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REGULAR COORDINATES AND REDUCTION OF DEFORMATION EQUATIONS FOR FUCHSIAN SYSTEMS

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Abstract. For a Fuchsian system

$$\frac{dY}{dx} = \Big(\sum_{j=1}^{p} \frac{A_j}{x-t_j}\Big)Y, \tag{F}$$

$t_1, t_2, \dots, t_p$ being distinct points in $\mathbb{C}$ and $A_1, A_2, \dots, A_p \in \mathrm{M}(n \times n; \mathbb{C})$, the number α of accessory parameters is determined by the spectral types $s(A_0), s(A_1), \dots, s(A_p)$, where $A_0 = -\sum_{j=1}^{p} A_j$. We call the set $z = (z_1, z_2, \dots, z_\alpha)$ of α parameters a regular coordinate if all entries of the A_j are rational functions in z. It is not yet known that, for any irreducibly realizable set of spectral types, a regular coordinate does exist. In this paper we study a process of obtaining a new regular coordinate from a given one by a coalescence of eigenvalues of the matrices A_j. Since a regular coordinate is a set of unknowns of the deformation equation for (F), this process gives a reduction of deformation equations. As an example, a reduction of the Garnier system to Painlevé VI is described in this framework.

1. Regular coordinates. We fix integers n and p. Let $\mathcal{O}_j$ $(0 \le j \le p)$ be a conjugacy class of $\mathrm{M}(n \times n; \mathbb{C})$. We assume that, for each $\mathcal{O}_j$, there is no integral difference between distinct eigenvalues. Moreover we assume

$$\sum_{j=0}^{p} \operatorname{tr} \mathcal{O}_j = 0. \tag{1}$$

We set

$$\mathcal{M} = \mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_p) = \Big\{(A_0, A_1, \dots, A_p) \in \mathcal{O}_0 \times \mathcal{O}_1 \times \dots \times \mathcal{O}_p\,;\ \sum_{j=0}^{p} A_j = O\Big\}/\sim,$$

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where $(A_0, A_1, \ldots, A_p) \sim (B_0, B_1, \ldots, B_p)$ if there is $P \in \mathrm{GL}(n; \mathbb{C})$ such that $A_j = PB_jP^{-1}$ $(0 \le j \le p)$. We denote $(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$ by $\vec{\mathcal{O}}$. We say that $\vec{\mathcal{O}}$ is *realizable* if $\mathcal{M}(\vec{\mathcal{O}}) \neq \emptyset$, and that $\vec{\mathcal{O}}$ is *irreducibly realizable* if there exists $[(A_0, A_1, \ldots, A_p)] \in \mathcal{M}(\vec{\mathcal{O}})$ such that the common invariant subspaces of $A_0, A_1, \ldots, A_p$ are trivial. To characterize the (irreducibly) realizable tuples $\vec{\mathcal{O}}$ is a fundamental problem, which is called Deligne–Simpson Problem (DSP) by Kostov. DSP is solved by Kostov [9], Crawley-Boevey [2] and Oshima [13].

The set $\mathcal{M}$ can be regarded as a moduli space of Fuchsian systems of differential equations. Let $t_1, t_2, \ldots, t_p$ be distinct points in $\mathbb{C}$, and $A_1, A_2, \ldots, A_p$ be matrices in $\mathrm{M}(n \times n; \mathbb{C})$. Consider the Fuchsian system

$$\frac{dY}{dx} = \Big(\sum_{j=1}^{p} \frac{A_j}{x - t_j}\Big) Y, \tag{2}$$

and set

$$A_0 = -\sum_{j=1}^{p} A_j.$$

The accessory parameters of the system (2) can be understood as a coordinate system of $\mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$, where $\mathcal{O}_j$ is the conjugacy class of A_j. The deformation of the system (2) is described by the system of partial differential equations

$$\begin{cases} \dfrac{\partial A_i}{\partial t_i} = -\displaystyle\sum_{k \neq i} \frac{[A_i, A_k]}{t_i - t_k}, \\ \dfrac{\partial A_j}{\partial t_i} = \dfrac{[A_i, A_j]}{t_i - t_j} \qquad (j \neq i) \end{cases}$$

for $(A_1, A_2, \ldots, A_p)$, where $[(A_0, A_1, \ldots, A_p)] \in \mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$ with A_0 normalized to the Jordan canonical form. Thus the deformation is a system of partial differential equations for the accessory parameters. Then, in order to describe the deformation equation explicitly, we have to find a coordinate system of $\mathcal{M}$, which is the theme of this article.

We want to find a good coordinate system. For the case $n = 2$ and $p = 3$, Okamoto [11] and Inaba–Iwasaki–Saito [6] constructed beautiful coordinate systems for the moduli space $\mathcal{M}$, which are fairly useful for the analysis of the Painlevé VI equation. Such constructions, however, are very hard even for this particular case, and then similar constructions for general $\mathcal{M}$ seem to be beyond our scope. We look for another kind of good coordinate systems.

Definition 1.1. Let α be the dimension of the moduli space $\mathcal{M} = \mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$. A coordinate system $(z_1, z_2, \ldots, z_\alpha)$ of $\mathcal{M}$ is called a *regular coordinate* if, for a representative $(A_0, A_1, \ldots, A_p)$ of a generic point of $\mathcal{M}$, all entries of the matrices $A_0, A_1, \ldots, A_p$ are rational functions of $(z_1, z_2, \ldots, z_\alpha)$.

Jimbo–Miwa–Môri–Sato [7] gave a set of variables for a tuple $(A_1, A_2, \ldots, A_p)$ of matrices such that all entries of the matrices are rational in the variables, and that the variables are canonical with respect to the Hamiltonian structure of the deformation equation. Thus these variables are good ones, but do not give a regular coordinate since the number of the variables exceeds the number α of the accessory parameters. Fuji–Suzuki [4] and Tsuda [15] obtained a same deformation equation in different ways. Their deformation equation is written in a coordinate obtained from JMMS variables by reducing the number of the variables. In the work [14] of classifying deformation equation of dimension 4, Sakai got 4 types of deformation equations which are also written in coordinates from JMMS variables. These coordinates are canonical coordinates with respect to the Hamiltonian structure, and we find that they are regular coordinates. Alday–Gaiotto–Tachikawa [1] conjectured the coincidence of the partition function of the four-dimensional gauge theory and the correlation function of the conformal field theory. In studying AGT conjecture Yamada [16] obtained deformation equations from Fuji–Suzuki–Tsuda equation by changing the spectral types of the matrices A_i.

Looking at these works, I noticed it important to study the regular coordinates in general extent. In this paper we consider two kinds of transformations of the tuple $(A_0, A_1, \ldots, A_p)$, one is Katz's operations and the other is coalescences of eigenvalues, and study the behavior of regular coordinates under these transformations. The former transformation keeps the deformation equations invariant ([5]), while the latter one gives a reduction of deformation equations. Explicit regular coordinates are also given for several particular cases.

2. Formulation of the problem. In the following we assume that the conjugacy classes $\mathcal{O}_j$ $(0 \leq j \leq p)$ are *semi-simple*. For a semi-simple conjugacy class $\mathcal{O}$ of $\mathrm{M}(n \times n; \mathbb{C})$, the partition of n which represents the multiplicities of the eigenvalues is called the *spectral type* of $\mathcal{O}$, and is denoted by $s(\mathcal{O})$. For a semi-simple conjugacy class $\mathcal{O}$ with $s(\mathcal{O}) = (m_1, m_2, \ldots, m_l)$, we set

$$z(\mathcal{O}) = \sum_{i=1}^{l} {m_i}^2, \tag{3}$$

which is the dimension of the centralizer of any representative of $\mathcal{O}$. For $A \in \mathcal{O}$, we also use the notation $s(A)$ and $z(A)$ in place of $s(\mathcal{O})$ and $z(\mathcal{O})$, respectively, and call $s(A)$ the spectral type of A. For a tuple $\vec{\mathcal{O}} = (\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$, the tuple $(s(\mathcal{O}_0), s(\mathcal{O}_1), \ldots, s(\mathcal{O}_p))$ of the spectral types is called the spectral type of $\vec{\mathcal{O}}$, and is denoted by $s(\vec{\mathcal{O}})$. If $\vec{\mathcal{O}} = (\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$ is irreducibly realizable, the dimension α of $\mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$ is given by

$$\alpha = (p-1)n^2 - \sum_{j=0}^{p} z(\mathcal{O}_j) + 2. \tag{4}$$

This is shown essentially in [8], where the index of rigidity is given by $2 - \alpha$. It is also known that α is an even integer.

