\nu})|_{\mathbb{R}^d} = T_\mu(\overline{\nu}|_{\mathbb{R}^d}) = T_\mu(\nu) = \kappa|_{\mathbb{R}^d}$$

by (10) and hence $S(\overline{\nu}, h) = \kappa$ for $T_\mu(\overline{\nu}) - \kappa =: h \in \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)'$ by (9).

"b) $\Rightarrow$ a)" For $u \in \mathcal{B}(\mathbb{R}^d)$ there is $\overline{u} \in \mathcal{R}(\mathbb{D}^d)$ such that $\overline{u}|_{\mathbb{R}^d} = u$ by (8). By assumption we can find $(\nu, h) \in \mathcal{R}(\mathbb{D}^d) \times \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)'$ such that $S(\nu,h) = \overline{u}$ and hence

$$u = \overline{u}|_{\mathbb{R}^d} = T_\mu(\nu)|_{\mathbb{R}^d} - h|_{\mathbb{R}^d} = T_\mu(\nu|_{\mathbb{R}^d})$$

by (9) and (10). This shows a).

"b) $\Leftrightarrow$ c)" Since $\mathcal{R}(\mathbb{D}^d) = \widetilde{P}_*(\mathbb{D}^d)'_b$ and $\mathcal{R}(\mathbb{D}^d \setminus \mathbb{R}^d) = \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)'_b$ are (FS)-spaces, S is surjective if and only if ${}^tS : \widetilde{P}_*(\mathbb{D}^d) \to \widetilde{P}_*(\mathbb{D}^d) \times \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)$ is injective with closed range if and only if B is bounded in $\widetilde{P}_*(\mathbb{D}^d)$ if ${}^tS(B)$ is bounded in $\widetilde{P}_*(\mathbb{D}^d) \times \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)$ (see [17, section 26]). Since ${}^tS(f) = (\check{\mu} * f, -f)$ for $f \in \widetilde{P}_*(\mathbb{D}^d)$ by (11) and since the Fourier transformation is a topological isomorphism this proves the claim by (12). ∎

3. The main results. In this section we always denote $\mu \in \mathcal{A}(\mathbb{R}^d)'$. We will prove the Main Theorem by means of two lemmata starting with the implication "b) $\Rightarrow$ c)":

Lemma 3.1. *$\widehat{\mu}$ satisfies (S) if T_μ admits an elementary solution $E \in \mathcal{B}(\mathbb{R}^d)$.*

Proof. Let $\overline{E} \in \mathcal{R}(\mathbb{D}^d)$ be an extension of E existing by (8). Then we have by (10)

$$T_\mu(\overline{E})|_{\mathbb{R}^d} = T_\mu(E) = \delta$$

and hence there is $H \in \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)'$ by (9) such that

$$\delta = T_\mu(\overline{E}) + H. \tag{13}$$

If (S) is not true there is $0 < \delta \le 1$ such that for any $l \in \mathbb{N}$ there is $t_l \in \mathbb{R}^d$ with $|t_l| \ge l$ such that

$$|\widehat{\mu}(-\zeta)| \le e^{-\delta|\zeta|} \quad \text{if } |\zeta - t_l| \le \delta |t_l|. \tag{14}$$

For $\eta, \zeta \in \mathbb{C}^d$ let $\langle \eta, \zeta\rangle := \sum \eta_k \zeta_k$ and set

$$g_l(z) := (2\pi/(\delta|t_l|))^{d/2} e^{-\langle z-t_l, z-t_l\rangle/(2\delta|t_l|)}, l \in \mathbb{N},$$

and notice that $g_l \in \widetilde{P}_*(\mathbb{D}^d)$ and

$$g_l = \widehat{f_l} \quad \text{for } f_l(z) := e^{i\langle z,t_l\rangle - \langle z,z\rangle\delta|t_l|/2} \in \widetilde{P}_*(\mathbb{D}^d).$$

Since $\overline{E} \in \mathcal{R}(\mathbb{D}^d) = \widetilde{P}_*(\mathbb{D}^d)'$ and $H \in \widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)'$, for any $j \in \mathbb{N}$ there is C_1 such that by (13), (11) and the Fourier inversion formula

$$\begin{aligned} 1 = f_l(0) &= |\langle T_\mu(\overline{E}) + H, f_l\rangle| = |(2\pi)^{-d}\langle \hat{\hat{\overline{E}}}, \widehat{\mu} g_l\rangle + \langle H, f_l\rangle| \\ &\le C_1(\|\widehat{\mu} g_l\|_j + \|f_l\|_{\infty,j}) \quad \text{for any } l \in \mathbb{N}. \end{aligned} \tag{15}$$

We will show however that both terms of the right hand side of (15) tend to 0 as $l \to \infty$ if j is large:

