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ON MOVABLE SINGULARITIES OF SELF-SIMILAR SOLUTIONS OF SEMILINEAR WAVE EQUATIONS

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Abstract. In this paper we analyze movable singularities of the solutions of the equation for self-similar profiles resulting from semilinear wave equation. We study local analytic solutions around two fixed singularity points of this equation—$\rho = 0$ and $\rho = 1$. The movable singularities of local analytic solutions at the origin will be connected with those of the Lane–Emden equation. The function describing approximately their position on the complex plane will be derived. For $\rho > 1$ some topological considerations will be presented that describe movable singularity of local analytic solution at $\rho = 1$. Numerical illustrations of the results will also be provided.

1. Introduction. This paper is the continuation of our studies on self-similar profiles of semilinear wave equations extended to higher space dimensions [14]. We studied the equation

$$(1-\rho^2)u'' + \Big(\frac{n-1}{\rho} - \frac{2(p+1)}{p-1}\,\rho\Big)u' - \frac{2(p+1)}{(p-1)^2}\,u + u^p = 0, \tag{1}$$

for self-similar profiles $u(\rho)$. We described conditions that lead to global analytic profiles on $\rho \in [0;1]$ for integer even $p > 2$ and integer $n \geq 3$. This equation is connected with the semilinear wave equation

$$\Phi_{tt} - \triangle\Phi - \Phi^p = 0, \qquad \Phi = \Phi(x,t), \qquad x \in R^n, \tag{2}$$

when we look for the solutions in the self-similar form

$$\Phi(t,r) = (T-t)^{-\alpha}u(\rho), \qquad \rho = \frac{r}{T-t}\,, \qquad \alpha = \frac{2}{p-1}\,, \tag{3}$$

where $r = |x|$. These solutions play an important role in time evolution of (2) as it was shown in many papers, see for example [3], [2] and the references therein. If the profile is

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regular then blowup develops when $t \to T$. As examples of simple self-similar solutions we present

$$\Phi_0(t) = \frac{b_0}{(T-t)^\alpha}, \qquad b_0 = \Big(\frac{2(p+1)}{(p-1)^2}\Big)^{1/(p-1)}, \tag{4}$$

which is obtained when we assume that the solution of (2) does not depend on spatial coordinates and solve the resulting ODE, and

$$\Phi_\infty(r) = b_\infty r^{-\alpha}, \qquad b_\infty = \Big(\frac{2(p(n-2)-n)}{(p-1)^2}\Big)^{1/(p-1)}, \tag{5}$$

which is spherically symmetric static solution. Corresponding profiles are $u_0(\rho) = b_0$ and $u_\infty(\rho) = b_\infty \rho^{-\alpha}$, respectively. The first solution is the asymptotics of generic blowup of (2) (see [2]) and the profile of the second solution is important in the proof of the existence of other global analytic profiles, see [3] and [14].

In [14] we showed that the local analytic solutions of (1) at the singular point $\rho = 0$ may be expanded in the convergent power series $u(\rho) = \sum_{l=0}^{\infty} a_l \rho^l$ with the coefficients given by the following recurrence

$$a_0 = c, \quad a_1 = 0, \quad a_{l+2} = \frac{\big(l(l-1) + l(2\alpha+2) + \alpha(\alpha+1)\big)a_l - c_l}{(l+2)(l+n)}, \tag{6}$$

where c_l coefficients are given by the Cauchy product [9]

$$\begin{aligned} \Big(\sum_{l=0}^{\infty} a_l (x-x_0)^l\Big)^p &= \sum_{l=0}^{\infty} c_l (x-x_0)^l, \\ c_0 = a_0^p, \qquad c_m &= \frac{1}{m a_0} \sum_{l=1}^{m} (lp - m + l) a_l c_{m-l}, \end{aligned} \tag{7}$$

for $m > 0$, and $a_0 = c$ is a free parameter. The Cauchy product simplifies power type nonlinearities and allows us to obtain a recurrence for the coefficients in compact form.

We also showed that at the other interesting singular point $\rho = 1$ there also exist local analytic solutions expressible by the convergent power series $u(y) = \sum_{l=0}^{\infty} a_l y^l$ with $y = 1 - \rho$ whose form is controlled by the parameter

$$k = \frac{(n-1)p - n - 3}{2(p-1)}. \tag{8}$$

For noninteger k the recurrence relation for the power series coefficients is of the form

$$\begin{aligned} a_0 &= b, \qquad a_1 = \frac{2(p+1)a_0 - (p-1)^2 c_0}{2(1-k)(p-1)^2}, \\ a_{l+1} &= \frac{\big(3l(l-1) + 2l\,\frac{2(p+1)}{p-1} + \frac{2(p+1)}{(p-1)^2}\big)a_l - c_l}{2(l+1)(l-k+1)} \\ &\quad + \frac{c_{l-1} - \big((l-1)(l-2) + (l-1)\frac{2(p+1)}{p-1} + \frac{2(p+1)}{(p-1)^2}\big)a_{l-1}}{2(l+1)(l-k+1)}, \end{aligned} \tag{9}$$

where b is a free parameter and c_l are the Cauchy product coefficients. However, when k is integer then the first k coefficients $\{a_0, \ldots, a_{k-1}\}$ have fixed numerical values taken from the set of solutions of nonlinear polynomial system presented in [14], the free parameter b

appears at y^k

$$u(y) = a_0 + \ldots + a_{k-1}y^{k-1} + by^k + f_1(b, a_0, \ldots, a_{k-1})y^{k+1} + \ldots, \tag{10}$$

and further coefficients result from the unique recurrence relation.

