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# ON $q$-ASYMPTOTICS FOR $q$-DIFFERENCE-DIFFERENTIAL EQUATIONS WITH FUCHSIAN AND IRREGULAR SINGULARITIES

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**Abstract.** This work is devoted to the study of a Cauchy problem for a certain family of $q$-difference-differential equations having Fuchsian and irregular singularities. For given formal initial conditions, we first prove the existence of a unique formal power series $\hat{X}(t,z)$ solving the problem. Under appropriate conditions, $q$-Borel and $q$-Laplace techniques (firstly developed by J.-P. Ramis and C. Zhang) help us in order to construct actual holomorphic solutions of the Cauchy problem whose $q$-asymptotic expansion in $t$, uniformly for $z$ in the compact sets of $\mathbb{C}$, is $\hat{X}(t,z)$. The small divisors phenomenon owing to the Fuchsian singularity causes an increase in the order of $q$-exponential growth and the appearance of a subexponential Gevrey growth in the asymptotics.

**1. Introduction.** This work is a slightly modified, abridged version of our paper [15], which has been published in J. Differential Equations. In particular, some basic hypothe-

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ses in that work have been simplified in our present arguments, so that results are now easier to state but less general, and some technical proofs have been omitted. Moreover, we include here some examples of the equations under consideration.

Partial differential equations of the form

$$t^{2r_2}\partial_t^{r_2}(z\partial_z)^{r_1}\partial_z^S u(t,z) = F(t,z,\partial_t,\partial_z)u(t,z), \tag{1}$$

where $S, r_1, r_2 \in \mathbb{N} := \{0,1,\dots\}$ and $F$ is some differential operator with polynomial coefficients, have been studied by the second author in [18, 17]. These equations belong to a class of partial differential equations with both irregular singularity at $t = 0$ in the sense of Mandai [21] (see also [6, 23]) and Fuchsian singularity at $z = 0$ (see, for example, [1, 4, 9, 11, 22, 31]).

Departing from 1-Borel summable formal initial data in some direction $d \in \mathbb{R}$

$$(\partial_z^j \hat{u})(t,0) = \hat{u}_j(t) \in \mathbb{C}[[t]], \quad 0 \le j \le S-1, \tag{2}$$

one can construct the formal solution $\hat{u}(t,z) = \sum_{m\ge 0}\hat{u}_m(t)z^m/m! \in (\mathbb{C}[[t]])[[z]]$ of (1), (2).

If $r_1 = 0$, $\hat{u}(t,z)$ is 1-Borel summable with respect to $t$ in the direction $d$, if this is well chosen, as a series with coefficients in the Banach space of holomorphic functions near the origin (in $z$) with the supremum norm, see [17]. Whereas if $r_1 \neq 0$, the Gevrey order with respect to $t$ suffers increasement, caused by the presence of small divisors introduced by the Fuchsian operator $(z\partial_z)^{r_1}$, see [18].

As a $q$-analog of the problem (1), (2) where $\partial_t$ is replaced by the operator $(f(qt) - f(t))/(qt - t)$ for $q \in \mathbb{C}$ (which formally tends to $\partial_t$ as $|q|$ tends to 1), we consider the $q$-difference-differential equation

$$\big((z\partial_z + 1)^{r_1}(t\sigma_q)^{r_2} + 1\big)\partial_z^S \hat{X}(t,z) = \sum_{k=0}^{S-1} b_k(z)(t\sigma_q)^{m_{0,k}}(\partial_z^k \hat{X})(t, zq^{-m_{1,k}}) \tag{3}$$

with given initial conditions

$$(\partial_z^j \hat{X})(t,0) = \hat{X}_j(t) \in \mathbb{C}[[t]], \qquad 0 \le j \le S-1, \tag{4}$$

where $S, m_{0,k}, m_{1,k}$ are nonnegative integers for $0 \le k \le S-1$ and $q \in \mathbb{C}$ with $|q| > 1$. $\sigma_q$ stands for the dilation operator $(\sigma_q \hat{X})(t,z) = \hat{X}(qt,z)$, and $b_k(z)$ are polynomials in $z$. As in previous works [19], [16], the map $(t,z) \mapsto (q^{m_{0,k}}t, zq^{-m_{1,k}})$ is assumed to be volume shrinking, meaning that the modulus of the Jacobian determinant $|q|^{m_{0,k}-m_{1,k}} < 1$. We will always assume that $r_1 \ge 0$, while $r_2 \ge 1$. The problem (3), (4) is studied under Assumption (A) (see Section 3) on the parameters involved in the equation. The following are some examples of equations solved in the present work:

Let $S = 2$, $m_{00} = 1$, $m_{10} = 5$, $m_{01} = 0$, $m_{11} = 2$ with $I_0 = \{2,3\}$ and $I_1 = \{1\}$. We fix $b_{02}, b_{03}, b_{11} \in \mathbb{C}$. If we take $r_1 = 0$ and $r_2 = 1$ the equation (3) turns into

$$t\partial_z^2 X(qt,z) + \partial_z^2 X(t,z) = (b_{02} + b_{03}z)z^2 tX(qt, zq^{-5}) + b_{11}z\partial_z X(t, zq^{-2}),$$

whilst for $r_1 = 1$ and $r_2 = 1$, the problem considered is

$$tz\partial_z^3 X(qt,z) + t\partial_z^2 X(qt,z) + \partial_z^2 X(t,z) = (b_{02} + b_{03}z)z^2 tX(qt, zq^{-5}) + b_{11}z\partial_z X(t, zq^{-2}).$$

Advanced/delayed partial differential equations have also been widely studied, see for example [12, 13, 14, 24, 30, 33]. Some authors have considered the use of special function transforms for the study of asymptotic properties of the solutions of $q$-difference-differential equations [10, 25]. Our present work is a contribution to this area.

This Cauchy problem (3), (4) has a unique formal solution $\hat{X}(t,z)=\sum_{h\geq 0}\hat{X}_h(t)\frac{z^h}{h!}$, where $\hat{X}_h(t)=\sum_{m\geq 0}f_{m,h}t^m\in\mathbb{C}[[t]]$, $h\geq 0$ (see Lemma 4.1). Our main result (Theorem 7.2) states the construction of an actual solution $X(t,z)$ which is asymptotically represented by $\hat{X}(t,z)$ in some sense to be described precisely later. For this purpose, we study the auxiliary Cauchy problem

$$\big((z\partial_z+1)^{r_1}\tau^{r_2}+1\big)\partial_z^S\hat{W}(\tau,z)=\sum_{k=0}^{S-1}b_k(z)\tau^{m_{0,k}}(\partial_z^k\hat{W})(\tau,zq^{-m_{1,k}}) \tag{5}$$

with initial conditions

$$(\partial_z^j\hat{W})(\tau,0)=\hat{W}_j(\tau)\in\mathbb{C}[[\tau]],\quad 0\leq j\leq S-1. \tag{6}$$

The $q$-Laplace transform is the key when reducing the study of (3), (4) to this auxiliary problem (see Lemma 4.2). The $q$-Laplace transform we consider was introduced by J.-P. Ramis and C. Zhang in [29], and in recent years it has been used with great success in the study of the asymptotic properties of solutions of $q$-difference equations, see [8], in much the same way as the classical Laplace–Borel transform has been applied to the asymptotic study of formal solutions to differential equations and singular perturbation problems in the complex domain (see the works of W. Balser [2, 3], B. Malgrange [20], J.-P. Ramis [26] or O. Costin [7]).

This new Cauchy problem (5), (6) is studied in two respects.

Firstly, we study the behavior of the solution when departing from initial data $W_j$ being holomorphic functions defined in a $q$-spiral $Vq^{\mathbb{Z}}=\{vq^h : v\in V,\ h\in\mathbb{Z}\}$, with $q$-exponential growth (or order 2). Here, $V\subseteq\mathbb{C}\setminus\{0\}$ is a well chosen bounded open set and $q$ is also well chosen. In Theorem 4.3 we prove there exists a unique solution of (5), (6),

$$W(\tau,z)=\sum_{h\geq 0}W_h(\tau)\,\frac{z^h}{h!}\,, \tag{7}$$

holomorphic on $Vq^{\mathbb{Z}}\times\mathbb{C}$ and of $q$-exponential growth (of order 1) in $\tau$, in the terminology of [29], uniformly for $z$ in any compact set of $\mathbb{C}$. The increase in the order may be seen as an effect of the small divisors appearing in the problem.