We are interested in finding a regular coordinate for a given irreducibly realizable $\vec{\mathcal{O}}$. The following lemmas give basic techniques.

Lemma 2.1. *For any generic pair A, B of $n \times n$-matrices, there exists a similar transformation which sends A to an upper triangular matrix and B to a lower triangular matrix simultaneously.*

Proof. First we assume that A, B are diagonalizable, and take eigenvectors $u_1, u_2, \dots, u_n$ (resp. $v_1, v_2, \dots, v_n$) of A (resp. B) such that $u_1, \dots, u_{n-1}, v_n$ are linearly independent. We set

$$Au_i = a_i u_i, \quad Bv_i = b_i v_i \qquad (1 \le i \le n).$$

We show that, for $i = 2, 3, \dots, n-1$, there is a vector u'_i such that

$$\begin{aligned} u'_i &\in \langle u_1, \dots, u_i \rangle, \\ Bu'_i &\in b_i u'_i + \langle u'_{i+1}, \dots, u'_{n-1}, v_n \rangle, \end{aligned}$$

Suppose that the assertion holds for $i+1, \dots, n-1$, and assume that $u_1, \dots, u_i, u'_{i+1}, \dots, u'_{n-1}, v_n$ are linearly independent. Then v_i can be written in these vectors

$$v_i = c_1 u_1 + \ldots + c_i u_i + c_{i+1} u'_{i+1} + \ldots + c_{n-1} u'_{n-1} + c_n v_n$$

with scalars $c_1, c_2, \dots, c_n$. We set

$$u'_i = c_1 u_1 + \ldots + c_i u_i.$$

Then clearly $u'_i \in \langle u_1, \dots, u_i \rangle$, and we have

$$\begin{aligned} Bu'_i &= B(v_i - c_{i+1} u'_{i+1} - \ldots - c_{n-1} u'_{n-1} - c_n v_n) \\ &\in b_i v_i + \langle u'_{i+1}, \dots, u'_{n-1}, v_n \rangle \\ &= b_i (u'_i + \langle u'_{i+1}, \dots, u'_{n-1}, v_n \rangle) + \langle u'_{i+1}, \dots, u'_{n-1}, v_n \rangle \\ &= b_i u'_i + \langle u'_{i+1}, \dots, u'_{n-1}, v_n \rangle, \end{aligned}$$

which shows the assertion for i. Thus, the similar transformation by the matrix

$$P = (u_1, u'_2, \dots, u'_{n-1}, v_n)$$

sends A and B to upper and lower triangular matrices, respectively.

In the above we assumed that $u_1, \dots, u_i, u'_{i+1}, \dots, u'_{n-1}, v_n$ are linearly independent in each step. We understand that the pair A, B is generic if these conditions are satisfied for some sets of eigenvectors $u_1, \dots, u_n$ and $v_1, \dots, v_n$.

The above proof can be modified to the case that A or B is not diagonalizable. ∎

LEMMA 2.2. *Let C be a diagonalizable $n \times n$-matrix with spectral type $(m_1, m_2, \dots, m_l)$.*

(i) *C can be parametrized by*

$$n^2 - \sum_{i=1}^{l} {m_i}^2 = n^2 - z(C)$$

parameters besides the eigenvalues.

(ii) *Denote the eigenvalue of multiplicity m_i by c_i. We set*

$$m'_i = n - m_1 - m_2 - \ldots - m_i$$

for $i = 1, 2, \dots, l$.

Then C can be generically parametrized as follows:

$$C = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_{m'_1} & P_1 \end{pmatrix},$$

$$C_1 + P_1 U_1 = c_2 - c_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} I_{m'_2} & P_2 \end{pmatrix},$$

$$C_2 + P_2 U_2 = c_3 - c_2 + \begin{pmatrix} C_3 \\ U_3 \end{pmatrix} \begin{pmatrix} I_{m'_3} & P_3 \end{pmatrix},$$

$$\vdots$$

$$C_{l-1} + P_{l-1} U_{l-1} = c_l - c_{l-1},$$

where C_i, U_i, P_i are $m'_i \times m'_i$, $m_i \times m'_i$ and $m'_i \times m_i$-matrices, respectively, and the scalars in the right hand sides are scalar matrices of appropriate sizes. The entries of P_i and U_i $(1 \le i \le l-1)$ are the parameters.

Proof. First we note that

$$\operatorname{rank}(C - c_1) = n - m_1 = m'_1.$$

Then, if the first m'_1 columns of $C - c_1$ are linearly independent, we get the decomposition

$$C - c_1 = \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_{m'_1} & P_1 \end{pmatrix}. \tag{5}$$

Using (5), we get

$$(C - c_1)(C - c_2) = \begin{pmatrix} C_1(C_1 + P_1U_1 - d_2) & C_1(C_1 + P_1U_1 - d_2)P_1 \\ U_1(C_1 + P_1U_1 - d_2) & U_1(C_1 + P_1U_1 - d_2)P_1 \end{pmatrix},$$

where $d_2 = c_2 - c_1$. Since

$$\operatorname{rank}((C - c_1)(C - c_2)) = m'_2 \quad \text{and} \quad \operatorname{rank} \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} = m'_1,$$

we have

$$\operatorname{rank}(C_1 + P_1U_1 - d_2) = m'_2.$$

Then we get the decomposition

$$C_1 + P_1U_1 - d_2 = \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} I_{m'_2} & P_2 \end{pmatrix},$$

if the first m'_2 columns of $C_1 + P_1U_1 - d_2$ are linearly independent. Continuing similar arguments, we get the parametrization in the assertion (ii). The parameters are given by the entries of P_i and U_i $(1 \le i \le l-1)$, and hence the number of the parameters is

$$\sum_{i=1}^{l-1} m'_i \cdot m_i + \sum_{i=1}^{l-1} m_i \cdot m'_i = 2 \sum_{i \ne j} m_i m_j = (m_1 + \ldots + m_l)^2 - \sum_{i=1}^{l} {m_i}^2,$$

which implies the assertion (i). ∎

We shall use the above lemmas to construct a regular coordinate for a given $\vec{\mathcal{O}} = (\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$. First assume that there are two $\mathcal{O}_j$ with spectral type (1^n). We may take

$$s(\mathcal{O}_0) = s(\mathcal{O}_p) = (1^n).$$

By Lemma 2.1, we can take a representative $(A_0, A_1, \ldots, A_p)$ of a generic point of $\mathcal{M}(\vec{\mathcal{O}})$ such that

$$A_0 = \begin{pmatrix} a_{01} & & O \\ & \ddots & \\ * & & a_{0n} \end{pmatrix}, \quad A_p = \begin{pmatrix} a_{p1} & & * \\ & \ddots & \\ O & & a_{pn} \end{pmatrix}, \tag{6}$$

where $a_{0i} \neq a_{0j}$ $(i \neq j)$ and $a_{pi} \neq a_{pj}$ $(i \neq j)$. We parametrize $A_1, \ldots, A_{p-1}$ according to Lemma 2.2(ii). By Lemma 2.2(i), we see the number of the parameters we use is

$$\sum_{j=1}^{p-1}(n^2 - z(A_j)) = (p-1)n^2 - \sum_{j=1}^{p-1} z(\mathcal{O}_j). \tag{7}$$

The maximal subgroup of $\mathrm{GL}(n;\mathbb{C})$ which leaves the form (6) invariant is

$$\mathrm{GL}(1)^n = \mathrm{GL}(1;\mathbb{C}) \times \mathrm{GL}(1;\mathbb{C}) \times \ldots \times \mathrm{GL}(1;\mathbb{C}).$$

Since the center $\mathbb{C}^\times$ acts trivially, the effective action is given by $\mathrm{GL}(1)^n/\mathbb{C}^\times \cong \mathrm{GL}(1)^{n-1}$. Then we can normalize $n-1$ off-diagonal entries of $A_1, \ldots, A_{p-1}$ to arbitrary values. This normalization is a system of algebraic equations for the parameters, which we call the system (N).