All these series above have nonzero radius of convergence, therefore, they are local solutions around expansion points. Moreover, it was proven that the singularities of the above solutions do not occur on the real axis for $\rho \in (0;1)$ when we do analytic continuation. Hence, we could match these asymptotics along the real axis which resulted in quantization conditions of initial data c and b at both endpoints $\rho = 0$ and $\rho = 1$, respectively—countable family of the initial data $\{c_l, b_l\}_{l=0}^{\infty}$ describes global analytic solutions on the interval $[0;1]$; see [3], [14] for details.

In this paper we examine the value of the radius of convergence for power series describing local solutions at $\rho = 0$ and $\rho = 1$. This radius is the distance from expansion point to the nearest singularity on the complex plane. The equation (1) has fixed singularities[1] connected with the singularities of the equation coefficients at $\rho = 0$, $\rho = \pm 1$ and ∞. Hence, the radius of convergence for the power series of local solutions at $\rho = 0$ and also at $\rho = 1$ should equal at most 1. However, it can be smaller because of additional movable singularities which correspond to the specific value of initial data c and b and move over the complex plane as initial conditions are varied. They are difficult to find because their location cannot be deduced from the equation but only from the solution. In practice, the solution is not known in closed form, but by a convergent Taylor series only. In this case one can calculate the circle around expansion point on which singularities are located, nonetheless, their exact location on this circle is usually a difficult task and numerical methods must be used.

The paper is organized as follows. In the first step we connect the movable singularities of local analytic solution at $\rho = 0$ with the corresponding singularities of the Lane–Emden equation. In the next step we examine the movable singularity of the local analytic solution at $\rho = 1$. Almost all results will be asymptotic ones, however, we check numerically their region of validity.

2. Local solutions at $\rho = 0$. In the study of the movable singularities of local analytic solutions at $\rho = 0$ we use vast theory of the Lane–Emden equation which can be found, for example, in [11] and in the references therein. The Lane–Emden equation (LE) is of the form

$$y'' + \frac{n-1}{x}y' + y^p = 0, \quad y = y(x), \tag{11}$$

with normalized initial conditions $y(0) = 1$, $y'(0) = 0$. This equation is used by astrophysicists to model static star structure, however, we use it because of other reasons. This equation is one of the simplest second order ODE with fixed singularities at 0, ∞ and power-type nonlinearity[2]. The equation (11) has the formal power series solution

[1] For the theory of ODEs in the complex domain see [12], [10] or [1]. Recent results on movable singularities can be found, e.g., in [8] in general setup and in [6] and [5] for Painlevé equations using Power Geometry.

[2] Moreover, first two terms in (11) are in fact the radial part of the Laplace operator $\triangle := \sum_{l=1}^{n} \partial^2_{x_l}$ which widely appears in the applications to physical systems.

$y(x) = \sum_{l=0}^{\infty} a_l x^l$ with [15]

$$a_0 = 1, \quad a_1 = 0, \quad a_{l+2} = \frac{-c_l}{(l+1)(l+n)} \quad \text{for} \quad l \geq 2, \tag{12}$$

where c_l can be computed using the Cauchy product (7). The proof for nonzero radius of convergence for the series can be conducted by the same line of argument as in [3], [14], i.e., by writing (11) as the system

$$\begin{cases} y' = v \\ xv' = -(n-1)v - xy^p, \end{cases} \tag{13}$$

from Proposition 1 in [4] the statement results. The radius of convergence of the solution given by (12) is usually finite [11], [15] and therefore there exist movable singularities that restrict the value of the radius of convergence. It is also well known that there is a pair of movable singularities located symmetrically around the origin on the imaginary axis. The position of these singularities on the complex plane are tabulated for $n = 3$ and different p in [11]. A solution outside the circle of convergence can be constructed using analytic continuation [15].