Secondly, if one departs from functions $W_j$, $0\leq j\leq S-1$, which are holomorphic near the origin, the coefficients in (7) turn out to be holomorphic functions in discs $D_h$ with radii tending to 0 as $h$ tends to infinity (see Theorem 6.1). Indeed,

$$\sup_{\tau\in\overline{D}_h}|\partial^n W_h(\tau)|\leq C_1\Big(\frac{1}{T}\Big)^n\Big(\frac{1}{X_1}\Big)^h n!h!(h+1)^{r_1n/r_2}|q|^{-h^2/2},\quad n,h\geq 0.$$

The constant $T>0$ is common for every element of the set of initial conditions in the auxiliary problem (14), (15) (see Theorem 3.7) and also for the ones in the auxiliary problem (22), (23) (see Theorem 4.3). A more general result concerning a wider choice of these constants is supplied in [15]. However, the corresponding constants related to $T$ in [15] suffer an increasement while they keep preserved here.

Departing from initial conditions under both assumptions, one can apply the $q$-Laplace transform on $W_h$ (see Proposition 7.1), obtaining holomorphic functions which are defined in a common domain $\mathcal{T}_{\lambda,q,\delta,r_0}$ (see (9) for its definition) for all $h \geq 0$.

The main result of this paper (Theorem 7.2) states that, if one departs from well chosen formal initial conditions $\hat{X}_j$, $0 \leq j \leq S-1$, one can find a solution of (3), (4)

$$X(t,z) = \sum_{h \geq 0} \mathcal{L}_q^{\lambda}(W_h)(t)\,\frac{z^h}{h!}\,,$$

which is holomorphic in $\mathcal{T}_{\lambda,q,\delta,r_0} \times \mathbb{C}$, and such that given $R > 0$, there exist constants $\tilde{C} > 0$, $\tilde{D} > 0$ such that for every $n \in \mathbb{N}$, $n \geq 1$, one has

$$\Big|X(t,z) - \sum_{h \geq 0}\sum_{m=0}^{n-1} f_{m,h} t^m \frac{z^h}{h!}\Big| \leq \tilde{C}\tilde{D}^n \Gamma\Big(\frac{r_1}{r_2}(n+1)\Big)|q|^{n(n-1)/2}|t|^n$$

for every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, $z \in D(0,R)$. Again one may note that the small divisors phenomenon has caused the appearance of the term $\Gamma(\frac{r_1}{r_2}(n+1))$.

The paper is organized as follows. Section 2 provides the information concerning the $q$-Laplace transform. Section 3 is devoted to the study of a first auxiliary Cauchy problem in suitable weighted Banach spaces of formal Laurent series. This is needed in the following section, more precisely, in the proof of Theorem 4.3. A second Cauchy problem in weighted Banach spaces of formal Taylor series is stated in Section 5, leading to Theorem 6.1. Finally, in Section 7, the solution of the main problem is constructed, giving asymptotic properties (see Theorem 7.2). Some final remarks on the nature of the solution in the special case that $r_1 = 0$, in which no small divisors appear, are remarked.

We fix some conventions. $\mathbb{C}^*$ stands for $\mathbb{C}\setminus\{0\}$, and $\mathbb{N}$ for the set $\{0,1,2,\dots\}$. $D(0,r)$ denotes the open disc with center 0 and radius $r > 0$. Given a set $V \subset \mathbb{C}$ and $q \in \mathbb{C}$, we define

$$Vq^{\mathbb{Z}} = \{vq^h : v \in V,\ h \in \mathbb{Z}\}, \quad Vq^{\mathbb{N}} = \{vq^h : v \in V,\ h \in \mathbb{N}\}.$$

**2. A $q$-analog of the Laplace transform and $q$-asymptotic expansion.** In [29] and [32], the authors introduce the concept of a $q$-analog of the Laplace transform. In this section, we recall this concept and some of its main properties. The proof of the next proposition is in the spirit of the one corresponding to Proposition 7.1, which can be found in [15].

PROPOSITION 2.1. *Let $q \in \mathbb{C}$ such that $|q| > 1$. Let $V$ be an open and bounded set in $\mathbb{C}^*$ and $D(0,\rho_0)$ a disc such that $V \cap D(0,\rho_0) \neq \emptyset$. Let $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ be a complex Banach space. We also fix a holomorphic function $\phi : Vq^{\mathbb{N}} \cup D(0,\rho_0) \to \mathbb{F}$ which satisfies the following estimates: there exist $C, M > 0$ such that*

$$\|\phi(xq^m)\|_{\mathbb{F}} \leq M|q|^{m^2/2}C^m \tag{8}$$

*for all $m \geq 0$, all $x \in V$. Let $\Theta$ be the Jacobi Theta function defined in $\mathbb{C}^*$ by*

$$\Theta(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n.$$

*Let $\delta > 0$ and $\lambda \in V \cap D(0,\rho_0)$. We denote by*

$$\mathcal{R}_{\lambda,q,\delta} = \Big\{ t \in \mathbb{C}^* : \Big|1 + \frac{\lambda}{tq^k}\Big| > \delta \ \forall k \in \mathbb{Z} \Big\}, \quad \mathcal{T}_{\lambda,q,\delta,r_1} = \mathcal{R}_{\lambda,q,\delta} \cap D(0,r_1). \tag{9}$$

*The $q$-Laplace transform of $\phi$ in the direction $\lambda q^{\mathbb{Z}}$ is defined by*

$$\mathcal{L}_q^{\lambda}(\phi)(t) := \sum_{m\in\mathbb{Z}} \phi(q^m\lambda)/\Theta\Big(\frac{q^m\lambda}{t}\Big)$$

*for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$, if $r_1 < |\lambda q^{1/2}|/C$. Moreover, $\mathcal{L}_q^{\lambda}(\phi)(t)$ defines a bounded holomorphic function on $\mathcal{T}_{\lambda,q,\delta,r_1}$ with values in $\mathbb{F}$ when $r_1 < |\lambda q^{1/2}|/C$. Assume that the function $\phi$ has the following Taylor expansion*

$$\phi(\tau) = \sum_{n\geq 0} \frac{f_n}{q^{n(n-1)/2}}\,\tau^n \tag{10}$$

*on $D(0,\rho_0)$, where $f_n \in \mathbb{F}$, $n \geq 0$. Then there exist two constants $D, B > 0$ such that*

$$\Big\|\mathcal{L}_q^{\lambda}(\phi)(t) - \sum_{m=0}^{n-1} f_m t^m\Big\|_{\mathbb{F}} \leq DB^n|q|^{n(n-1)/2}|t|^n \tag{11}$$

*for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$.*

Remark. In the situation described by (11) it is said that $\mathcal{L}_q^{\lambda}(\phi)$ admits the series $\sum_{m=0}^{\infty} f_m t^m$ as $q$-Gevrey asymptotic expansion of order 1 (whenever the exponent of $|q|$ in the bounds is $n(n-1)/(2r)$ the order is said to be $r$). Analogously, a function that satisfies estimates such as (8) is said to have $q$-exponential growth of order 1 in $Vq^{\mathbb{N}}$.

If $\phi(z) = \sum_{n\geq 0} a_n z^n$ is an entire function such that there exists $C > 0$ such that

$$|a_n| \leq C\exp(-(n-\alpha)^2/2)$$

for all $n \geq 0$ and some $\alpha \geq 0$, then $\phi$ satisfies the estimates (8). For a reference, see [27].

Remark. It is worth noticing that Theta Jacobi function $\Theta(x)$ satisfies the $q$-difference equation $\Theta(qx) = qx\Theta(x)$ for all $x \in \mathbb{C}^*$, so that it turns out to be a useful tool in the framework of $q$-difference equations.

In general, one has $\Theta(q^m\lambda/t) = q^{m(m+1)/2}(\lambda/t)^m\Theta(\lambda/t)$ for all $t \in \mathbb{C}^*$. Moreover, from Lemma 4.6 of [28], there exists $K_1 > 0$ such that

$$|\Theta(q^m\lambda/t)| \geq K_1\delta\sum_{n\in\mathbb{Z}} |q|^{-n(n-1)/2}\Big|\frac{q^m\lambda}{t}\Big|^n,$$

for all $t \in \mathcal{R}_{\lambda,q,\delta}$, all $m \in \mathbb{Z}$. This growth property is implicitly exploited in the present work.