Next we look at the relation

$$\sum_{j=0}^{p} A_j = O. \tag{8}$$

The diagonal entries of (8) give n relations

$$\sum_{j=1}^{p-1}((i,i)\text{-entry of } A_j) = -a_{0i} - a_{pi} \qquad (1 \leq i \leq n). \tag{9}$$

If we take a sum of these n relations, we get

$$\sum_{j=0}^{p} \operatorname{tr} A_j = 0,$$

which is a relation for the eigenvalues and is already assumed in (1). Then we have $n-1$ independent relations among (9), which we call the system (D).

Thus we have $2(n-1)$ relations (N) and (D) for the parameters of $A_1, \ldots, A_{p-1}$. If these relations are independent and solvable, the number of the parameters is reduced from (7) to

$$(p-1)n^2 - \sum_{j=1}^{p-1} z(\mathcal{O}_j) - 2(n-1) = (p-1)n^2 - \sum_{j=1}^{p-1} z(\mathcal{O}_j) - z(\mathcal{O}_0) - z(\mathcal{O}_p) + 2 = \alpha.$$

Note that the off-diagonal entries of A_0 and A_p are written linearly in terms of the entries of $A_1, \ldots, A_{p-1}$ by the relation (8). Hence, if the system (N) and (D) is independent and solvable, and if the solution of the system can be written rationally in α parameters, the parameters make a regular coordinate for $\mathcal{M}(\vec{\mathcal{O}})$.

This method can be directly applied to the following particular case.

PROPOSITION 2.3. *In the case*

$$s(\mathcal{O}_0) = s(\mathcal{O}_1) = \ldots = s(\mathcal{O}_p) = (1^n),$$

we have a regular coordinate for $\mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$.

Proof. In this case we have

$$\alpha = (p-1)n^2 - (p+1)n + 2.$$

We may assume that A_0 and A_p are of the form (6). We denote the eigenvalues of A_j by $a_{j1}, a_{j2}, \ldots, a_{jn}$ for $0 \le j \le p$. According to Lemma 2.2(ii), we can parametrize $A_1, \ldots, A_{p-1}$ as

$$\begin{aligned} A_j &= a_{j1} + \begin{pmatrix} C_1^j \\ U_1^j \end{pmatrix} \begin{pmatrix} I_{n-1} & P_1^j \end{pmatrix}, \\ C_1^j + P_1^j U_1^j &= a_{j2} - a_{j1} + \begin{pmatrix} C_2^j \\ U_2^j \end{pmatrix} \begin{pmatrix} I_{n-2} & P_2^j \end{pmatrix}, \\ &\vdots \\ C_{n-1}^j + P_{n-1}^j U_{n-1}^j &= a_{jn} - a_{j,n-1}, \end{aligned} \tag{10}$$

for $1 \le j \le p-1$. Here, for $1 \le k \le n-1$, U_k^j and P_k^j are $1 \times (n-k)$ and $(n-k) \times 1$-matrices, respectively, and so we set

$$U_k^j = \begin{pmatrix} (u_k^j)_1 & \ldots & (u_k^j)_{n-k} \end{pmatrix}, \quad P_k^j = \begin{pmatrix} (p_k^j)_1 \\ \vdots \\ (p_k^j)_{n-k} \end{pmatrix}.$$

By the action of $\mathrm{GL}(1)^{n-1}$, we can normalize U_1^1 to

$$U_1^1 = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}. \tag{11}$$

Namely we have $(u_1^1)_1 = \ldots = (u_1^1)_{n-1} = 1$. By (10), the i-th diagonal entry of A_j is given by

$$a_{j,n-i+1} - \sum_{k=1}^{n-i} (p_k^j)_i (u_k^j)_i + U_{n-i+1}^j P_{n-i+1}^j$$

for $1 \le i \le n$. We put this into the relation (9) for $1 \le i \le n-1$ to obtain

$$\sum_{j=1}^{p-1} \sum_{k=1}^{n-i} (p_k^j)_i (u_k^j)_i - \sum_{j=1}^{p-1} U_{n-i+1}^j P_{n-i+1}^j = a_{0i} + \sum_{j=1}^{p-1} a_{j,n-i+1} + a_{pi}.$$

Since we have normalized U_1^1 as (11), this relation can be written as

$$(p_1^1)_i + \sum_{\substack{1 \le j \le p-1,\, 1 \le k \le n-i \\ (j,k) \ne (1,1)}} (p_k^j)_i (u_k^j)_i - \sum_{j=1}^{p-1} U_{n-i+1}^j P_{n-i+1}^j = a_{0i} + \sum_{j=1}^{p-1} a_{j,n-i+1} + a_{pi}. \tag{12}$$

Hence every entry of P_1^1 is a polynomial of the entries of U_k^j and P_k^j with $(j,k) \ne (1,1)$. The number of these entries is α, and hence they make a regular coordinate. ∎

Now we relax the condition $s(\mathcal{O}_0) = s(\mathcal{O}_p) = (1^n)$. Let us consider the case

$$s(\mathcal{O}_0) = (m, 1^{n-m}), \quad s(\mathcal{O}_p) = (1^n)$$

with $1 < m < n$. If we take a representative $(A_0, A_1, \ldots, A_p)$ with A_0, A_p of the form (6), and if we assume $a_{01} = \ldots = a_{0m}$, we should have

$$A_0 = \left(\begin{array}{ccc|ccc} a_{01} & & O & & & \\ & \ddots & & & O & \\ O & & a_{01} & & & \\ \hline & & & a_{0,m+1} & & O \\ & * & & & \ddots & \\ & & & * & & a_{0n} \end{array}\right),$$

because A_0 is diagonalizable. This form of A_0 is left invariant by the action of $\mathrm{GL}(m) \times \mathrm{GL}(1)^{n-m}$. Then, by using this action, we can normalize the principal $m \times m$ part of A_p to a diagonal matrix. Thus we have

$$A_p = \left(\begin{array}{ccc|ccc} a_{p1} & & O & & & \\ & \ddots & & & * & \\ O & & a_{pm} & & & \\ \hline & & & a_{p,m+1} & & * \\ & O & & & \ddots & \\ & & & O & & a_{pn} \end{array}\right).$$

Then the principal $m \times m$ part of the relation (8) becomes a system of algebraic equations for the parameters of $A_1, \ldots, A_{p-1}$. In this way, we can increase the number of the equations of the system (D) by $m^2 - m$, which is just the difference of $z(\mathcal{O}_0)$ for $s(\mathcal{O}_0) = (1^n)$ and for $s(\mathcal{O}_0) = (m, 1^{n-m})$, and hence the difference of α.