Now we describe some applications of the theory of the LE equation to the semilinear wave equation. We will show that the structure of movable singularities for local solutions of (1) at $\rho = 0$ is similar to those of the LE equation. In addition we will show the connection between the radius of convergence of the series solution of the Lane–Emden equation R_{LE} and the radius of convergence for power series solutions of semilinear wave equation $R_W(c)$ for fixed n, p and the initial value c at $\rho = 0$. We develop some hints and calculations presented in [14] and interpret the results in different way arriving at the mentioned result. We start, as in [14], from the power series solution at $\rho = 0$ with the coefficients (6). Using the rescaling of the form

$$\rho = \frac{x}{c^{(p-1)/2}}, \tag{14}$$

$$y = u/c, \tag{15}$$

this power series solution at $\rho = 0$ transforms as

$$u(\rho) = cy\Big(\frac{x}{c^{(p-1)/2}}\Big) = c\Big[\Big(1 - \frac{1}{2n}x^2 + O(x^4)\Big) + o\Big(\frac{1}{c}\Big)\Big] \tag{16}$$

and the equation (1) as ($' = \frac{d}{dx}$)

$$y'' + \frac{n-1}{x}y' + y^p = \frac{1}{c^{p-1}}\big(x^2y'' + (2\alpha+2)xy' + \alpha(\alpha+1)y\big), \tag{17}$$

where, as previously, $\alpha = 2/(p-1)$. For large values of c the leading order asymptotics of the power series (16) fulfils (17) with vanishing right hand side, i.e., the LE equation (11). This suggests that for given n, p and large initial data c the singularities of the local solutions of (1) at $\rho = 0$ correspond to the singularity of the LE equation. This correspondence is easy to obtain—using again (14) we have

$$R_{LE} \approx R_W c^{(p-1)/2} \tag{18}$$

or

$$R_{LE}^2 \approx R_W^2 c^{p-1} \tag{19}$$

for the squares of the both radii of convergence. The last formula is more convenient to use because the series have expansion in even powers of variable. In this relation for fixed

n and p the value of R_{LE} is also fixed, hence $R_W = R_W(c)$, which gives an approximate relation describing the change of the radius of convergence as the movable singularity moves on the complex plane. This connection suggests that movable singularities of the local solutions of semilinear wave equation are also located as for LE solutions. In fact, as numerical scanning of the complex plane indicates, there are two such singularities located on the imaginary axis symmetrically around the origin[3] and move only along this axis, see Figure 1. For small values of c movable singularities are outside the unit circle

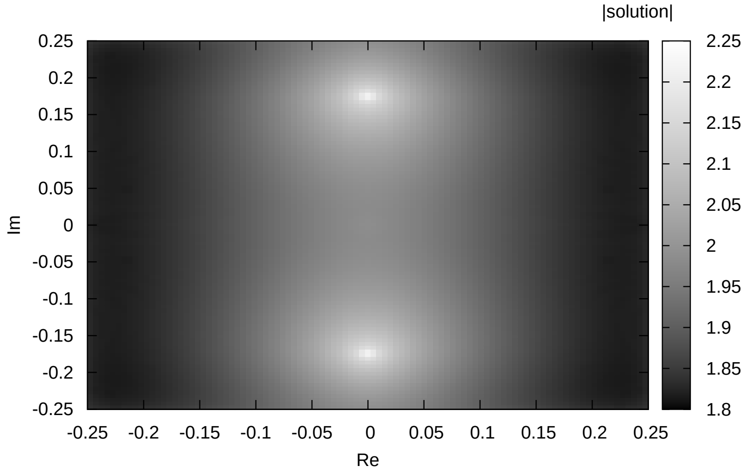

Fig. 1. Density plot of the modulus of the solution for $n = 3$, $p = 7$, $c = 2$. Singularities are located approximately at $\rho = \pm 0.17487i$.

around the origin on the complex plane (Figure 2), therefore, power series expansion at $\rho = 0$ has the radius of convergence equal to 1—the distance from the expansion point to the nearest fixed singularities at $\rho = \pm 1$. However, for given n, p, when $c \to \infty$ the movable singularities tend to the fixed singularity at $\rho = 0$, therefore, for large c these singularities enter the unit circle and this results in the decrease of the radius of convergence of power series solution at $\rho = 0$. In that sense these movable singularities may be called the confluent or coalescent ones. This result, although simple, is important because even if we do not know the closed form of the function expressible by the power series (6), we obtain the approximation to the location of its singular points.

To illustrate the accuracy of (19) we use the following simple method to estimate the square of the radius of convergence. For given n and p, using (12), we can calculate coefficients of the power series. Then the ratio $R_{LE}^2(l) = a_{2l}/a_{2l+2}$ in the limit $l \to \infty$ gives the estimation for R_{LE}^2. This is a standard ratio test adopted to the series even in the powers of x, hence square. However, this limit cannot be performed using numerical methods, so next, we fit $f(l) = \overline{R_{LE}^2} + a/l^b$ for $R_{LE}^2(l)$ and large values of l. This determines numerical estimation of the square of the radius of convergence $\overline{R_{LE}^2}$. The same procedure

[3]This is also suggested by the fact that the power series solutions for both equations are even ones.