It is straightforward to check the following

Proposition 2.2. *Let $V$ be an open and bounded set in $\mathbb{C}^*$ and $D(0,\rho_0)$ be a disc such that $V \cap D(0,\rho_0) \neq \emptyset$. Let $\phi$ be a holomorphic function on $Vq^{\mathbb{N}} \cup D(0,\rho_0)$ with values in $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ which satisfies the estimates: There exist $C, K > 0$ such that $\|\phi(xq^m)\|_{\mathbb{F}} \leq K|q|^{m^2/2}C^m$ for all $m \geq 0$, all $x \in V$. Then the function $M\phi(\tau) = \tau\phi(\tau)$ is holomorphic on $Vq^{\mathbb{N}} \cup D(0,\rho_0)$ and satisfies estimates of the form (8). Let $\lambda \in V \cap D(0,\rho_0)$. We have* *the equality*

$$\mathcal{L}_q^\lambda(M\phi)(t) = t\mathcal{L}_q^\lambda(\phi)(qt)$$

*for all* $t \in \mathcal{T}_{\lambda,q,\delta,r_1}$, *if* $r_1 < |\lambda q^{1/2}|/(C|q|)$.

For convenience, we recall the following concepts.

Definition 2.3. A series $\hat{f}(t) = \sum_{n\geq 0} f_n t^n \in \mathbb{C}[[t]]$ is said to be *q-Gevrey of order* 1 if its so-called *formal q-Borel transform of order* 1,

$$\hat{\mathcal{B}}_q \hat{f}(\tau) = \sum_{n\geq 0} \frac{f_n}{q^{n(n-1)/2}} \tau^n,$$

converges (i.e. it has positive radius of convergence).

The *formal q-Laplace transform of order* 1 of a series $\hat{g}(\tau) = \sum_{n\geq 0} g_n \tau^n \in \mathbb{C}[[\tau]]$ is defined as

$$\hat{\mathcal{L}}_q \hat{g}(t) = \sum_{n\geq 0} q^{n(n-1)/2} g_n t^n,$$

so that these formal transforms are inverse of each other.

It is immediate to check that, in agreement with Proposition 2.2, for every $\hat{g} \in \mathbb{C}[[\tau]]$,

$$\hat{\mathcal{L}}_q(\tau\hat{g})(t) = t\hat{\mathcal{L}}_q\hat{g}(qt). \tag{12}$$

**3. A Cauchy problem in a weighted Banach space of formal Laurent series.** With the help of the $q$-Laplace transform we will change our initial problem (3), (4) into an equivalent one (5), (6), whose study will require the consideration of two auxiliary Cauchy problems. The first of them, which we are going to present in this section, will be crucial in the study of the $q$-exponential growth of the coefficients of a solution of (5), (6). Although our equation involves a complex number $q$ with $|q| > 1$, in this section and in Section 5 we will be only concerned with the value $|q|$, so we directly work with a real value $q > 1$.

Definition 3.1. We consider the vector space $\mathbb{E}_{q,(T,X)}$ of formal Laurent power series

$$V(\xi, x) = \sum_{l\in\mathbb{Z}, h\geq 0} v_{l,h}\xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi, \xi^{-1}, x]] \tag{13}$$

such that

$$\|V(\xi, x)\|_{(T,X)} := \sum_{l\in\mathbb{Z}, h\geq 0} \frac{|v_{l,h}|}{q^{P(l,h)}} T^l \frac{X^h}{h!} < \infty,$$

where $T, X > 0$, $q > 1$ are positive real numbers and where

$$P(l,h) = \begin{cases} \frac{1}{4}l^2 + \frac{1}{2}lh - \frac{1}{2}h^2 & \text{if } l \geq 0,\ h \geq 0, \\ -\frac{1}{2}h^2 & \text{if } l \leq 0,\ h \geq 0. \end{cases}$$

The space $(\mathbb{E}_{q,(T,X)}, \|\cdot\|_{(T,X)})$ is a Banach space.

Remark. Notice that $(\mathbb{E}_{q,(T,X')}, \|\cdot\|_{(T,X')}) \hookrightarrow (\mathbb{E}_{q,(T,X)}, \|\cdot\|_{(T,X)})$ is a continuous inclusion when $0 < X \leq X'$.

We consider the integration operator $\partial_x^{-1}$ defined on $\mathbb{C}[[\xi,\xi^{-1},x]]$ by

$$\partial_x^{-1}(V(\xi,x)) := \sum_{l\in\mathbb{Z},h\geq 1} v_{l,h-1}\xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi,\xi^{-1},x]].$$

The main result in this section, Theorem 3.7, rests on the following technical lemmas whose proofs are omitted for simplicity.

Lemma 3.2. *Let $m_1, s, h_1, h_2 \geq 0$ be nonnegative integers. Let $T, X > 0$. Assume that $s+h_2 \geq 2h_1$, $m_1 \geq s+h_2$. Then there exist $C > 0$ (depending on $q$, $s, h_1, h_2, m_1$) such that $\left\|x^s(\partial_x^{-h_2}V)(q^{h_1}\xi, x/q^{m_1})\right\|_{(T,X)} \leq CX^{(s+h_2)}\left\|V(\xi,x)\right\|_{(T,X)}$ for all $V(\xi,x) \in \mathbb{E}_{q,(T,X)}$.*

Lemma 3.3. *Let $s, h_1 \geq 0$ and $T_0, X_0 > 0$. Then there exists $0 < X_1 \leq X_0 q^{-s}$ and for all $T_1 > 0$ satisfying $q^{-h_1}T_0 \leq T_1 \leq T_0 q^{s/2-h_1}$, there exists a constant $C_1 > 0$ (depending on $q, s, h_1, T_0, X_0$) such that $\left\|x^s V(q^{h_1}\xi, x)\right\|_{(T_1,X_1)} \leq C_1\|V(\xi,x)\|_{(T_0,X_0)}$ for all $V(\xi,x) \in \mathbb{E}_{q,(T_0,X_0)}$.*

Lemma 3.4. *Let $h_2 \geq 0$ and $T_0, X_0 > 0$. Then there exists $0 < X_1 < X_0$ and for all $T_1$ satisfying $T_0 \leq T_1 \leq T_0 q^{h_2/2}$, there exists a constant $C_2 > 0$ (depending on $q, h_2, T_0, X_0$) such that $\left\|\partial_x^{-h_2}V(\xi,x)\right\|_{(T_1,X_1)} \leq C_2\left\|V(\xi,x)\right\|_{(T_0,X_0)}$ for all $V(\xi,x) \in \mathbb{E}_{q,(T_0,X_0)}$.*

Let $S, m_{0,k}, m_{1,k}$, $0 \leq k \leq S-1$ be positive integers. Let $\mathcal{D}$ be the linear operator from $\mathbb{C}[[\xi,\xi^{-1},x]]$ into $\mathbb{C}[[\xi,\xi^{-1},x]]$ defined by

$$\mathcal{D}(V(\xi,x)) := \partial_x^S V(\xi,x) - \sum_{k=0}^{S-1} a_k(x)(\partial_x^k V)(q^{m_{0,k}}\xi, x/q^{m_{1,k}}),$$

for all $V \in \mathbb{C}[[\xi,\xi^{-1},x]]$, where $a_k(x) = \sum_{s\in I_k} a_{ks}x^s \in \mathbb{C}[x]$, with $I_k$ being a finite subset of $\mathbb{N}$, for $0 \leq k \leq S-1$.

We make the following hypothesis.

Assumption (A). For all $0 \leq k \leq S-1$, for all $s \in I_k$, we have

$$s - k \geq 2m_{0,k}, \quad m_{1,k} \geq s + S - k.$$

Remark. This assumption can be weakened to be

$$s + S - k \geq 2m_{0,k}, \quad m_{1,k} \geq s + S - k,$$

for all $0 \leq k \leq S-1$, for all $s \in I_k$ if one asks additional conditions on the growth properties of the initial conditions in (4). These constraints would be related to the constant $T$ in $\mathbb{E}_{q,(T,X)}$. See [15] for details.