For the case

$$s(\mathcal{O}_0) = (m, 1^{n-m}), \quad s(\mathcal{O}_p) = (m', 1^{n-m'})$$

with $1 < m' \le m < n$, the above argument holds without any modification. In this case, if we take $a_{p1} = \ldots = a_{pm'}$, the action of $\mathrm{GL}(m') \times \mathrm{GL}(1)^{n-m'}$ leaves the normalized forms of A_0 and A_p invariant. Then we can normalize m'^2 entries of $A_1, \ldots, A_{p-1}$ by this action, which increases the number of the equations of the system (N) by ${m'}^2 - m'$. Just as above, the last number coincides with the difference of $z(\mathcal{O}_p)$ and hence of α.

To consider more complicated cases, we use the following lemma.

LEMMA 2.4. *Let A and B be a generic pair of diagonalizable $n \times n$-matrices of spectral types (m_1, m_2) and (n_1, n_2), respectively, with $m_1 > n_1$. Then there exists $P \in \mathrm{GL}(n; \mathbb{C})$ such that*

$$P^{-1}AP = \begin{pmatrix} a_1 I_{n_1} & O & O \\ O & a_1 I_{m_1-n_1} & O \\ * & O & a_2 I_{m_2} \end{pmatrix}, \quad P^{-1}BP = \begin{pmatrix} b_1 I_{n_1} & O & * \\ O & b_2 I_{m_1-n_1} & O \\ O & O & b_2 I_{m_2} \end{pmatrix},$$

where a_i (resp. b_i) is the eigenvalue of A (resp. B) of multiplicity m_i (resp. n_i) for $i = 1, 2$.

Proof. We set $n_1 = k$, $m_1 - n_1 = l$ and $m_2 = m$. We may assume that A and B are of lower and upper triangular form, respectively. Since A and B are diagonalizable, we have

$$A = \begin{pmatrix} a_1 I_k & O & O \\ O & a_1 I_l & O \\ A_{31} & A_{32} & a_2 I_m \end{pmatrix}, \quad B = \begin{pmatrix} b_1 I_k & B_{12} & B_{13} \\ O & b_2 I_l & O \\ O & O & b_2 I_m \end{pmatrix},$$

where A_{31}, A_{32}, B_{12} and B_{13} are $m \times k$, $m \times l$, $k \times l$ and $k \times m$-matrices, respectively. We transform A and B by a matrix P of the form

$$P = \begin{pmatrix} I_k & P_{12} & O \\ O & I_l & O \\ O & P_{32} & I_m \end{pmatrix}.$$

Then we have

$$P^{-1}AP = \begin{pmatrix} a_1 I_k & O & O \\ O & a_1 I_l & O \\ A_{31} & X_{32} & a_2 I_m \end{pmatrix}, \quad P^{-1}BP = \begin{pmatrix} b_1 I_k & X_{12} & B_{13} \\ O & b_2 I_l & O \\ O & O & b_2 I_m \end{pmatrix},$$

with

$$X_{12} = B_{12} + (b_1 - b_2)P_{12} + B_{13}P_{32},$$
$$X_{32} = A_{32} + A_{31}P_{12} + (a_2 - a_1)P_{32}.$$

From the relations $X_{12} = O$ and $X_{32} = O$, we obtain the linear equation

$$\begin{pmatrix} A_{31} & (a_2 - a_1)I_m \\ (b_1 - b_2)I_k & B_{13} \end{pmatrix} \begin{pmatrix} P_{12} \\ P_{32} \end{pmatrix} = - \begin{pmatrix} A_{32} \\ B_{12} \end{pmatrix}$$

for P_{12} and P_{32}. The determinant of the matrix in the left hand side does not vanish for a generic pair (A, B), and hence we find a matrix P in the assertion of the lemma. ∎

By using Lemma 2.4 repeatedly, we obtain the following assertion.

PROPOSITION 2.5. *Let A and B be a generic pair of diagonalizable $n \times n$-matrices of spectral types*

$$s(A) = (m_1, m_2, \ldots, m_p), \quad s(B) = (n_1, n_2, \ldots, n_q).$$

We set

$$M_k = \sum_{i=1}^{k} m_i, \quad N_l = \sum_{j=1}^{l} n_j$$

for $1 \le k < p$ and $1 \le l < q$, and set $M_0 = N_0 = 0$. Then there exists $P \in \mathrm{GL}(n;\mathbb{C})$ such that

$$P^{-1}AP = \begin{pmatrix} a_1 I_{m_1} & O & \ldots & O \\ * & a_2 I_{m_2} & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & a_p I_{m_p} \end{pmatrix}, \quad P^{-1}BP = \begin{pmatrix} b_1 I_{n_1} & * & \ldots & * \\ O & b_2 I_{n_2} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & b_2 I_{n_q} \end{pmatrix},$$

where, among the lower off-diagonal entries of $P^{-1}AP$, the (i, j)-entry is 0 if $N_{l-1} + 1 \le j < i \le N_l$ for some l, and among the upper off-diagonal entries of $P^{-1}BP$, the (i, j)-entry is 0 if $M_{k-1} + 1 \le i < j \le M_k$ for some k. The maximal subgroup G of $\mathrm{GL}(n;\mathbb{C})$ which leaves these forms of $P^{-1}AP$ and $P^{-1}BP$ invariant is

$$G = \mathrm{GL}(l_1) \times \mathrm{GL}(l_2 - l_1) \times \ldots \times \mathrm{GL}(l_r - l_{r-1}) \times \mathrm{GL}(n - l_r),$$

where $(l_1, l_2, \dots, l_r)$ *is the increasing sequence of integers determined by*

$$\{M_1, M_2, \dots, M_{p-1}, N_1, N_2, \dots, N_{q-1}\} = \{l_1, l_2, \dots, l_r\}.$$

Example 2.6. Let A, B be a generic pair of diagonalizable 10×10-matrices of spectral types

$$s(A) = (3, 3, 3, 1), \quad s(B) = (2, 2, 2, 2, 2).$$

Then, by Proposition 2.5, we can send A, B by some $P \in \mathrm{GL}(10; \mathbb{C})$ to

$$P^{-1}AP = \left(\begin{array}{ccc|ccc|ccc|c} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a_{41} & a_{42} & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ \hline a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_3 & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & 0 & a_3 & 0 & 0 \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & 0 & 0 & a_3 & 0 \\ \hline a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & 0 & a_4 \end{array} \right),$$

$$P^{-1}BP = \left(\begin{array}{cc|cc|cc|cc|cc} b_1 & 0 & 0 & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} & b_{19} & b_{1,10} \\ 0 & b_1 & 0 & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} & b_{29} & b_{2,10} \\ \hline 0 & 0 & b_2 & 0 & b_{35} & b_{36} & b_{37} & b_{38} & b_{39} & b_{3,10} \\ 0 & 0 & 0 & b_2 & 0 & 0 & b_{47} & b_{48} & b_{49} & b_{4,10} \\ \hline 0 & 0 & 0 & 0 & b_3 & 0 & b_{57} & b_{58} & b_{59} & b_{5,10} \\ 0 & 0 & 0 & 0 & 0 & b_3 & b_{67} & b_{68} & b_{69} & b_{6,10} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & b_4 & 0 & 0 & b_{7,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 & 0 & b_{8,10} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_5 \end{array} \right).$$

The maximal subgroup of $\mathrm{GL}(10; \mathbb{C})$ which leaves the forms of these matrices invariant is

$$\mathrm{GL}(2) \times \mathrm{GL}(1) \times \mathrm{GL}(1) \times \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(1) \times \mathrm{GL}(1).$$

Now we can formulate our problem in general. Let an irreducibly realizable tuple $\vec{\mathcal{O}} = (\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_p)$ be given. We can take a representative $(A_0, A_1, \dots, A_p)$ of a generic point of $\mathcal{M}(\vec{\mathcal{O}})$ such that a pair of two matrices, say A_0, A_p, is normalized as in Proposition 2.5. We read $A_0 = A$ and $A_p = B$, and use the same notation as in the proposition. We parametrize the other matrices $A_1, \dots, A_{p-1}$ according to Lemma 2.2(ii). Note that the number of the parameters is given by (7). Let G be the maximal subgroup of $\mathrm{GL}(n; \mathbb{C})$ which leaves the normalized forms of A_0 and A_p invariant. By the action of G, we can normalize $\dim G - 1$ entries of $A_1, \dots, A_{p-1}$, which gives a system (N) of algebraic equations for the parameters of $A_1, \dots, A_{p-1}$. On the other hand, by the normalization, the diagonal entries of A_0 and A_p are the eigenvalues, and both of the off-diagonal (i, j)-entries of A_0 and A_p are 0 if $M_{k-1} < i, j \le M_k$ for some k or $N_{l-1} < i, j \le N_l$ for some l. Then, for these (i,j), the (i,j)-entries of the relation (8) give a system (D) of algebraic equations for the parameters of $A_1,\dots,A_{p-1}$.