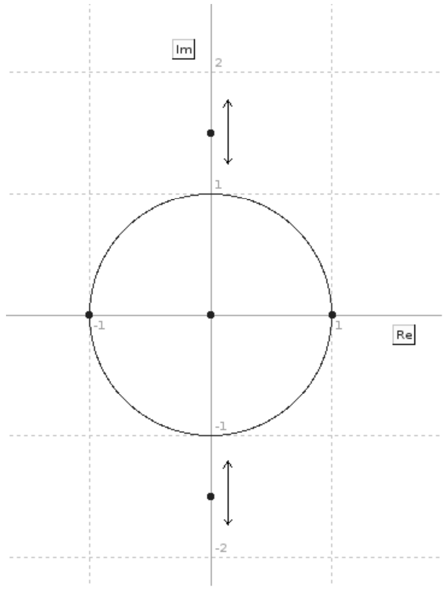

Fig. 2. Singularities of local analytic solution around $\rho = 0$ on the complex plane of ρ. Fixed singularities at $\rho = 0$ and $\rho = \pm 1$ are marked as well as the movable singularities of the power series solution at $\rho = 0$ which are located symmetrically around the origin on the imaginary axis and move along this axis. The unit circle is also drawn.

can be applied for the estimation of $\overline{R^2_W}$. We choose $n = 3$, $p = 7$ case as an example. In the first step of the verification of (19) we estimate the radius of convergence of the LE solution (Figure 3), with the result $\overline{R^2_{LE}} = 1.92916$. Using this estimation and (19)

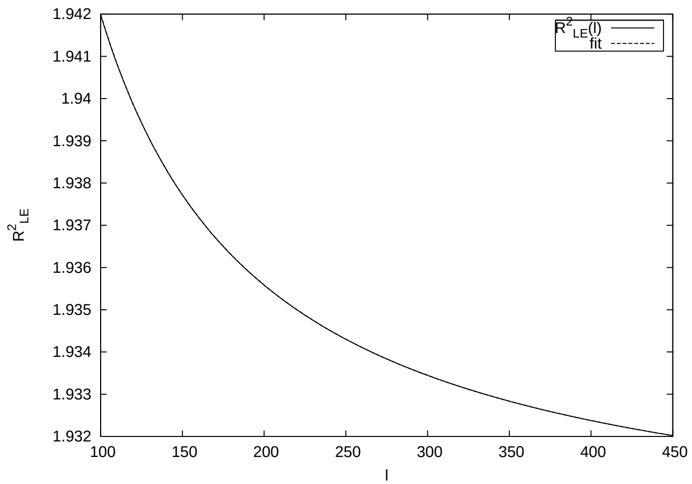

Fig. 3. $n = 3$, $p = 7$ ratio test for the Lane–Emden equation.

we obtain Table 1, from which one can see that the approximate equality (19) valid for large c is also fulfilled for small values of c. In Figure 4 we presented graphically the values from the table, and the curve (19).

c	$\overline{R_W^2}$	R_W^2-theory	$\frac{\overline{R_W^2}-R_W^2}{R_W^2}$
0.5	1.0	NA	NA
1.0	1.0	NA	NA
1.5	1.838680E−001	1.693638E−001	8.56391E−002
2.0	3.058030E−002	3.014313E−002	1.45033E−002
2.5	7.931650E−003	7.901839E−003	3.77262E−003
3.0	2.649650E−003	2.646310E−003	1.26213E−003
3.5	1.049970E−003	1.049446E−003	4.99574E−004
4.0	4.710920E−004	4.709863E−004	2.24363E−004
4.5	2.323490E−004	2.323235E−004	1.09705E−004
5.0	1.234740E−004	1.234662E−004	6.28512E−005

Table 1. $n = 3$, $p = 7$; R_W^2—theoretical estimation of the radius of convergence based on (19). The cells with the values NA indicate that movable singularities are outside the unit circle and the ratio test cannot be applied to locate them. In this case the method must be generalized by analytic continuation of local power series at $\rho = 0$ along the imaginary axis and then determination of the border for this continuation—movable singularities. We do not do this here because we only want to check the impact of movable singularities on the value of the radius of convergence of local analytic solution at $\rho = 0$.

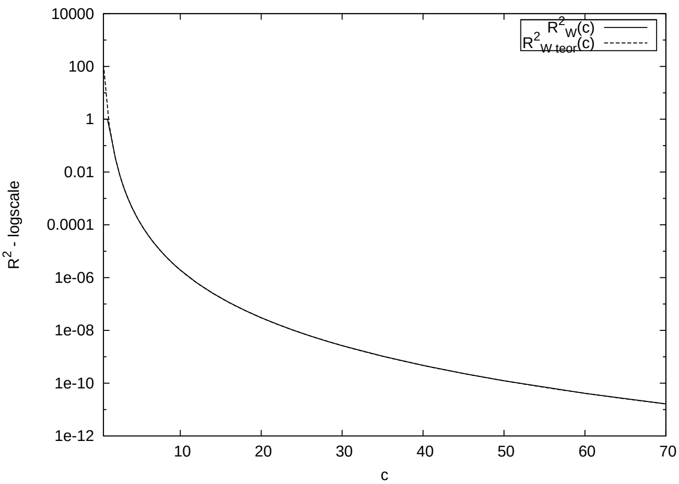

Fig. 4. $n = 3$, $p = 7$, comparison of numerical estimation and (19) for squares of the radius of convergence.