We consider the operator $\mathcal{A}$ from $\mathbb{C}[[\xi,\xi^{-1},x]]$ into $\mathbb{C}[[\xi,\xi^{-1},x]]$ defined by

$$\mathcal{A}(V(\xi,x)) = V(\xi,x) - \mathcal{D}(\partial_x^{-S}V(\xi,x)) = \sum_{k=0}^{S-1} a_k(x)(\partial_x^{k-S}V)(q^{m_{0,k}}\xi, x/q^{m_{1,k}})$$

for all $V \in \mathbb{C}[[\xi,\xi^{-1},x]]$.

From Lemma 3.2, we deduce the following

Lemma 3.5. *Let $T>0$. Then there exists $X>0$ such that $\mathcal{A}$ is a linear bounded operator from $(\mathbb{E}_{q,(T,X)}, \|\cdot\|_{(T,X)})$ into itself. Moreover,*

$$\|\mathcal{A}(V(\xi,x))\|_{(T,X)} \leq \frac{1}{2}\|V(\xi,x)\|_{(T,X)},$$

*for all $V \in \mathbb{E}_{q,(T,X)}$.*

From Lemma 3.5, we deduce

Corollary 3.6. *Let $T>0$. Then there exists $X>0$ such that $\mathcal{D}\circ\partial_x^{-S}$ is an invertible linear operator from $(\mathbb{E}_{q,(T,X)}, \|\cdot\|_{(T,X)})$ into itself. In particular, there exists $C>0$ such that*

$$\|\mathcal{D}(\partial_x^{-S} b(\xi,x))\|_{(T,X)} \leq C\|b(\xi,x)\|_{(T,X)}$$

*for all $b(\xi,x) \in \mathbb{E}_{q,(T,X)}$.*

Theorem 3.7. *Let $S \geq 1$ be an integer. For all $0 \leq k \leq S-1$, let $m_{0,k}, m_{1,k}$ be positive integers and $a_k(x) = \sum_{s\in I_k} a_{ks}x^s \in \mathbb{C}[x]$. Let Assumption* (A) *be satisfied.*

*Consider the functional equation*

$$\partial_x^S V(\xi,x) = \sum_{k=0}^{S-1} a_k(x)(\partial_x^k V)(q^{m_{0,k}}\xi, x/q^{m_{1,k}}) \tag{14}$$

*with initial conditions*

$$(\partial_x^j V)(\xi,0) = \phi_j(\xi), \quad 0 \leq j \leq S-1. \tag{15}$$

*We assume that $\phi_j(\xi) \in \mathbb{E}_{q,(T,X_0)}$ for $0\leq j\leq S-1$, where $X_0>0$ and $T>0$. Then there exists $X>0$ such that the problem* (14), (15) *has a unique solution $V(\xi,x) \in \mathbb{E}_{(T,X)}$. Moreover, there exists $C>0$ (depending on $S, q, a_k(x), m_{0,k}, m_{1,k}$ for $0\leq k\leq S-1$ and $X_0$, $T$) such that*

$$\|V(\xi,x)\|_{(T,X)} \leq C\sum_{j=0}^{S-1}\|\phi_j(\xi)\|_{(T,X_0)}.$$

*Proof.* A formal series $V(\xi,x) \in \mathbb{C}[[\xi,\xi^{-1},x]]$ which satisfies (15) can be written in the form $V(\xi,x) = \partial_x^{-S}U(\xi,x) + I(\xi,x)$ where

$$I(\xi,x) = \sum_{j=0}^{S-1}\phi_j(\xi)\,\frac{x^j}{j!}$$

and $U(\xi,x) \in \mathbb{C}[[\xi,\xi^{-1},x]]$. A formal series $V(\xi,x) \in \mathbb{C}[[\xi,\xi^{-1},x]]$ is a solution of the problem (14), (15) if and only if $U(\xi,x)$ satisfies the equation

$$\mathcal{D}(\partial_x^{-S}U(\xi,x)) = -\mathcal{D}(I(\xi,x)). \tag{16}$$

By construction,

$$-\mathcal{D}(I(\xi,x)) = \sum_{k=0}^{S-1}\sum_{j=k}^{S-1}\sum_{s\in I_k}\frac{a_{ks}}{q^{m_{1,k}(j-k)}(j-k)!}\,x^{s+j-k}\phi_j(q^{m_{0,k}}\xi).$$

From Lemma 3.3 (taking $T_0 = T_1 := T$) and Assumption (A), there exists $X_1>0$ such that $x^{s+j-k}\phi_j(q^{m_{0,k}}\xi) \in \mathbb{E}_{q,(T,X_1)}$ for all $0\leq k\leq S-1$, all $k\leq j\leq S-1$, all $s\in I_k$.

Moreover, there exists $C_1 > 0$ (depending on $I_k, j, m_{0,k}, X_0, T$) such that

$$\|x^{s+j-k}\phi_j(q^{m_{0,k}}\xi)\|_{(T,X_1)} \leq C_1\|\phi_j(\xi)\|_{(T,X_0)}. \tag{17}$$

We deduce that $\mathcal{D}(I(\xi,x)) \in \mathbb{E}_{q,(T,X_1)}$ and from (17) there exists a constant $C_1' > 0$ (depending on $q, a_k(x), m_{0,k}, m_{1,k}$ for $0 \leq k \leq S-1$ and $X_0, T$) such that

$$\|\mathcal{D}(I(\xi,x))\|_{(T,X_1)} \leq C_1' \sum_{j=0}^{S-1} \|\phi_j(\xi)\|_{(T,X_0)}. \tag{18}$$

From Corollary 3.6, we deduce that equation (16) has a unique solution $U(\xi,x) \in \mathbb{E}_{q,(T,X_1)}$. Moreover, there exists a constant $C_2 > 0$ (depending on $q, a_k(x), m_{0,k}, m_{1,k}$ for $0 \leq k \leq S-1$) such that

$$\|U(\xi,x)\|_{(T,X_1)} \leq C_2\|\mathcal{D}(I(\xi,x))\|_{(T,X_1)}. \tag{19}$$

Take $T_1 = T_0 := T$ in Lemma 3.4. We derive there exists $X_2 < X_1$ such that $\partial_x^{-S}U(\xi,x) \in \mathbb{E}_{q,(T,X_2)}$. Moreover, there exists a constant $C_3 > 0$ (depending on $q, S, T, X_1$) such that

$$\|\partial_x^{-S}U(\xi,x)\|_{(T,X_2)} \leq C_3\|U(\xi,x)\|_{(T,X_1)}. \tag{20}$$

From Lemma 3.3 (with $T_0 = T_1 := T$), there exists $X_3 < X_2$ such that $I(\xi,x) \in \mathbb{E}_{q,(T,X_3)}$. Moreover, there exists a constant $C_4 > 0$ (depending on $S, q, T, X_0$) such that

$$\|I(\xi,x)\|_{(T,X_3)} \leq C_4 \sum_{j=0}^{S-1} \|\phi_j(\xi)\|_{(T,X_0)}. \tag{21}$$

Finally, the formal series $V(\xi,x) = \partial_x^{-S}U(\xi,x) + I(\xi,x)$, solution of problem (14), (15), belongs to $\mathbb{E}_{q,(T,X_3)}$. Moreover, from the inequalities (18), (19), (20) and (21), we get a constant $C_5$ (depending on $S, q, a_k(x), m_{0,k}, m_{1,k}$ for $0 \leq k \leq S-1$ and $X_0, T$) such that

$$\|V(\xi,x)\|_{(T,X_3)} \leq C_5 \sum_{j=0}^{S-1} \|\phi_j(\xi)\|_{(T,X_0)}. \ \blacksquare$$

**4. A Cauchy problem in analytic spaces of $q$-exponential growth.** Let $S \geq 1$, $r_1, r_2 \geq 0$ be integers. For all $0 \leq k \leq S-1$, let $m_{0,k}, m_{1,k}$ be positive integers and $b_k(z) = \sum_{s\in I_k} b_{ks}z^s$ be a polynomial in $z$, where $I_k$ is a subset of $\mathbb{N}$. The next lemma is concerned with formal solution of the problem (3), (4).