Our problem is to parametrize the solutions of (N) and (D). If the solutions are expressed rationally in α parameters, the parameters make a regular coordinate for $\mathcal{M}(\vec{\mathcal{O}})$.

The problem can be regarded as a uniformization of a system of algebraic equations. Also it can be regarded as a construction problem of representations of quivers.

3. Katz operations

Definition 3.1 ([8], [3]). Let $(A_1,A_2,\dots,A_p)$ be a tuple of $n\times n$-matrices.

(i) Let $(a_1,a_2,\dots,a_p)$ be a point in $\mathbb{C}^p$. The operation

$$(A_1,A_2,\dots,A_p)\mapsto(A_1+a_1,A_2+a_2,\dots,A_p+a_p)$$

is called the *addition* with parameters $(a_1,a_2,\dots,a_p)$.

(ii) Let λ be a point in $\mathbb{C}$. Define $pn\times pn$-matrices $G_1,G_2,\dots,G_p$ by

$$G_i=\sum_{j=1}^{p}E_{ij}\otimes(A_j+\delta_{ij}\lambda)\qquad(1\le i\le p),$$

where E_{ij} is the $p\times p$-matrix with the only nonzero entry 1 at (i,j)-th position $(1\le i,j\le p)$. Let $\mathcal{K}$ and $\mathcal{L}$ be the subspaces of $\mathbb{C}^{pn}$ defined by

$$\mathcal{K}=\left\{\begin{pmatrix}v_1\\ \vdots\\ v_p\end{pmatrix}: v_i\in\operatorname{Ker}A_i\ (1\le i\le p)\right\},\qquad \mathcal{L}=\operatorname{Ker}(G_1+G_2+\ldots+G_p).$$

It is easy to see that $\mathcal{K}$ and $\mathcal{L}$ are invariant subspaces for $(G_1,G_2,\dots,G_p)$. Then $(G_1,G_2,\dots,G_p)$ induces the action $(\bar{G}_1,\bar{G}_2,\dots,\bar{G}_p)$ on the quotient space $\mathbb{C}^{pn}/(\mathcal{K}+\mathcal{L})$. The operation

$$(A_1,A_2,\dots,A_p)\mapsto(\bar{G}_1,\bar{G}_2,\dots,\bar{G}_p)$$

is called the *middle convolution* with parameter λ.

The addition and the middle convolution are called the Katz operations. The Katz operations can be uniquely extended to operations for tuples $(A_0,A_1,\dots,A_p)$ with sum zero. Moreover it is easy to see that the Katz operations induce maps from the moduli space $\mathcal{M}(\mathcal{O}_0,\mathcal{O}_1,\dots,\mathcal{O}_p)$ to other moduli spaces by sending $[(A_0,A_1,\dots,A_p)]$ to $[(A_0+a_0,A_1+a_1,\dots,A_p+a_p)]$ and to $[(\bar{G}_0,\bar{G}_1,\dots,\bar{G}_p)]$. It is shown that the Katz operations do not change the number of accessory parameters and the irreducibility.

In general, for any matrix A, we have a basis $\{v_1,v_2,\dots,v_l\}$ of $\operatorname{Ker}A$ such that every entry of v_i $(1\le i\le l)$ is a rational function of the entries of A. Noting this fact, we obtain the following result.

Theorem 3.2. *If a moduli space $\mathcal{M}(\mathcal{O}_0,\mathcal{O}_1,\dots,\mathcal{O}_p)$ has a regular coordinate, the images of the Katz operations also have regular coordinates.*

The moduli spaces $\mathcal{M}(\vec{\mathcal{O}})$ for the irreducibly realizable tuples $\vec{\mathcal{O}}$ are classified by the Katz operations, which induces the classification of the spectral type $s(\vec{\mathcal{O}})$. A spectral type $s(\vec{\mathcal{O}})$ is called *basic* if the rank is minimum among the class it belongs.

Thanks to Theorem 3.2, for finding regular coordinates, we have only to consider the moduli spaces corresponding to basic spectral types. It is shown by Oshima [13] that, for every value of α, there are only finitely many basic spectral types. The basic spectral types with $\alpha = 2$ are classified by Kostov [10], and those with $\alpha = 4$ are classified by Oshima [12]. Here we give the list of those spectral types.

The case $\alpha = 2$:

$$(11, 11, 11, 11),\ (1^3, 1^3, 1^3),\ (22, 1^4, 1^4),\ (33, 222, 1^6).$$

The case $\alpha = 4$:

$$\begin{aligned}&(11, 11, 11, 11, 11),\ (21, 21, 1^3, 1^3),\ (31, 22, 22, 1^4),\ (22, 22, 22, 211),\\ &(211, 1^4, 1^4),\ (221, 221, 1^5),\ (32, 1^5, 1^5),\ (2^3, 2^3, 2211),\ (33, 2211, 1^6),\\ &(44, 2^4, 22211),\ (44, 332, 1^8),\ (55, 3331, 2^5),\ (66, 444, 2^5 11).\end{aligned}$$

Among these spectral types, owing to Proposition 2.3, we already know that regular coordinates exist for the cases $(11, 11, 11, 11), (11, 11, 11, 11, 11)$ and $(1^3, 1^3, 1^3)$. For the other cases, we find regular coordinates except the cases $(44, 332, 1^8), (55, 3331, 2^5)$ and $(66, 444, 2^5 11)$. We note the results.

<u>$(22, 1^4, 1^4)$</u>

$$A_1 = \begin{pmatrix} a_1 & & & O \\ & a_2 & & \\ & * & a_3 & \\ & & & a_4 \end{pmatrix},\quad A_2 = \begin{pmatrix} b_1 & & * & \\ & b_2 & & \\ & O & b_3 & \\ & & & b_4 \end{pmatrix},$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_2 & P_1 \end{pmatrix},\quad C_1 + P_1 U_1 = c_2 - c_1,$$

where

$$U_1 = \begin{pmatrix} 1 & u_{12} \\ 1 & 1 \end{pmatrix},\quad P_1 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Then (u_{12}, p_{21}) is a regular coordinate.

<u>$(33, 222, 1^6)$</u>

$$A_1 = \left(\begin{array}{cc|cc|cc} a_1 & & & & & \\ & a_1 & \multicolumn{2}{c|}{O} & \multicolumn{2}{c}{O} \\ \hline & & a_2 & & & \\ \multicolumn{2}{c|}{*} & & a_2 & \multicolumn{2}{c}{O} \\ \hline & & & & a_3 & \\ \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & & a_3 \end{array}\right),\quad A_2 = \left(\begin{array}{cc|cc|cc} b_1 & & & & & \\ & b_2 & \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{*} \\ \hline & & b_3 & & & \\ \multicolumn{2}{c|}{O} & & b_4 & \multicolumn{2}{c}{*} \\ \hline & & & & b_5 & \\ \multicolumn{2}{c|}{O} & \multicolumn{2}{c|}{O} & & b_6 \end{array}\right),$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_3 & P_1 \end{pmatrix},\quad C_1 + P_1 U_1 = c_2 - c_1,$$

where

$$U_1 = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 1 & u_{22} & u_{23} \\ 1 & 1 & 1 \end{pmatrix},\quad P_1 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Then (p_{11}, p_{21}) is a regular coordinate.