I) If $|z - t_l| \le \delta|t_l|/2$ then

$$|\Im(z)| \le |z - t_l| \le \delta|t_l|/2 \quad \text{and} \quad 2|t_l| \ge |z| \ge (1 - \delta/2)|t_l| \ge |t_l|/2$$

since $\delta \le 1$. We thus get for large l by (14)

$$\begin{aligned} |\widehat{\mu}(-z) g_l(z)| e^{|z|/j} &\le C_2 e^{(-\delta + 1/j)|z| + \Im(z)^2/(2|t_l|\delta)} \\ &\le C_2 e^{-\delta|z|/2 + \delta|t_l|/8} \le C_2 e^{-\delta|t_l|/8} \end{aligned}$$

if $j \ge 2/\delta$.

II) Let $z \in W_j$ and $|\Re(z)| \ge 1$ (and hence $|\Im(z)| \le 2|\Re(z)|/j$). If $|z - t_l| \ge \delta|t_l|/2$ we get by (3) (for $\varepsilon := 1/j$)

$$\begin{aligned} |\widehat{\mu}(-z) g_l(z)| e^{|z|/j} &\le C_3 e^{K|\Im(z)| + 2|z|/j + |\Im(z)|^2/(|t_l|\delta) - |z-t_l|^2/(2\delta|t_l|)} \\ &\le C_3 e^{2(K+1)|z|/j + 4|z|^2/(j^2|t_l|\delta) - |z-t_l|^2/(2\delta|t_l|)} \\ &\le C_3 e^{(2K+2+8/(j\delta))|t_l|/j + 2(K+1)|z-t_l|/j + (8j^{-2} - 1/2)|z-t_l|^2/(|t_l|\delta)} \\ &\le C_3 e^{(2K+2+8/(j\delta))|t_l|/j + 2(K+1)|z-t_l|/j - |z-t_l|^2/(4|t_l|\delta)} \\ &\le C_3 e^{[(2+\delta)(K+1)/j + 8/(j^2\delta) - \delta/16]|t_l|} \le C_3 e^{-\delta|t_l|/32} \end{aligned} \tag{16}$$

if j is large. If $z \in W_j$ and $|\Re(z)| \le 1$ then $|z| \le 2$ and the right hand side of (16) tends to 0 as $l \to \infty$. We have thus shown that $\|\widehat{\mu} g_l\|_j \to 0$ as $l \to \infty$ if j is large.

III) Let $z \in W_{\infty,j}$ (and hence $j \le 2|\Re(z)|$ and $|\Im(z)| \le 2|\Re(z)|/j$ if j is large). Then

$$\begin{aligned}|f_l(z)| &\le e^{[|\Im(z)|+|\Im(z)|^2\delta/2-|\Re(z)|^2\delta/2]|t_l|}\\ &\le e^{[2|\Re(z)|/j+(4/j^2-1/2)\delta|\Re(z)|^2]|t_l|}\\ &\le C_4 e^{-j^2\delta|t_l|/8} \quad \text{if } j \text{ is large.}\end{aligned}$$

This shows that $\|f_l\|_{\infty,j} \to 0$ if j is large. ■

To show that (S) is sufficient for the surjectivity of T_μ we need the following variant of Harnack's inequality (see [7, 3.1]):

LEMMA 3.2. *Let $F(z), G(z)$ and $F(z)/G(z)$ be holomorphic when $|z| < R$, $z \in \mathbb{C}^d$. For $|z| < R$ we then get*

$$|F(z)/G(z)| \le \sup_{|\eta|<R} |F(\eta)|(\sup_{|\eta|<R} |G(\eta)|)^{2|z|/(R-|z|)}|G(0)|^{-(R+|z|)/(R-|z|)}.$$

LEMMA 3.3. *T_μ is surjective in $\mathcal{B}(\mathbb{R}^d)$ if $\widehat{\mu}$ satisfies (S).*

Proof. a) We will show the criterion of 2.1c). Let $B \subset \widetilde{P}_*(\mathbb{D}^d)$ such that B is bounded in $\widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)$ and $\widehat{\mu}\widehat{B}$ is bounded in $\widetilde{P}_*(\mathbb{D}^d)$. Since $\widetilde{P}_*(\mathbb{D}^d)$ is a compact injective inductive spectrum (and hence regular), there is j_1 such that any germ $g = \widehat{\mu}\widehat{f} \in \widehat{\mu}\widehat{B}$ has a holomorphic extension $g_1 \in \mathcal{H}(W_{j_1})$ contained in a bounded set in $\widetilde{P}_{*,j_1}$. Since B is bounded in $\widetilde{P}_*(\mathbb{D}^d \setminus \mathbb{R}^d)$ (which is also a regular inductive spectrum) there is j_2 such that $B|_{W_{j_2}} \subset \widetilde{P}_{*,\infty,j_2}$ and therefore there is j_3 such that $\widehat{f}$ has an extension $f_1 \in \mathcal{H}(W_{j_3})$ for any $f \in B$ (see the proof of [19, 4.1.2]). We may assume that $j_1 = j_3$. By the identity theorem this implies that any germ $g = \widehat{\mu}\widehat{f} \in \widehat{\mu}\widehat{B}$ has a holomorphic extension $g_1 = \widehat{\mu}f_1$ contained in a bounded set in $\widetilde{P}_{*,j_1}$ where $f_1 \in \mathcal{H}(W_{j_1})$. Condition (S) and Lemma 3.2 now easily imply that $\widehat{B}$ (and hence B) is bounded in $\widetilde{P}_*(\mathbb{D}^d)$. The argument is included for the convenience of the reader:

Fix $t \in \mathbb{R}^d$ with large $|t|$ and choose ζ for $\delta = 1/(8j_2)$ by (S) where $j_2 \ge j_1$ is to be determined later. Then $F := f_1(\zeta + \ \cdot\ )\widehat{\mu}(\zeta + \ \cdot\ )$ and $G := \widehat{\mu}(\zeta + \ \cdot\ )$ satisfy the assumptions of 3.2 for $R := |t|/(4j_2)$. If $|\xi - t| \le |t|/(16j_2)$ then $z_0 := \xi - \zeta$ satisfies

$$|z_0| \le 3|t|/(16j_2) \text{ and } \frac{R+|z_0|}{R-|z_0|} \le 7 \text{ and } \frac{2|z_0|}{R-|z_0|} \le 6$$

and by 3.2 and (3) we thus get C_1 and C_2 (independent of j_2) such that

$$|f_1(\xi)| = |f_1(\zeta + z_0)| \le C_1 e^{-|\xi|/(4j_1)+C_2|\xi|/j_2} \le C_1 e^{-|\xi|/(8j_1)} \quad \text{if } j_2 \ge 8C_2 j_1$$

since $\widehat{\mu}f_1$ is contained in a bounded set in $\widetilde{P}_{*,j_1}$. This proves the claim. ■

Combining Lemma 3.1 and Lemma 3.3 we have proved the Main Theorem since it is evident that a) implies b).

It is known that (S) is satisfied for all $0 \ne \mu \in \mathcal{A}(\{0\})'$ and for all ultradistributions $\mu \ne 0$ with compact support (see [18]). For $\mu \in \mathcal{A}(\mathbb{R}^d)'$ however this does not hold as we will prove now:

THEOREM 3.4. *For any $d \in \mathbb{N}$ there is $0 \ne \mu \in \mathcal{A}(\mathbb{R}^d)'$ such that $T_\mu : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ is not surjective.*

Proof. We reason by contradiction, assuming that there is $d \in \mathbb{N}$ such that $T_\mu : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ is surjective for any $0 \neq \mu \in \mathcal{A}(\mathbb{R}^d)'$.

a) We may assume w.l.o.g. that $d = 1$.

Proof: Since T_μ is surjective on $\mathcal{B}(\mathbb{R}^d)$ for any $0 \neq \mu \in \mathcal{A}(\mathbb{R}^d)'$ this also holds for $\mu := \nu(x_1) \otimes \delta(x')$ where $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $0 \neq \nu \in \mathcal{A}(\mathbb{R})'$. Hence, $\widehat{\mu}(z) = \widehat{\nu}(z_1)$ satisfies (S) (on $\mathbb{C}^d$) by the Main Theorem and therefore $\widehat{\nu}(z_1)$ satisfies (S) (on $\mathbb{C}$): in fact, we may apply (S) for $t := (t_1, 0)$ and notice that ζ then satisfies $|\zeta| \leq |\zeta_1|/(1-\delta)$. Thus, T_ν is surjective on $\mathcal{B}(\mathbb{R})$ for any $0 \neq \nu \in \mathcal{A}(\mathbb{R})'$ by the Main Theorem.

b) Let T_μ be surjective on $\mathcal{B}(\mathbb{R})$ for any $0 \neq \mu \in \mathcal{A}(\mathbb{R})'$. Then

$$\text{any } 0 \neq f \in \mathcal{F}(\mathcal{A}([-1,1])') =: E \text{ satisfies } (S) \tag{17}$$

by the Main Theorem, where

$$E = \{g \in \mathcal{H}(\mathbb{C}) \mid \forall \delta > 0 : p_\delta(g) := \sup_{\mathbb{C}} |g(z)| e^{-|\Im(z)| - \delta|z|} < \infty\}$$

by the Paley-Wiener theorem (see e.g. [8]).

Let $E_1 := \{g \in E \mid g(0) = 1\}$. E_1 is a complete metrizable space since E_1 is a closed subset of the Fréchet space E.