LEMMA 4.1. *For every choice of formal series $\hat{X}_j \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, the Cauchy problem (3), (4) has a unique solution in the form of a formal power series $\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t)\frac{z^h}{h!}$, where $\hat{X}_h \in \mathbb{C}[[t]]$ for every $h \geq 0$.*

With the help of the $q$-Laplace transform, we reformulate our problem. Consider the Cauchy problem

$$\big((z\partial_z + 1)^{r_1}\tau^{r_2} + 1\big)\partial_z^S \hat{W}(\tau,z) = \sum_{k=0}^{S-1} b_k(z)\tau^{m_{0,k}}(\partial_z^k\hat{W})(\tau, zq^{-m_{1,k}}) \tag{22}$$

with initial conditions

$$(\partial_z^j\hat{W})(\tau,0) = \hat{W}_j(\tau) \in \mathbb{C}[[\tau]], \quad 0 \leq j \leq S-1. \tag{23}$$

LEMMA 4.2. *The formal series* $\hat{X}(t,z)=\sum_{h\geq 0}\hat{X}_h(t)\frac{z^h}{h!}$, *where* $\hat{X}_h\in\mathbb{C}[[t]]$ *for every* $h\geq 0$, *satisfies the Cauchy problem* (3), (4) *if and only if the formal series* $\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{\mathcal{B}}_q\hat{X}_h(\tau)\frac{z^h}{h!}$ *satisfies the Cauchy problem* (22), (23) *with* $W_j(\tau)=\hat{\mathcal{B}}_q\hat{X}_j$, $0\leq j<S$.

*Conversely,* $\hat{W}(\tau,z)=\sum_{h\geq 0}\hat{W}_h(\tau)\frac{z^h}{h!}$, *with* $\hat{W}_h\in\mathbb{C}[[\tau]]$ *for every* $h\geq 0$, *satisfies the Cauchy problem* (22), (23) *if and only if the formal series* $\hat{X}(t,z)=\sum_{h\geq 0}\hat{\mathcal{L}}_q\hat{W}_h(t)\frac{z^h}{h!}$ *satisfies the Cauchy problem* (3), (4) *with* $\hat{X}_j(t)=\hat{\mathcal{L}}_q\hat{W}_j(t)$ *for* $0\leq j\leq S-1$.

*Proof.* It suffices to insert each series in the corresponding Cauchy problem and apply (12). ■

Let $V$ be an open and bounded set in $\mathbb{C}^*$, and $q\in\mathbb{C}$ with $|q|>1$. In the following result we study the $q$-exponential growth of the coefficients of a solution to the Cauchy problem (22), (23). We will depart from initial conditions $W_j$, $0\leq j\leq S-1$, holomorphic in $Vq^{\mathbb{Z}}$. We make Assumption (A) in the previous section, so that we may apply Theorem 3.7, and we also suitably choose $q$ and $V$ in order to deal with a small divisors problem.

THEOREM 4.3. *Let Assumption* (A) (*of Section* 3) *be fulfilled by the sets* $I_k$ *and the integers* $m_{0,k},m_{1,k}$, *for* $0\leq k\leq S-1$.

1) *We make the following assumptions on* $q$ *and on the open set* $V$: $q$ *is of the form* $q=|q|e^{i\theta}$, *with* $\theta=2\pi/(br_2)$ *for some* $b\in\mathbb{N}$, $b\geq 1$. *If* $V^{r_2}=\{x^{r_2}:x\in V\}$, *we assume that there exists* $\varepsilon\in(0,\min\{\pi/b,\pi/2\})$ *such that*
$$V^{r_2}\cap\Big(\bigcup_{l=0}^{b-1}S\Big(-\pi+\frac{2\pi l}{b},2\varepsilon\Big)\Big)=\emptyset,$$
*where* $S(d,\varphi)$ *stands for the unbounded sector in* $\mathbb{C}$ *with vertex at* 0, *bisected by direction* $d$ *and with opening* $\varphi$.
2) *The following assumptions on the initial conditions hold*: *Let* $T>0$. *There exists a constant* $K_0>0$ *such that*
$$\sup_{x\in V}|W_j(xq^l)|\leq K_0|q|^{l^2/4}\Big(\frac{1}{T}\Big)^l\frac{1}{1+l^2},\quad \sup_{x\in V}|W_j(xq^{-l})|\leq K_0T^l\frac{1}{1+l^2}\tag{24}$$
*for all* $0\leq j\leq S-1$, *all* $l\geq 0$.

*Then there exists a unique solution of* (22), (23)
$$(\tau,z)\mapsto W(\tau,z)=\sum_{h\geq 0}W_h(\tau)\frac{z^h}{h!}$$
*which is holomorphic on* $Vq^{\mathbb{Z}}\times\mathbb{C}$. *Moreover, for all* $\rho>0$, *there exists* $C>0$ (*depending on* $\rho,S,|q|,b_k(z),m_{0,k},m_{1,k}$ *for* $0\leq k\leq S-1$ *and* $T$) *such that*
$$\sup_{x\in V,z\in D(0,\rho)}|W(xq^l,z)|\leq CK_0|q|^{l^2/2}\Big(\frac{1}{T}\Big)^l,\qquad \sup_{x\in V,z\in D(0,\rho)}|W(xq^{-l},z)|\leq CK_0T^l\tag{25}$$
*for all* $l\geq 0$ (*where* $K_0>0$ *is defined in* (24)).

*Proof.* From the hypothesis 1) in the statement, there exists $\delta>0$ such that
$$\big|(h+1)^{r_1}x^{r_2}q^{r_2l}+1\big|>\delta\tag{26}$$
for all $l\in\mathbb{Z}$, all $h\geq 0$, all $x\in V$.

Remark. Condition 1) in the previous statement could be replaced by a more general condition, namely: Let $q$ and $V$ be such that (26) is satisfied for some $\delta > 0$ and for all $l \in \mathbb{Z}$, all $h \geq 0$, all $x \in V$. However, we prefer to use 1) because of its easy geometrical interpretation.

We consider the sequence of functions $W_h(\tau)$, $h \geq S$, defined as follows

$$\frac{W_{h+S}(xq^l)}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{b_{kh_1} x^{m_{0,k}} q^{m_{0,k}l}}{((h+1)^{r_1} x^{r_2} q^{r_2 l} + 1)} \frac{W_{h_2+k}(xq^l)}{h_2! q^{m_{1,k}h_2}} \tag{27}$$

for all $h \geq 0$, all $l \in \mathbb{Z}$, all $x \in V$. One checks that the sequence $W_h(\tau)$, $h \geq 0$, of holomorphic functions on $Vq^{\mathbb{Z}}$, satisfies the recursion (27) if and only if the formal series $W(\tau, z) = \sum_{h \geq 0} W_h(\tau) \frac{z^h}{h!}$ in the $z$ variable, satisfies the problem (22), (23). From this we deduce that the solution $W$, if exists, is unique.

According to (24) and (27), we can recursively prove that the sequence $(w_{l,h})_{l \in \mathbb{Z}, h \geq 0}$ defined by

$$w_{l,h} = \sup_{x \in V} |W_h(xq^l)|, \tag{28}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$, consists of positive real numbers. Due to (26), the sequence $(w_{l,h})_{l \in \mathbb{Z}, h \geq 0}$ satisfies the following inequalities: There exists $r > 0$ (depending on $m_{0,k}$, $V$) such that

$$\frac{w_{l,h+S}}{h!} \leq \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}|\, r\, |q|^{m_{0,k}l}}{\delta} \frac{w_{l,h_2+k}}{h_2! |q|^{m_{1,k}h_2}}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$.

We consider the sequence of real numbers $(v_{l,h})_{l \in \mathbb{Z}, h \geq 0}$ defined by the recursion

$$\frac{v_{l,h+S}}{h!} = \sum_{k=0}^{S-1} \sum_{h_1+h_2=h, h_1 \in I_k} \frac{|b_{kh_1}|\, r\, |q|^{m_{0,k}l}}{\delta} \frac{v_{l,h_2+k}}{h_2! |q|^{m_{1,k}h_2}} \tag{29}$$

with initial conditions $v_{l,j} = w_{l,j}$, for $0 \leq j \leq S-1$, all $l \in \mathbb{Z}$. By construction, we deduce that

$$w_{l,h} \leq v_{l,h} \tag{30}$$

for all $l \in \mathbb{Z}$, all $h \geq 0$.