$(21,21,1^3,1^3)$

$$A_2=\begin{pmatrix} a_1 & & O\\ & a_2 & \\ * & & a_3\end{pmatrix},\quad A_3=\begin{pmatrix} b_1 & & *\\ & b_2 & \\ O & & b_3\end{pmatrix},$$

$$A_0=c_1+\begin{pmatrix}C_1\\U_1\end{pmatrix}\begin{pmatrix}I_2 & P_1\end{pmatrix},\quad C_1+P_1U_1=c_2-c_1,$$

$$A_1=d_1+\begin{pmatrix}D_1\\V_1\end{pmatrix}\begin{pmatrix}I_2 & Q_1\end{pmatrix},\quad D_1+Q_1V_1=d_2-d_1,$$

where

$$U_1=\begin{pmatrix}1 & 1\end{pmatrix},\quad P_1=\begin{pmatrix}p_1\\p_2\end{pmatrix},\quad V_1=\begin{pmatrix}v_1 & v_2\end{pmatrix},\quad Q_1=\begin{pmatrix}q_1\\q_2\end{pmatrix}.$$

Then (p_1,p_2,v_1,v_2) is a regular coordinate.

$(31,22,22,1^4)$

$$A_2=\left(\begin{array}{cc|cc} a_1 & & & O\\ & a_1 & & \\ \hline & & a_2 & \\ * & & & a_2\end{array}\right),\quad A_3=\left(\begin{array}{cc|cc} b_1 & & & *\\ & b_2 & & \\ \hline & & b_3 & \\ O & & & b_4\end{array}\right),$$

$$A_0=c_1+\begin{pmatrix}C_1\\U_1\end{pmatrix}\begin{pmatrix}I_3 & P_1\end{pmatrix},\quad C_1+P_1U_1=c_2-c_1,$$

$$A_1=d_1+\begin{pmatrix}D_1\\V_1\end{pmatrix}\begin{pmatrix}I_2 & Q_1\end{pmatrix},\quad D_1+Q_1V_1=d_2-d_1,$$

where

$$U_1=\begin{pmatrix}1 & 1 & 1\end{pmatrix},\quad P_1=\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix},\quad V_1=\begin{pmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{pmatrix},\quad Q_1=\begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}.$$

Then $(v_{21},v_{22},q_{21},q_{22})$ is a regular coordinate.

$(22,22,22,211)$

$$A_2=\left(\begin{array}{cc|cc} a_1 & & & O\\ & a_1 & & \\ \hline & & a_2 & \\ * & & & a_2\end{array}\right),\quad A_3=\left(\begin{array}{cc|cc} b_1 & & & *\\ & b_1 & & \\ \hline & & b_2 & \\ O & & & b_3\end{array}\right),$$

$$A_0=c_1+\begin{pmatrix}C_1\\U_1\end{pmatrix}\begin{pmatrix}I_2 & P_1\end{pmatrix},\quad C_1+P_1U_1=c_2-c_1,$$

$$A_1=d_1+\begin{pmatrix}D_1\\V_1\end{pmatrix}\begin{pmatrix}I_2 & Q_1\end{pmatrix},\quad D_1+Q_1V_1=d_2-d_1,$$

where

$$U_1=I_2,\quad P_1=\begin{pmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix},\quad V_1=\begin{pmatrix}v_{11} & v_{12}\\ 1 & v_{22}\end{pmatrix},\quad Q_1=\begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}.$$

Then $(q_{12},q_{21},q_{22},v_{22})$ is a regular coordinate.

$\underline{(211, 1^4, 1^4)}$

$$A_1 = \begin{pmatrix} a_1 & & & O \\ & a_2 & & \\ & & a_3 & \\ * & & & a_4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_1 & & & * \\ & b_2 & & \\ & & b_3 & \\ O & & & b_4 \end{pmatrix},$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_2 & P_1 \end{pmatrix},$$

$$C_1 + P_1 U_1 = c_2 - c_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} 1 & P_2 \end{pmatrix}, \quad C_2 + P_2 U_2 = c_3 - c_2,$$

where

$$U_1 = \begin{pmatrix} 1 & u_{12} \\ 1 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad U_2 = (v), \quad P_2 = (q).$$

Then (u_{12}, p_{12}, v, q) is a regular coordinate.

$\underline{(221, 221, 1^5)}$

$$A_1 = \left(\begin{array}{cc|cc|c} a_1 & & & & \\ & a_1 & \multicolumn{2}{c|}{O} & O \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{*}} & a_2 & & O \\ & & & a_2 & \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & a_3 \end{array}\right), \quad A_2 = \left(\begin{array}{cc|cc|c} b_1 & & & & \\ & b_2 & \multicolumn{2}{c|}{*} & * \\ \hline \multicolumn{2}{c|}{O} & b_3 & & * \\ & & & b_4 & \\ \hline \multicolumn{2}{c|}{O} & \multicolumn{2}{c|}{O} & b_5 \end{array}\right),$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_3 & P_1 \end{pmatrix},$$

$$C_1 + P_1 U_1 = c_2 - c_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} 1 & P_2 \end{pmatrix}, \quad C_2 + P_2 U_2 = c_3 - c_2,$$

where

$$U_1 = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 1 & 1 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \end{pmatrix}, \quad U_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} q_1 & q_2 \end{pmatrix}.$$

Then $(u_{12}, p_{21}, v_2, q_2)$ is a regular coordinate.

$\underline{(32, 1^5, 1^5)}$

$$A_1 = \begin{pmatrix} a_1 & & O \\ & \ddots & \\ * & & a_5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_1 & & * \\ & \ddots & \\ O & & b_5 \end{pmatrix},$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_2 & P_1 \end{pmatrix}, \quad C_1 + P_1 U_1 = c_2 - c_1,$$

where

$$U_1 = \begin{pmatrix} 1 & u_{12} \\ 1 & u_{22} \\ 1 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{pmatrix}.$$

Then $(u_{12}, u_{22}, p_{21}, p_{22})$ is a regular coordinate.

$\underline{(2^3, 2^3, 2211)}$

$$A_1 = \left(\begin{array}{c|c|c} \begin{matrix} a_1 & \\ & a_1 \end{matrix} & O & O \\ \hline * & \begin{matrix} a_2 & \\ & a_2 \end{matrix} & O \\ \hline * & * & \begin{matrix} a_3 & \\ & a_3 \end{matrix} \end{array}\right), \quad A_2 = \left(\begin{array}{c|c|c} \begin{matrix} b_1 & \\ & b_1 \end{matrix} & * & * \\ \hline O & \begin{matrix} b_2 & \\ & b_2 \end{matrix} & * \\ \hline O & O & \begin{matrix} b_3 & \\ & b_4 \end{matrix} \end{array}\right),$$

$$\begin{aligned} &A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_2 & P_1 \end{pmatrix}, \\ &C_1 + P_1 U_1 = c_2 - c_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} I_2 & P_2 \end{pmatrix}, \quad C_2 + P_2 U_2 = c_3 - c_2, \end{aligned}$$

where

$$U_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \\ p_{41} & p_{42} \end{pmatrix},$$

$$U_2 = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, \quad P_2 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$

Then $(v_{11}, v_{12}, v_{21}, q_{22})$ is a regular coordinate.