For any $\eta > 0$ there are $g \in E_1$, $\varepsilon > 0$ and $C_1 > 1$ such that for any $f \in E_1$ with $p_\varepsilon(f - g) < \varepsilon$ we have: for any $t \in \mathbb{R}$ with $|t| \geq C_1$ there is $\zeta \in \mathbb{C}$ such that

$$|\zeta - t| \leq \eta|t| \quad \text{and} \quad |f(\zeta)| \geq e^{-\eta|\zeta|}. \tag{18}$$

Proof: For fixed $\eta > 0$ let

$$S_k := \{f \in E_1 \mid f \text{ satisfies (18) for } C_1 := k\}.$$

S_k is closed in E_1 (since the topology of E_1 is stronger than locally uniform convergence) and $E_1 = \bigcup_{k \in \mathbb{N}} S_k$ by (17) since $0 \notin E_1$. Hence, there is k_0 by Baire's theorem such that S_{k_0} has an interior point. This proves the claim.

c) In the following we will fix $0 < \eta < 1/4$ such that

$$4\pi\eta \leq -\ln(\pi\eta) \quad \text{and} \quad |\sin(z)| \leq 2 \text{ if } |z - \pi/2| \leq \eta\pi/2 \tag{19}$$

and choose $g \in E_1$, $\varepsilon > 0$ and $C_1 > 1$ by b).

We may suppose that g is a polynomial. In fact, the polynomials are dense in E (notice that the set $\{\delta^{(j)} \mid j \in \mathbb{N}_0\}$ is total in $\mathcal{A}([-1,1])'_b$ and that the Fourier transformation is a topological isomorphism from $\mathcal{A}([-1,1])'_b$ onto E). Thus, the polynomials P with $P(0) = 1$ are also dense in E_1.

Let $h_n(z) := (\cos(z/n))^n$. Then $h_n(0) = 1$ and

$$|h_n(z)| \leq e^{|\Im(z)|} \qquad \text{for any } n \in \mathbb{N} \tag{20}$$

and hence $h_n \in E_1$. Moreover,

$$h_n \to 1 \quad \text{in } E_1. \tag{21}$$

Proof: Fix $1 \geq \gamma > 0$ and let $|z| \geq A := \frac{1}{\gamma}\ln(2/\gamma)$. Then $1 \leq \gamma e^{\gamma|z|}/2$ and

$$|h_n(z) - 1| \leq e^{|\Im(z)|} + 1 \leq \gamma e^{|\Im(z)| + \gamma|z|}$$

by (20). For $|z| \le A$ we get by (19) since $|\cos(z)| \le e^{|\Im(z)|}$

$$|h_n(z) - 1| \le |z| \sup_{|\xi| \le A} |\cos(\xi/n)|^{n-1} |\sin(\xi/n)|$$
$$\le Ae^A \sup_{|\zeta| \le A/n} |\sin(\zeta)| \le \gamma \le \gamma e^{|\Im(z)| + \gamma|z|} \quad \text{if } n \text{ is large.}$$

d) Since g is a polynomial in E_1 and $h_n \in E_1$, also $h_n g \in E_1$. (21) implies that for any $\varepsilon > 0$ there is B_ε such that

$$|g(z) - h_n(z)g(z)| \le B_\varepsilon e^{\varepsilon|z|/2} |h_n(z) - 1| \le \varepsilon e^{|\Im(z)| + \varepsilon|z|} \quad \text{if } n \in \mathbb{N} \text{ is large.} \tag{22}$$

Since $h_n g \in E_1$ we thus get by (22) and (18) for large n: for any $t \in \mathbb{R}$ with $|t| \ge C_1$ there is $\zeta \in \mathbb{C}$ such that $|\zeta - t| \le \eta|t|$ and

$$e^{-\eta|\zeta|} \le |h_n(\zeta)g(\zeta)| \le B_\eta e^{\eta|\zeta|} |h_n(\zeta)|$$

and hence

$$|h_n(\zeta)| \ge e^{-2\eta|\zeta|}/B_\eta. \tag{23}$$

e) Let $t_n := \pi n/2$ and let $|\zeta - t_n| \le \eta|t_n| = \eta\pi n/2$. Then $|\zeta/n - t_n/n| \le \eta\pi/2$ and therefore

$$|h_n(\zeta)| = |\cos(\zeta/n)|^n = |\cos(\zeta/n) - \cos(t_n/n)|^n$$
$$\le 2^n |\zeta/n - t_n/n|^n \le (\eta\pi)^n$$
$$\le (\eta\pi)^{n/2} e^{-2\eta\pi n} \le e^{-2\eta|\zeta|}/(2B_\eta) \quad \text{if } n \text{ is large} \tag{24}$$

by (19), (24) contradicts (23). ■

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