In the following, we put $a_k(x) = \sum_{s \in I_k} (|b_{ks}| r/\delta) x^s$ for $0 \leq k \leq S-1$ and we consider the formal Laurent series $V(\xi, x) = \sum_{l \in \mathbb{Z}, h \geq 0} v_{l,h} \xi^l \frac{x^h}{h!}$. From the recursion (29), we get that $V(\xi, x)$ satisfies the following Cauchy problem

$$\partial_x^S V(\xi, x) = \sum_{k=0}^{S-1} a_k(x) (\partial_x^k V)(\xi |q|^{m_{0,k}}, x/|q|^{m_{1,k}}) \tag{31}$$

with initial conditions

$$(\partial_x^j V)(\xi, 0) = \phi_j(\xi) := \sum_{l \in \mathbb{Z}} w_{l,j} \xi^l, \quad 0 \leq j \leq S-1. \tag{32}$$

From the hypothesis (24), we get that for any $0 \leq j \leq S-1$, $\phi_j(\xi)$ belongs to $\mathbb{E}_{|q|,(T,X_0)}$, for all $X_0 > 0$. By hypothesis, Assumption (A) holds for the sets $I_k$ and the numbers $m_{0,k}, m_{1,k}$. From Theorem 3.7, we deduce that the unique solution $V(\xi, x)$ of the problem

(31), (32) satisfies $V(\xi,x) \in \mathbb{E}_{|q|,(T,X)}$ for a real number $X > 0$. Moreover, there exists a constant $C > 0$ (depending on $S, |q|, a_k(x), m_{0,k}, m_{1,k}$ for $0 \le k \le S-1$ and $X_0, T$) such that

$$\|V(\xi,x)\|_{(T,X)} \le C \sum_{j=0}^{S-1} \|\phi_j(\xi)\|_{(T,X_0)}. \tag{33}$$

From the inequality $P(l,h) \le \frac{l^2}{2} - \frac{h^2}{4}$, for all $l \in \mathbb{Z}$, $h \ge 0$, and (33) we get that there exists a constant $C' > 0$ (depending on $S, |q|, a_k(x), m_{0,k}, m_{1,k}$ for $0 \le k \le S-1$ and $X_0$, $T$) such that

$$|v_{l,h}| \le K_0 C' |q|^{l^2/2} |q|^{-h^2/4} h! \Big(\frac{1}{T}\Big)^l \Big(\frac{1}{X}\Big)^h, \quad |v_{-l,h}| \le K_0 C' |q|^{-h^2/2} T^l h! \Big(\frac{1}{X}\Big)^h \tag{34}$$

for all $l \ge 0$, all $h \ge 0$, where $K_0$ is the constant introduced in (24). From the inequalities (30) and (34), we get that

$$\sup_{x \in V, z \in D(0,\rho)} |W(xq^l, z)| \le K_0 C' |q|^{l^2/2} \Big(\frac{1}{T}\Big)^l \Big(\sum_{h \ge 0} |q|^{-h^2/4} \Big(\frac{\rho}{X}\Big)^h\Big),$$

$$\sup_{x \in V, z \in D(0,\rho)} |W(xq^{-l}, z)| \le K_0 C' T^l \Big(\sum_{h \ge 0} |q|^{-h^2/2} \Big(\frac{\rho}{X}\Big)^h\Big)$$

for all $l \ge 0$, all $\rho > 0$. So the estimates (25) hold. ■

**5. Second auxiliary Cauchy problem.** Our second approach to the auxiliary problem is to assume the initial conditions $W_h$, $0 \le h \le S-1$, of (22), (23) are holomorphic functions in suitably small neighborhoods of 0. Our purpose is to obtain information on the rate of decreasing of the derivatives of the functions $W_h$, $h \ge 0$, coefficients of the solution constructed in Theorem 4.3, near the origin. This will be done in the next section, where we will need the second auxiliary Cauchy problem we deal with in this section.

DEFINITION 5.1. Let $q > 1$ be given. Let us consider the space $\mathbb{H}_{(T,X)}$ of formal power series

$$V(\xi,x) = \sum_{l \ge 0, h \ge 0} v_{l,h} \xi^l \frac{x^h}{h!} \in \mathbb{C}[[\xi,x]]$$

such that

$$|V(\xi,x)|'_{(T,X)} := \sum_{l \ge 0, h \ge 0} |v_{l,h}| T^l q^{h^2/2} \frac{X^h}{h!} < \infty,$$

where $T, X$ are positive real numbers.

The space $(\mathbb{H}_{(T,X)}, |\cdot|'_{(T,X)})$ is a Banach algebra.

REMARK. We have a continuous inclusion $(\mathbb{H}_{(T,X')}, |\cdot|'_{(T,X')}) \hookrightarrow (\mathbb{H}_{(T,X)}, |\cdot|'_{(T,X)})$ whenever $0 < X \le X'$.

The procedure followed in this section matches point by point with the one used in Section 3 so details are omitted. In this section, only the second inequality of Assumption (A) must hold. It is worthy to point out that the series $R(\xi) := \sum_{\ell \ge 0} 2^{\ell+\xi} \xi^\ell$, involved in the following result, belongs to $\mathbb{H}_{(T,X)}$ if and only if $T < 1/2$.

THEOREM 5.2. *Let us consider the Cauchy problem*

$$\partial_x^S V(\xi,x) = \sum_{k=0}^{S-1} c_k(x) R(\xi)(\partial_x^k V)(\xi, x/|q|^{m_{1,k}}) \tag{35}$$

*with initial conditions*

$$(\partial_x^j V)(\xi,0) = \phi_j(\xi), \qquad 0 \le j \le S-1, \tag{36}$$

*and assume that* $\phi_j(\xi) \in \mathbb{H}_{(T,X_0)}$, $0 \le j \le S-1$, *where* $X_0 > 0$ *and* $0 < T < 1/2$. *Then there exists* $X_1 > 0$ *such that the problem* (35), (36) *has a unique solution* $V(\xi,x) \in \mathbb{H}_{(T,X_1)}$. *Moreover, there exists* $C > 0$ (*depending on* $S, q, X_0, T$, *and* $c_k(x), m_{1,k}$ *for* $0 \le k \le S-1$) *such that*

$$|V(\xi,x)|'_{(T,X_1)} \le C \sum_{j=0}^{S-1} |\phi_j(\xi)|'_{(T,X_0)}.$$

**6. Estimates for the derivatives of $W_j$ near the origin.** In the Cauchy problem (22), (23) we consider initial conditions $W_j$ which are holomorphic functions respectively defined in open sets containing the closed disc

$$\overline{D}_j = \{\tau : |\tau| \le 1/(2(j+1)^{r_1/r_2})\}, \quad 0 \le j \le S-1,$$

(for the sake of brevity, we say that $W_j$ is holomorphic in $\overline{D}_j$). Then Cauchy's integral formula for the derivatives allows us to obtain constants $A_j > 0$ such that for every natural number $n \ge 0$ we have $\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \le A_j^n n!$. So, the assumptions in the following result are not restrictive. Its proof is not significantly simplified with respect to the corresponding one in [15], Theorem 4, which provides a generalization, so we omit it and refer to [15] for the details.

THEOREM 6.1. *Consider the Cauchy problem* (22), (23). *Suppose* $W_j(\tau)$, $0 \le j \le S-1$, *are holomorphic functions in* $\overline{D}_j$ *such that there exist constants* $T, K > 0$ *such that*

$$\max_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \le K \left(\frac{1}{T}\right)^n \frac{n!}{1+n^2}, \quad n \ge 0, \quad j = 0,1,\dots,S-1.$$

*Then there exists a formal solution of* (22), (23), $W(\tau,z) = \sum_{h\ge0} W_h(\tau)\frac{z^h}{h!}$, *where* $W_h$ *is a holomorphic function in* $\overline{D}_h = \{\tau : |\tau| \le 1/(2(h+1)^{r_1/r_2})\}$, $h \ge S$. *Moreover, there exists a constant* $X_1 > 0$ *such that*

$$\sup_{\tau \in \overline{D}_j} |\partial^n W_j(\tau)| \le C_1 \left(\frac{1}{T}\right)^n \left(\frac{1}{X_1}\right)^j n! j! (j+1)^{r_1 n/r_2} |q|^{-j^2/2}, \tag{37}$$

*for every* $n, j \ge 0$, *where* $C_1$ *is a positive constant* (*depending on* $S, q, T, b_k(z)$ *and* $m_{1,k}$ *for* $0 \le k \le S-1$).