$\underline{(33, 2211, 1^6)}$

$$A_1 = \left(\begin{array}{c|c|c} \begin{matrix} a_1 & \\ & a_1 \end{matrix} & O & O \\ \hline * & \begin{matrix} a_2 & \\ & a_2 \end{matrix} & O \\ \hline * & * & \begin{matrix} a_3 & \\ * & a_4 \end{matrix} \end{array}\right), \quad A_2 = \left(\begin{array}{c|c|c} \begin{matrix} b_1 & \\ & b_2 \end{matrix} & * & * \\ \hline O & \begin{matrix} b_3 & \\ & b_4 \end{matrix} & * \\ \hline O & O & \begin{matrix} b_5 & * \\ & b_6 \end{matrix} \end{array}\right),$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_3 & P_1 \end{pmatrix}, \quad C_1 + P_1 U_1 = c_2 - c_1,$$

where

$$U_1 = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 1 & u_{22} & u_{23} \\ 1 & 1 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Then $(u_{22}, u_{23}, p_{22}, p_{23})$ is a regular coordinate.

<u>$(44, 2^4, 22211)$</u>

$$A_1 = \left(\begin{array}{c|c|c|c} \begin{matrix} a_1 & \\ & a_1 \end{matrix} & O & O & O \\ \hline * & \begin{matrix} a_2 & \\ & a_2 \end{matrix} & O & O \\ \hline * & * & \begin{matrix} a_3 & \\ & a_3 \end{matrix} & O \\ \hline * & * & * & \begin{matrix} a_4 & \\ & a_4 \end{matrix} \end{array}\right),$$

$$A_2 = \left(\begin{array}{c|c|c|c} \begin{matrix} b_1 & \\ & b_1 \end{matrix} & * & * & * \\ \hline O & \begin{matrix} b_2 & \\ & b_2 \end{matrix} & * & * \\ \hline O & O & \begin{matrix} b_3 & \\ & b_3 \end{matrix} & * \\ \hline O & O & O & \begin{matrix} b_4 & \\ & b_5 \end{matrix} \end{array}\right),$$

$$A_0 = c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_4 & P_1 \end{pmatrix}, \quad C_1 + P_1 U_1 = c_2 - c_1,$$

where

$$U_1 = \begin{pmatrix} 1 & 0 & u_{13} & u_{14} \\ 0 & 1 & 1 & u_{24} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}.$$

Then $(u_{13}, u_{14}, p_{41}, p_{42})$ is a regular coordinate.

It is not yet known whether regular coordinates exist for the remaining cases $(44, 332, 1^8), (55, 3331, 2^5)$ and $(66, 444, 2^5 11)$.

4. Coalescence of eigenvalues. The spectral type of a diagonalizable matrix changes when two eigenvalues coalesce. Then any spectral type is obtained from (1^n) by an iteration of coalescences of the eigenvalues.

Since we have a regular coordinate for $\mathcal{M}(\vec{\mathcal{O}})$ with spectral type $(1^n, 1^n, \ldots, 1^n)$ by Proposition 2.3, we may have regular coordinates for other spectral types by coalescences of eigenvalues. In some particular cases, we can actually have regular coordinates in this way.

Proposition 4.1. *We have a regular coordinate for $\mathcal{M}(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$ with spectral type*

$$(21^{n-2}, 1^n, \ldots, 1^n).$$

Proof. We regard $\vec{\mathcal{O}} = (\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_p)$ as a result of a coalescence of a tuple $\vec{\mathcal{C}} = (\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_p)$ of spectral type $(1^n, 1^n, \ldots, 1^n)$. Let $(A_0, A_1, \ldots, A_p)$ be a representative of a generic point of $\mathcal{M}(\vec{\mathcal{C}})$. We parametrize $(A_0, A_1, \ldots, A_p)$ by using the regular coordinate as in the proof of Proposition 2.3. We consider the coalescence $a_{02} \to a_{01}$. As explained in Section 2, this induces two relations

$$\sum_{j=1}^{p-1}((1,2)\text{-entry of } A_j) = 0, \qquad \sum_{j=1}^{p-1}((2,1)\text{-entry of } A_j) = 0. \tag{13}$$

If $n \geq 4$ and $p \geq 3$, (13) becomes a system of linear equations in $(u_2^1)_1, (u_2^2)_2$, and the coefficient matrix is

$$\begin{pmatrix} (p_2^1)_1 & -(p_2^2)_1 \\ -(p_2^1)_2 & (p_2^2)_2 \end{pmatrix},$$

which is generically non-singular. Thus we obtain a regular coordinate from one for $\mathcal{M}(\vec{\mathcal{C}})$ by eliminating $(u_2^1)_1$ and $(u_2^2)_2$.

If $n = 3$ and $p \geq 3$, (13) becomes a system of linear equation in $(u_2^1)_1, (u_2^2)_1$ with non-singular coefficient matrix, and hence we obtain a regular coordinate by eliminating them. Similarly, if $n \geq 4$ and $p = 2$, we can eliminate $(p_2^1)_1, (p_2^1)_2$. In the case $n = 3$ and $p = 2$, the result of the coalescence is rigid, and hence $\mathcal{M}(\vec{\mathcal{O}})$ becomes a point.

When $n = 2$ and $p \geq 3$, we use another normalization. We set

$$A_0 = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}, \quad A_p = \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix},$$
$$A_j = c_{j1} + \begin{pmatrix} d_j \\ v_j \end{pmatrix} \begin{pmatrix} 1 & q_j \end{pmatrix}, \quad d_j + q_j v_j = c_{j2} - c_{j1} \quad (1 \leq j \leq p-1),$$

and normalize $q_1 = 1$ by the action of $\mathrm{GL}(1)$. Then v_1 is determined by the trace condition, and hence the system (13) becomes

$$\begin{cases} \displaystyle\sum_{j=2}^{p-1}(q_j - 1)v_j = a_1 + b_1 + \sum_{j=1}^{p-1} c_{j2}, \\ \displaystyle\sum_{j=2}^{p-1} q_j(q_j - 1)v_j = \sum_{j=2}^{p-1}(c_{j2} - c_{j1})q_j - a_1 - b_1 - c_{11} - \sum_{j=2}^{p-1} c_{j2}, \end{cases} \tag{14}$$

which is a system of linear equations in $v_2, \ldots, v_{p-1}$. If $p \geq 4$, we can solve this system in v_2, v_3, and hence get a regular coordinate after the coalescence. If $p = 3$, the result of the coalescence is rigid. Finally the case $n = 2$ and $p = 2$ is rigid. ∎

Definition 4.2. Suppose that $\mathcal{M}(\vec{\mathcal{C}})$ has a regular coordinate, and that we obtain $\mathcal{M}(\vec{\mathcal{O}})$ from $\mathcal{M}(\vec{\mathcal{C}})$ by a coalescence of eigenvalues. If $\mathcal{M}(\vec{\mathcal{O}})$ has a regular coordinate and if the regular coordinate is rational in the regular coordinate for $\mathcal{M}(\vec{\mathcal{C}})$, we say that the coalescence induces a *good reduction* from $\mathcal{M}(\vec{\mathcal{C}})$ to $\mathcal{M}(\vec{\mathcal{O}})$.

Since a regular coordinate becomes the unknowns of the deformation equation, a good reduction gives a reduction formula of deformation equations for $\mathcal{M}(\vec{\mathcal{C}})$ to the deformation equation for $\mathcal{M}(\vec{\mathcal{O}})$.