We are now in the position to derive the asymptotic of the solution close to movable singularities. We use the simple test-power test [10] and check its output numerically.

A similar method for the LE equation was applied in [11]. First, we use the fact that the power series solution at $\rho = 0$ is even in ρ, therefore, we change the independent variable $\xi = \rho^2$ in (1) and we obtain

$$4\xi(1-\xi)\frac{d^2u}{d\xi^2} + \Big(2n - 2\Big(1 + \frac{2(p+1)}{p-1}\Big)\xi\Big)\frac{du}{d\xi} - \frac{2(p+1)}{(p-1)^2}u + u^p = 0. \tag{20}$$

Next, we employ the knowledge of the fact that singularity is located on the imaginary axis and we again change variable $\zeta = -\xi$ to obtain the following approximate equation near the singularity $\rho_0 = i\sqrt{\zeta_0}$

$$-4\zeta_0(1+\zeta_0)\frac{d^2u}{d\zeta^2} \approx u^p. \tag{21}$$

Using the power type function $u(\zeta) = A(\zeta_0 - \zeta)^\beta$ as a trial solution we obtain the following singularity asymptotics

$$u(\rho) \approx \frac{(4(1+\zeta_0)\zeta_0)^{1/(p-1)}b_0}{(\zeta_0 + \rho^2)^{2/(p-1)}}, \tag{22}$$

where b_0 is given by (4). This asymptotics perfectly fits numerical solution as it was presented in Figure 5. This numerical verification, despite the fact it is approximate

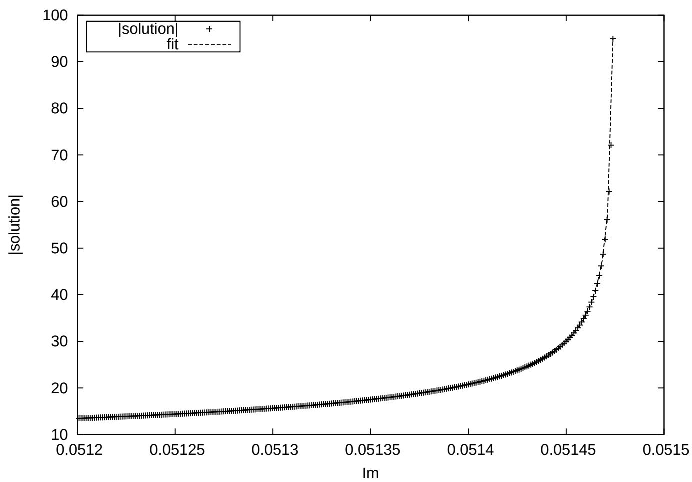

Fig. 5. Plot of the modulus of the solution for $n = 3$, $p = 7$, $c = 3$ along the imaginary axis near the movable singularity. The data were fitted with $\bar{a}/(\bar{b}^2 - x^2)^{\bar{c}}$. From the fit the values $\bar{a} = 0.409609$, $\bar{b} = 0.0514747$ and $\bar{c} = 0.333333 \approx 1/3$ were obtained in agreement with (22). The parameter $\bar{a}$, according to (22), depends on $\bar{b}$, which gives an additional method to verify the applicability of the fit of that form.

check, is vital part of the procedure. It is because the test-power test, although simple, is conclusive only when the power type ansatz fails. Otherwise, if the test gives an outcome then this result should be carefully examined, see the discussion and the examples of this ambiguity in [10], pages 90–91 or in [8].

Fit of the form (22) may be also used to determine a quite good approximation to the position of movable singularities. The consistency of the fit can be checked because in the numerator and the denominator of (22) there is a parameter connected with the position of singularity ζ_0, and other coefficients are known—they are n, p dependent.

3. Local solutions at $\rho = 1$. In this section we determine the remaining singularity structure connected with local analytic solution at $\rho = 1$ on the real axis for $\rho > 1$. To this end we show that there exist topological areas of the values of solution that determine existence of singularities. The results of this section can be visualized if we start from $\rho = 1$ and integrate local analytic solution along the real axis in the positive direction. We start with a few propositions.