**7. Analytic solutions of the Cauchy problem with Fuchsian and irregular singularities.** Let $W_h$ be the initial data in the Cauchy problem (22), (23), and suppose they are subject to the hypotheses of Theorem 4.3 and to the hypotheses in Theorem 6.1. Those results give us a sequence of functions $\{W_h\}_{h\ge0}$, holomorphic in $Vq^{\mathbb{Z}} \cup D_h$ for each

$h \geq 0$, and such that the series

$$W(\tau, z) = \sum_{h \geq 0} W_h(\tau) \frac{z^h}{h!}$$

defines a holomorphic function on $Vq^{\mathbb{Z}} \times \mathbb{C}$ which solves the Cauchy problem.

Moreover, from (28), (30) and (34) in the proof of Theorem 4.3 we know that

$$\sup_{x \in V} |W_h(xq^l)| \leq K_0 C' |q|^{l^2/2} |q|^{-h^2/4} h! \Big(\frac{1}{T}\Big)^l \Big(\frac{1}{X}\Big)^h \tag{38}$$

for all $l, h \geq 0$.

Let us choose $\lambda \in V$ and $\delta > 0$. By (38) we see that every $W_h$ satisfies estimates as those in (8). If we choose an integer $n(h)$ in such a way that $\lambda q^{n(h)} \in D_h$, then, according to Proposition 2.1, the $q$-Laplace transform of $W_h$ in the direction $\lambda q^{n(h)} q^{\mathbb{Z}}$, which clearly equals $\lambda q^{\mathbb{Z}}$, is given by

$$\mathcal{L}_q^{\lambda q^{n(h)}}(W_h)(t) = \sum_{m \in \mathbb{Z}} \frac{W_h(q^m \lambda q^{n(h)})}{\Theta\big(q^m \lambda q^{n(h)}/t\big)} = \sum_{m \in \mathbb{Z}} \frac{W_h(q^m \lambda)}{\Theta(q^m \lambda / t)},$$

so that it deserves to be denoted by $\mathcal{L}_q^{\lambda}(W_h)(t)$. This function is well defined and holomorphic in the set $\mathcal{T}_{\lambda q^{n(h)}, q, \delta, r(h)}$, which is equal to $\mathcal{T}_{\lambda, q, \delta, r(h)}$, whenever $r(h) < |\lambda q^{n(h)} q^{1/2}| T$. We will show that these radii $r(h)$ can be taken independent of $h$, equal to $r_0 = |\lambda q^{1/2}| T / |q| = |\lambda q^{-1/2}| T$ for every $h \geq 0$, and we will obtain precise estimates for the corresponding $q$-asymptotic expansions.

Let us assume that the function $W_h$ has the following Taylor expansion at 0,

$$W_h(\tau) = \sum_{n \geq 0} \frac{f_{n,h}}{q^{n(n-1)/2}} \tau^n, \tag{39}$$

where $f_{n,h} \in \mathbb{C}$, $n, h \geq 0$, and $\tau \in \overline{D}_h$.

The proof for the next result can entirely be reproduced regarding Proposition 3 in [15], so its demonstration is omitted. We only point out that the estimates (37) in Theorem 6.1, the properties of Theta Jacobi function (see the second remark after Proposition 2.1) and the estimates in (38) are taken into account in its proof which also rests on an appropriate modification of the corresponding one in Proposition 2.1, providing the estimates in (11).

Proposition 7.1. *In the situation assumed in this section, there exist $B(h), D(h) > 0$ with*

$$B(h) = A_1 (h+1)^{r_1/r_2}, \qquad D(h) = A_2 (h+1)^{r_1/r_2} h! A_3^h |q|^{-h^2/4}, \tag{40}$$

*where $A_1, A_2$ and $A_3$ are positive constants that do not depend on $h$, such that*

$$\Big| \mathcal{L}_q^{\lambda}(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h} t^m \Big| \leq D(h) B(h)^n |q|^{n(n-1)/2} |t|^n \tag{41}$$

*for all $n \geq 1$, for all $t \in \mathcal{T}_{\lambda, q, \delta, r_0}$.*

We are ready to obtain our main result.

Theorem 7.2. *Suppose* $\hat{X}_j(t) = \sum_{m\geq 0} f_{m,j}t^m \in \mathbb{C}[[t]]$, $0 \leq j \leq S-1$, *are given initial conditions for the Cauchy problem* (3), (4), *and let*

$$\hat{X}(t,z) = \sum_{h\geq 0} \hat{X}_h(t)\,\frac{z^h}{h!} = \sum_{h\geq 0}\sum_{m\geq 0} f_{m,h}t^m\,\frac{z^h}{h!}$$

*be the only formal series solution of the problem* (*see Lemma* 4.1). *We suppose that the series* $\hat{X}_j(t)$, $0 \leq j \leq S-1$, *are* $q$*-Gevrey of order* 1, *and that their formal* $q$*-Borel transforms of order* 1, $W_j(\tau) = \hat{\mathcal{B}}_q\hat{X}_j(\tau)$, *which are holomorphic functions around* 0, *indeed satisfy the assumptions of Theorems* 4.3 *and* 6.1. *We also assume that the remaining hypotheses of Theorem* 4.3 *are satisfied. Let* $W(\tau,z) = \sum_{h\geq 0} W_h(\tau)\,\frac{z^h}{h!}$ *be the solution of the Cauchy problem* (22), (23), *corresponding to the initial conditions* $W_j$, $0 \leq j \leq S-1$. *Then*

1) *The function* $X(t,z) = \sum_{h\geq 0} \mathcal{L}_q^{\lambda}(W_h)(t)\frac{z^h}{h!}$ *is holomorphic in* $\mathcal{T}_{\lambda,q,\delta,r_0} \times \mathbb{C}$.
2) *The function* $X(t,z)$ *solves the Cauchy problem* (3), (4).
3) *If* $r_1 \geq 1$, *given* $R > 0$ *there exist constants* $\tilde{C}, \tilde{D} > 0$ *such that for every* $n \in \mathbb{N}\setminus 0$,

$$\Big|X(t,z) - \sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m\,\frac{z^h}{h!}\Big| \leq \tilde{C}\tilde{D}^n\Gamma\Big(\frac{r_1}{r_2}(n+1)\Big)|q|^{n(n-1)/2}|t|^n \tag{42}$$

*for every* $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, $z \in D(0,R)$.

*If* $r_1 = 0$, *given* $R > 0$ *there exist constants* $\tilde{C}, \tilde{D} > 0$ *such that for every* $n \in \mathbb{N}$, $n \geq 1$,

$$\Big|X(t,z) - \sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m\,\frac{z^h}{h!}\Big| \leq \tilde{C}\tilde{D}^n|q|^{n(n-1)/2}|t|^n \tag{43}$$

*for every* $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$, $z \in D(0,R)$.

Remark. Due to the estimates (42) and (43), we may say that the function $X(t,z)$ admits the series $\sum_{h\geq 0}\sum_{m\geq 0} f_{m,h}t^m\,\frac{z^h}{h!}$ as $q$-asymptotic expansion of order 1 in $t$, uniformly for $z$ in the compact subsets of $\mathbb{C}$. It may be noted that, because of the small divisors problem we have dealt with, a new factor appears in the estimates, in terms of the Eulerian Gamma function. The value $r_1/r_2$ may be thought of as a sub-order, or a second-level order, in the asymptotic expansion.

*Proof.* 1) In view of (41), for $n = 1$, and (40) we have that

$$|\mathcal{L}_q^{\lambda}(W_h)(t) - f_{0,h}| \leq D(h)B(h)|t| \leq A_1(h+1)^{2r_1/r_2}A_2h!A_3^h|q|^{-h^2/4}r_0$$

for every $h \geq 0$, every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$. According to the estimates (37) we have

$$|f_{0,h}| \leq C_1\Big(\frac{1}{X_1}\Big)^h h!|q|^{-h^2/2}$$

for every $h \geq 0$. So, we conclude that there exist $A_4, A_5 > 0$ such that

$$|\mathcal{L}_q^{\lambda}(W_h)(t)| \leq A_4A_5^h h!|q|^{-h^2/4}$$

for every $h \geq 0$, every $t \in \mathcal{T}_{\lambda,q,\delta,r_0}$. Then for $z \in D(0,R)$

$$\Big|\sum_{h\geq 0} \mathcal{L}_q^{\lambda}(W_h)(t)\,\frac{z^h}{h!}\Big| \leq \sum_{h\geq 0} A_4(A_5R)^h|q|^{-h^2/4} < \infty,$$

so that the series converges and the function it defines is holomorphic in $\mathcal{T}_{\lambda,q,\delta,r_0} \times \mathbb{C}$.