Example 4.3. We consider the sequence of coalescences

$$(1^6, 1^6, 1^6) \to (31^3, 1^6, 1^6) \to (33, 1^6, 1^6).$$

We take a regular coordinate for $(1^6, 1^6, 1^6)$ according to Proposition 2.3:

$$A_0 = \begin{pmatrix} a_1 & & O \\ & \ddots & \\ * & & a_6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_1 & & * \\ & \ddots & \\ O & & b_6 \end{pmatrix},$$

$$\begin{aligned}
A_1 &= c_1 + \begin{pmatrix} C_1 \\ U_1 \end{pmatrix} \begin{pmatrix} I_5 & P_1 \end{pmatrix}, \\
C_1 + P_1 U_1 &= c_2 - c_1 + \begin{pmatrix} C_2 \\ U_2 \end{pmatrix} \begin{pmatrix} I_4 & P_2 \end{pmatrix}, \\
C_2 + P_2 U_2 &= c_3 - c_2 + \begin{pmatrix} C_3 \\ U_3 \end{pmatrix} \begin{pmatrix} I_3 & P_3 \end{pmatrix}, \\
C_3 + P_3 U_3 &= c_4 - c_3 + \begin{pmatrix} C_4 \\ U_4 \end{pmatrix} \begin{pmatrix} I_2 & P_4 \end{pmatrix}, \\
C_4 + P_4 U_4 &= c_5 - c_4 + \begin{pmatrix} C_5 \\ U_5 \end{pmatrix} \begin{pmatrix} 1 & P_5 \end{pmatrix}, \\
C_5 + P_5 U_5 &= c_6 - c_5.
\end{aligned}$$

We normalize $U_1 = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}$, and determine P_1 by (12). Then the entries of $U_2, P_2, U_3, P_3, U_4, P_4, U_5$ and P_5 make a regular coordinate.

We consider the first coalescence $(1^6, 1^6, 1^6) \to (31^3, 1^6, 1^6)$ by taking $c_5, c_6 \to c_4$. Then

$$U_4 = O, \quad P_4 = O, \quad U_5 = O, \quad P_5 = O,$$

and the entries of U_2, P_2, U_3 and P_3 make a regular coordinate after the first coalescence. Thus the first coalescence induces a good reduction.

Next we consider the second coalescence $(31^3, 1^6, 1^6) \to (33, 1^6, 1^6)$ by taking $c_2, c_3 \to c_1$. After the second coalescence, we obtain the parametrization

$$A_1 = c_1 + \begin{pmatrix} D_1 \\ V_1 \end{pmatrix} \begin{pmatrix} I_3 & Q_1 \end{pmatrix}, \qquad D_1 + Q_1 V_1 = c_4 - c_1,$$

where

$$V_1 = \begin{pmatrix} v_{11} - (a_4 + b_4 + c_1) & v_{12} - (a_4 + b_4 + c_1) & v_{13} - (a_4 + b_4 + c_1) \\ v_{21} - (a_5 + b_5 + c_1) & v_{22} - (a_5 + b_5 + c_1) & v_{23} - (a_5 + b_5 + c_1) \\ 1 & 1 & 1 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

with the relation

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_{11} \\ q_{21} \\ q_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} v_{21} & v_{22} & v_{23} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_{12} \\ q_{22} \\ q_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$q_{13} = a_1 + b_1 + c_4 + (a_4 + b_4 + c_1)q_{11} + (a_5 + b_5 + c_1)q_{12} - q_{11}v_{11} - q_{12}v_{21},$$
$$q_{23} = a_2 + b_2 + c_4 + (a_4 + b_4 + c_1)q_{21} + (a_5 + b_5 + c_1)q_{22} - q_{21}v_{12} - q_{22}v_{22},$$
$$q_{33} = a_3 + b_3 + c_4 + (a_4 + b_4 + c_1)q_{31} + (a_5 + b_5 + c_1)q_{32} - q_{31}v_{13} - q_{32}v_{23}.$$

Thus $(v_{11}, v_{12}, v_{21}, v_{22}, q_{11}, q_{21}, q_{12}, q_{22})$ makes a regular coordinate after the second coalescence. This regular coordinate can be written in terms of the regular coordinate before the second coalescence as follows. We set

$$U_2 = \begin{pmatrix} u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix}, \quad U_3 = \begin{pmatrix} u_{31} & u_{32} & u_{33} \end{pmatrix},$$
$$P_2 = \begin{pmatrix} p_{21} \\ p_{22} \\ p_{23} \\ p_{24} \end{pmatrix}, \quad P_3 = \begin{pmatrix} p_{31} \\ p_{32} \\ p_{33} \end{pmatrix}.$$

Then we have

$$\begin{aligned}
v_{11} &= (1 - p_{31})(u_{31} - u_{33}) - p_{32}(u_{32} - u_{33}) - p_{24}(u_{21} - u_{24}),\\
v_{12} &= -p_{31}(u_{31} - u_{33}) + (1 - p_{32})(u_{32} - u_{33}) - p_{24}(u_{22} - u_{24}),\\
v_{13} &= -p_{31}(u_{31} - u_{33}) - p_{32}(u_{32} - u_{33}) - p_{24}(u_{23} - u_{24}),\\
v_{21} &= (1 - p_{21})(u_{21} - u_{24}) - p_{22}(u_{22} - u_{24}) - p_{23}(u_{23} - u_{24}),\\
v_{22} &= -p_{21}(u_{21} - u_{24}) + (1 - p_{22})(u_{22} - u_{24}) - p_{23}(u_{23} - u_{24}),\\
v_{23} &= -p_{21}(u_{21} - u_{24}) - p_{22}(u_{22} - u_{24}) + (1 - p_{23})(u_{23} - u_{24}),\\
q_{11} &= p_{31},\\
q_{21} &= p_{32},\\
q_{31} &= p_{33},\\
q_{12} &= p_{21} + p_{31}p_{24},\\
q_{22} &= p_{22} + p_{32}p_{24},\\
q_{32} &= p_{23} + p_{33}p_{24}.
\end{aligned}$$

Thus the second coalescence also induces a good reduction.

Example 4.4. In the proof of Proposition 4.1, we see that the coalescence $(11, 11, \dots, 11) \to (2, 11, \dots, 11)$ induces a good reduction. We shall show that, when $p = 4$, this good reduction gives a reduction formula of the Garnier system in two variables to the sixth Painlevé equation.

Take $p = 4$ and retain the notation in the proof of Proposition 4.1, so that $s(\vec{\mathcal{C}}) = (11, 11, 11, 11, 11)$, $s(\vec{\mathcal{O}}) = (2, 11, 11, 11, 11)$. We have a regular coordinate (v_2, v_3, q_2, q_3) for $\mathcal{M}(\vec{\mathcal{C}})$. By a coalescence $a_2 \to a_1$, we get $\vec{\mathcal{O}}$, and have a regular coordinate (q_2, q_3) for $\mathcal{M}(\vec{\mathcal{O}})$. By solving (14) in v_2 and v_3, we have

$$\begin{aligned}
v_2 &= \frac{a_1 + b_1 + c_{11} + c_{22} + c_{32} + (c_{21} - c_{22})q_2 + (a_1 + b_1 + c_{12} + c_{22} + c_{31})q_3}{(q_2 - 1)(q_3 - q_2)},\\
v_3 &= \frac{a_1 + b_1 + c_{11} + c_{22} + c_{32} + (a_1 + b_1 + c_{12} + c_{21} + c_{32})q_2 + (c_{31} - c_{32})q_3}{(q_3 - 1)(q_2 - q_3)}.
\end{aligned} \tag{15}$$

The deformation equation for $\mathcal{M}(\vec{\mathcal{C}})$ is the Garnier system in two variables, and that for $\mathcal{M}(\vec{\mathcal{O}})$ is the sixth Painlevé equation. Then we can regard (v_2, v_3, q_2, q_3) as the unknowns of the Garnier system, and (q_2, q_3) as the unknowns of the sixth Painlevé equation. Hence, if we put (15) into the Garnier system and set $a_2 = a_1$, we obtain the sixth Painlevé equation. In this way, we get the reduction formula.

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