PROPOSITION 3.1. *Assume that there exist ρ_0—a regular point of equation* (1) *at which $u'(\rho_0) = u''(\rho_0) = 0$. Then the solution must be $u(\rho) = 0$ or $u(\rho) = \pm b_0$.*

Proof. Using the assumptions of the proposition and the equation (1) at ρ_0 we obtain $u(\rho_0)(b_0^{p-1} - u^{p-1}(\rho_0)) = 0$ with the solutions $u(\rho_0) = 0$, $u(\rho_0) = \pm b_0$. As ρ_0 is a nonsingular point of (1), we may use uniqueness to conclude that the only solutions are $u(\rho) = 0$ or $u(\rho) = \pm b_0$, depending on $u(\rho_0)$. ■

PROPOSITION 3.2. *There is no minimum in the area $\rho > 1$, $b_0 > u(\rho) > 0$.*

Proof. Let us assume that there is a minimum of $u(\rho)$ at $\rho_0 > 1$, which means that $u'(\rho_0) = 0$ and $u''(\rho_0) > 0$. Then from (1) we get $(1-\rho_0^2)u''(\rho_0) = u(\rho_0)(b_0^{p-1} - u^{p-1}(\rho_0))$. However, according to the assumptions, $(1 - \rho_0^2) < 0$ and $u(\rho_0)(b_0^{p-1} - u^{p-1}(\rho_0)) > 0$, hence $u''(\rho_0) < 0$, which leads to contradiction. ■

In the same way we can prove the following

PROPOSITION 3.3. *There is no maximum in the area $\rho > 1$, $b_0 < u(\rho)$.*

PROPOSITION 3.4. *There is no maximum in the area $\rho > 1$, $-b_0 < u(\rho) < 0$.*

PROPOSITION 3.5. *There is no minimum in the area $\rho > 1$, $-b_0 > u(\rho)$.*

The areas from the above propositions are presented in Figure 6. As a conclusion from

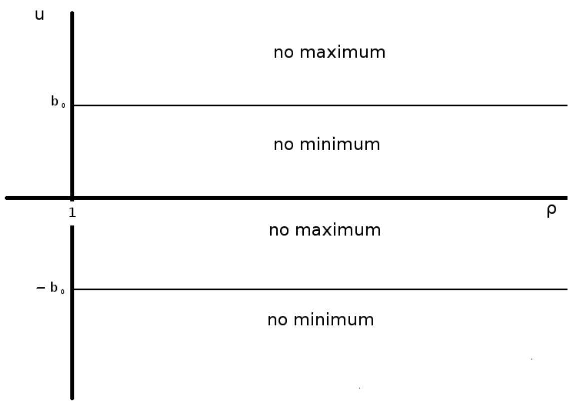

Fig. 6. “Topology” of solution for $\rho > 1$.

the propositions (see also Proposition 3.1 in [13]) we have that if $u(\rho)$ increase ($u'(\rho) > 0$) in the area $\rho > 1$, $b_0 < u(\rho)$, then singularity at some ρ_0 may appear when we continue analytically power series solution at $\rho = 1$. This leads to the conjecture that singularity may appear on the real axis for some values of initial data and may move when initial data is varied. Symmetrically, the same description applies to the area $\rho > 1$, $-b_0 > u(\rho)$ and decreasing solution.

Now we prove a few simple facts about the sign of $u'(1)$. We start from

PROPOSITION 3.6. *At $\rho = 1$ for $k < 1$ and $0 < u(1) < b_0$ we have $u'(1) < 0$.*

Proof. At $\rho = 1$ from (1) we obtain

$$2(k-1)u'(1) = u(1)(b_0^{p-1} - u^{p-1}(1)). \tag{23}$$

Assumptions imply $2(k-1) < 0$ and $u(1)(b_0^{p-1} - u^{p-1}(1)) > 0$, therefore, we get $u'(1) < 0$ as claimed. ∎

In the same manner, using (23), one can prove the following

PROPOSITION 3.7. *At $\rho = 1$ for $k < 1$ and $b_0 < u(1)$ we have $u'(1) > 0$.*

PROPOSITION 3.8. *At $\rho = 1$ for $k < 1$ and $-b_0 < u(1) < 0$ we have $u'(1) > 0$.*

PROPOSITION 3.9. *At $\rho = 1$ for $k < 1$ and $u(1) < -b_0$ we have $u'(1) < 0$.*

For $k > 1$ the sign of $u'(1)$ will reverse in the last four propositions according to (23). We must only examine $k = 1$. In this case from (23) we get $u(1) = 0$ or $u(1) = \pm b_0$ but no answer about the sign of $u'(1)$. However, as far as we consider local analytic solutions at $\rho = 1$ we can use the power series (10) to get $u'(y = 0) = -b$, where we used $\frac{d}{d\rho} = -\frac{d}{dy}$. For $b > 0$ (but the solution has also reflection symmetry) we have $u'(1) < 0$ and we get that initially the solution aims into the area where there is no minimum, therefore, there is a possibility that this solution may be continued to infinity for some values of b, namely, when it does not go outside the interval $-b_0 \leq u \leq b_0$. Nonetheless, if $|u(1)|$ is big enough then numerical examples indicate that singularity appears on the real axis and it arrives from infinity to $\rho = 1$ when $|u(1)|$ is increased—this behavior will be numerically illustrated below. In addition, when we consider only global (on $[0; 1]$) analytic solutions we have [14], [3] that $u(1)$ for noninteger k accumulate around b_∞ and the relation between b_∞ and b_0 with respect to k is as follows

- $b_\infty < b_0$ for $k < 1$;
- $b_\infty = b_0$ for $k = 1$;
- $b_\infty > b_0$ for $k > 1$.

The above results and conjectures are supported by Figures 7 which present solutions for $\rho > 1$ for a few initial values b and different values of k. They were obtained by numerical integration of the equation (1) from the vicinity of $\rho = 1$ to infinity along the real axis using as initial data the local power series solution at $\rho = 1$. The numerical integration is (approximately) equivalent to analytic continuation of the series solution [7], [16]. Numerical experiments suggest that global analytic solutions which result from the matching of local analytic solutions at $\rho = 0$ and $\rho = 1$ have values of the parameter b such that the movable singularity for $\rho > 1$ does not exist on the real axis, i.e., they are finite for $\rho > 1$ and stay in the area $0 < u(\rho) < b_0$, i.e., they are positive.

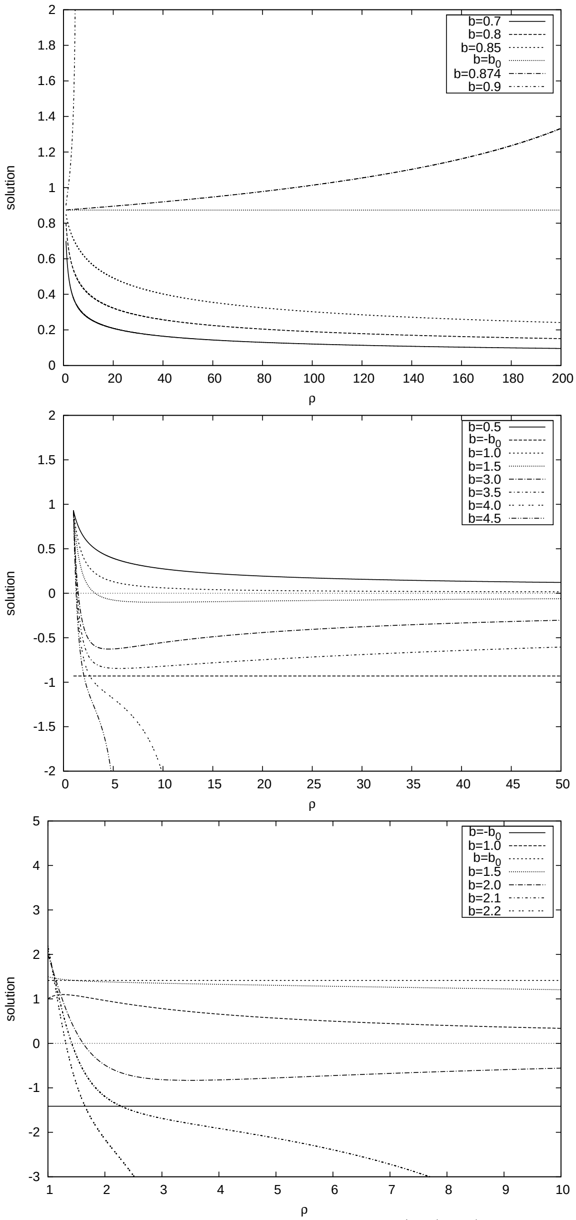

Fig. 7. Local solutions for $\rho \geq 1$ for $n = 3$, $p = 7$, $k = 2/3$ (top); $n = 4$, $p = 5$, $k = 1$ (center); $n = 6$, $p = 3$, $k = 3/2$ (bottom).

Now we focus on the asymptotics of solutions. First, we derive asymptotic for bounded solutions as $\rho \to \infty$. Assuming asymptotic solution of the form $u(\rho) \approx 1/\rho^a$, where $a > 0$, from (1) we have the approximate equality

$$-\rho^2 u'' - \Big(\frac{2(p+1)}{p-1}\,\rho\Big)u' - \frac{2(p+1)}{(p-1)^2}\,u \approx 0, \tag{24}$$

which has two solutions of the assumed form

$$u(\rho) \approx \rho^{-2/(p-1)}, \tag{25}$$

and

$$u(\rho) \approx \rho^{-(p+1)/(p-1)}. \tag{26}$$

For large ρ both approximate solutions are in the area $0 < u(\rho) < b_0$ and monotonically tend to zero. An appropriate numerical example that confirms that these asymptotics are valid for bounded solutions is presented in Figure 8. This behavior is also common for global analytic solutions as numerical data suggest.

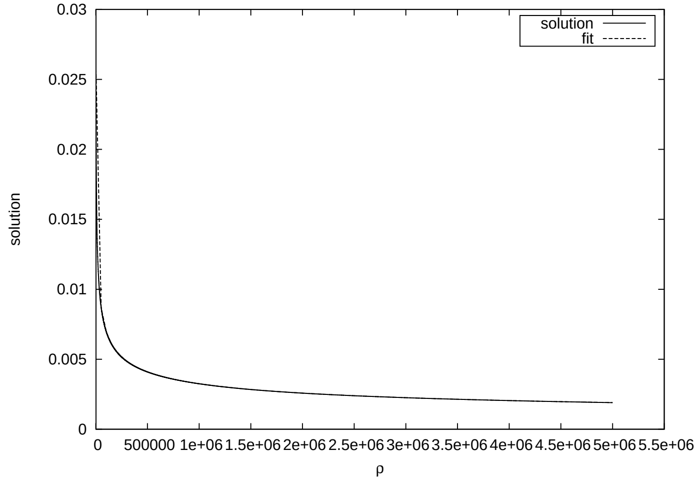

Fig. 8. Asymptotic for $\rho \to \infty$ for $n = 3$, $p = 7$, $b = 0.5$ when the solution is bounded. Fit with $f(\rho) = \bar{a} + \bar{b}/\rho^{\bar{c}}$ and $\bar{a} = 10^{-30}$, $\bar{b} = 0.324967$ and $\bar{c} = 0.333334 \approx 1/3$, which is in agreement with (25).

Next, we derive the asymptotics near the movable singularity. From the previous considerations we conclude that the singularity is on the real axis at $\rho_0 > 1$ for appropriate initial data. To determine solution behavior near this ρ_0 we change independent variable $y = \rho - 1$ in (1) to obtain

$$-y(2+y)u'' + \Big(\frac{n-1}{1+y} - \frac{2(p+1)}{p-1}(1+y)\Big)u' - \frac{2(p+1)}{(p-1)^2}\,u + u^p = 0. \tag{27}$$

In the vicinity of the singularity at $y_0 = \rho_0 - 1$ we can write the approximate equation

$$y_0(2+y_0)u'' \approx u^p. \tag{28}$$

As previously, searching for the solutions of the power type $u(y) \approx A(y_0 - y)^a$ we have the following asymptotics

$$u(\rho) \approx \frac{\big((\rho_0 - 1)(\rho_0 + 1)\big)^{1/(p-1)} b_0}{(\rho_0 - \rho)^{2/(p-1)}}\,, \tag{29}$$

where again b_0 is a constant from (4). Numerical verification of this formula is presented in Figure 9.

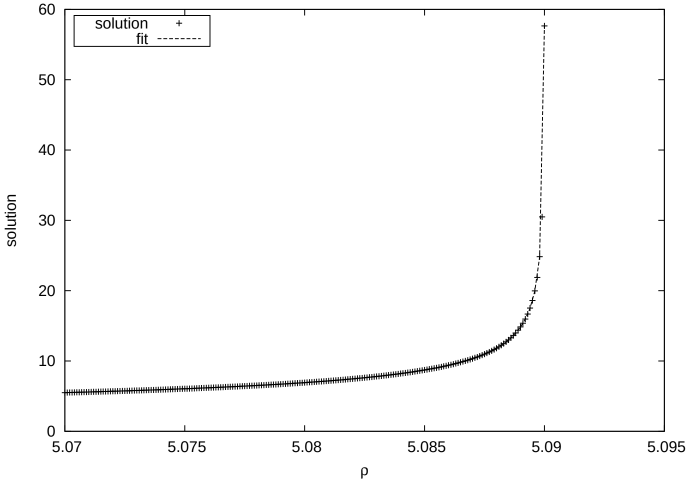

Fig. 9. Asymptotics for $n = 3$, $p = 7$, $b = 0.9$ near the singularity located on the real axis. Fit with $\bar{a}/(\bar{b} - x)^{\bar{c}}$ and $\bar{a} = 1.49289$, $\bar{b} = 5.09002$, $\bar{c} = 0.333333 \approx 1/3$. This result agrees with (29). Here, the consistency of the fit can be also checked employing the fact that the parameter $\bar{a}$ should depend on $\bar{b}$ if the prediction (29) is valid.

This section ends our examination of movable singularities for the solutions of the equation for the profiles of semilinear wave equation. We were unable to spot any further movable singularities on the complex plane.

4. Conclusions. This paper shows that movable singularities of the Lane–Emden equation are closely connected with movable singularities of the local solutions of the equation for the profiles of semilinear wave equation around the origin. These two ODEs describe phenomena which result from PDEs that contain part of the Laplace operator in spherical symmetry. This coincidence can be intuitively explained by the fact that the Lane–Emden equation is the simplest second order ODE with power type nonlinearity and simple fixed singularity structure. Moreover, it suggests the general way of treatment of such types of ODEs which often appears in physical problems. For the profiles of semilinear wave equation the two types of movable singularities connected with local power series solutions at $\rho = 0$ and at $\rho = 1$ are in some sense decoupled, that we could do our analysis independently for both local analytic solutions. However, if we take appropriate initial data we can obtain global analytic solution on $[0; 1]$, see [3], [14]. These global analytic solutions do not possess singularities on the real axis for $\rho > 1$ as numerics indicates and therefore they are important in applications as regular self-similar solutions of PDE [2].

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