2) Since the series $\sum_{h\geq 0} W_h(\tau)\frac{z^h}{h!}$ is a solution of (22), (23), one can guarantee that $X(t,z)$ is a solution of the Cauchy problem (3), (4) by Proposition 2.2.

3) For every $n \geq 1$, every $(t,z) \in \mathcal{T}_{\lambda,q,\delta,r_0} \times D(0,R)$, the sum

$$\sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m \frac{z^h}{h!}$$

is convergent, as we see from the estimates in (37). One may take into account (41) and (40) and write

$$\begin{aligned}\Big|X(t,z) - \sum_{h\geq 0}\sum_{m=0}^{n-1} f_{m,h}t^m \frac{z^h}{h!}\Big| &\leq \sum_{h\geq 0}\Big|\mathcal{L}_q^\lambda(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h}t^m\Big| \frac{R^h}{h!}\\ &\leq A_2A_1^n|q|^{n(n-1)/2}|t|^n \sum_{h\geq 0}(h+1)^{r_1(n+1)/r_2}(A_3R)^h|q|^{-h^2/4}\\ &= \frac{A_2}{A_3R}A_1^n|q|^{n(n-1)/2}|t|^n \sum_{h\geq 1} h^{r_1(n+1)/r_2}(A_3R)^h|q|^{-(h-1)^2/4}. \quad (44)\end{aligned}$$

In case $r_1 = 0$, the conclusion easily follows, since the last sum is convergent and independent of $n$. In case $r_1 \geq 1$, we follow an idea of B. Braaksma and L. Stolovitch [5]. Let $\varepsilon > 0$, and let $\gamma$ be a contour that goes from $\infty e^{-i\pi}$ to $-\varepsilon$ along the negative real axis, then it turns once around 0 in the positive sense, and it goes from $-\varepsilon$ to $\infty e^{i\pi}$ again along the negative real axis. For $\mu := \frac{r_1(n+1)}{r_2} > 0$, Hankel's formula allows us to write $\frac{h^\mu}{\Gamma(\mu+1)} = \frac{1}{2\pi i}\int_\gamma e^{hs}s^{-\mu-1}\,ds$, so that the sum in (44) may be written as

$$\begin{aligned}&\frac{\Gamma(\mu+1)}{2\pi i}\sum_{h\geq 1}(A_3R)^h|q|^{-(h-1)^2/4}\int_\gamma e^{hs}s^{-\mu-1}\,ds\\ &\qquad = \frac{\Gamma(\mu+1)}{2\pi i}\sum_{h\geq 1}\int_\gamma s^{-\mu-1}|q|^{-(h-1)^2/4}(A_3Re^s)^h\,ds. \quad (45)\end{aligned}$$

We consider now the entire function $F(z) = \sum_{h\geq 1}|q|^{-(h-1)^2/4}z^h$, $z \in \mathbb{C}$. The series converges uniformly in every closed disc. Observe that as $s$ runs over $\gamma$, its real part remains bounded above, and the same is valid for the modulus of $A_3Re^s$. So, we may write $F(A_3Re^s) = \sum_{h\geq 1}|q|^{-(h-1)^2/4}(A_3Re^s)^h$ uniformly in $\gamma$, and the dominated convergence theorem ensures that

$$\begin{aligned}\sum_{h\geq 1}\int_\gamma s^{-\mu-1}|q|^{-(h-1)^2/4}(A_3Re^s)^h\,ds &= \int_\gamma s^{-\mu-1}\sum_{h\geq 1}|q|^{-(h-1)^2/4}(A_3Re^s)^h\,ds\\ &= \int_\gamma s^{-\mu-1}F(A_3Re^s)\,ds. \quad (46)\end{aligned}$$

Moreover, $F(A_3Re^s)$ remains bounded as $s$ runs over $\gamma$, say by $M > 0$, and it is easy to obtain, estimating on each of the three parts of $\gamma$, that

$$\Big|\int_\gamma s^{-\mu-1}F(A_3Re^s)\,ds\Big| \leq 2\frac{M}{\mu\varepsilon^\mu} + \frac{2\pi M}{\varepsilon^\mu} \leq \frac{\tilde{M}^\mu}{\mu\varepsilon^\mu}, \quad (47)$$

where $\tilde{M} > 0$ is some suitable constant independent of $h$. Gathering (44), (45), (46) and (47) and from the definition of $\mu$ one can conclude. ■

Remark. The case $r_1 = 0$, as it may be seen in the last theorem, deserves some attention, since the Fuchsian singularity at $z = 0$ does not appear any more. The most important consequence of this fact is the disappearance of the small divisors phenomenon we had in general.

Moreover, the condition 1) in Theorem 4.3, concerning the argument of $q$ and the set $V$, can be relaxed. Indeed, the estimates (26) hold if one assumes that there exists $\delta > 0$ such that $\mathrm{dist}(V^{r_2}q^{r_2\mathbb{Z}}, \{-1\}) > \delta$, where dist is the Euclidean distance between two sets in $\mathbb{C}$. For example, suppose $V$ is such that there exist $R_1, R_2$ with $0 < R_1 \le |x^{r_2}| \le R_2$ for all $x \in V$, and suppose that $R_2 < |q|R_1$ and $|q|^{r_2 j}R_2 < 1 < |q|^{r_2(j+1)}R_1$ for some $j \in \mathbb{Z}$. In Theorem 6.1 all the functions $W_h$ are holomorphic in a common disc, say $\overline{D}$, and there exists a constant $X_1 > 0$ such that

$$\sup_{\tau\in\overline{D}} |\partial^n W_j(\tau)| \le C_1\Big(\frac{1}{T}\Big)^n\Big(\frac{1}{X_1}\Big)^j n!j!|q|^{-j^2/2}$$

for every $n, j \ge 0$. The proof of Proposition 7.1 admits some simplification, and one obtains that

$$\Big|\mathcal{L}_q^\lambda(W_h)(t) - \sum_{m=0}^{n-1} f_{m,h}t^m\Big| \le A_2 h! A_3^h |q|^{-h^2/4} A_1^n |q|^{n(n-1)/2}|t|^n,$$

for every $h \ge 0$, $n \ge 1$. Finally, no sub-order appears in the $q$-asymptotic expansion of the solution $X(t,z)$.

## References


[1] S. Alinhac, *Problèmes de Cauchy pour des opérateurs singuliers*, Bull. Soc. Math. France 102 (1974), 289–315.

[2] W. Balser, *From Divergent Power Series to Analytic Functions. Theory and Application of Multisummable Power Series*, Lecture Notes in Math. 1582, Springer, Berlin, 1994.

[3] W. Balser, *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*, Universitext, Springer, New York, 2000.

[4] A. Bove, J. E. Lewis, C. Parenti, *Propagation of Singularities for Fuchsian Operators*, Lecture Notes in Math. 984, Springer, Berlin, 1983.

[5] B. Braaksma, L. Stolovitch, *Small divisors and large multipliers*, Ann. Inst. Fourier (Grenoble) 57 (2007), 603–628.

[6] H. Chen, Z. Luo, H. Tahara, *Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity*, Ann. Inst. Fourier (Grenoble) 51 (2001), 1599–1620.

[7] O. Costin, *Asymptotics and Borel Summability*, Chapman Hall/CRC Monogr. Surv. Pure Appl. Math. 141, CRC Press, Boca Raton, FL, 2009.

[8] L. Di Vizio, J.-P. Ramis, J. Sauloy, C. Zhang, *Équations aux q-différences*, Gaz. Math. 96 (2003), 20–49.

[9] R. Gérard, H. Tahara, *Singular Nonlinear Partial Differential Equations*, Aspects Math., Friedr. Vieweg and Sohn, Braunschweig, 1996.

[10] C.-L. Ho, *On the use of Mellin transform to a class of q-difference-differential equations*, Phys. Lett. A 268 (2000), 217–223.

[11] K. Igari, *Removable singularities of holomorphic solutions of linear partial differential equations*, J. Math. Soc. Japan 56 (2004), 87–113.

[12] T. Kato, *Asymptotic behavior of solutions of the functional differential equation* $y'(x) = ay(\lambda x) + by(x)$, in: Delay and Functional Differential Equations and their Applications (Park City, 1972), Academic Press, New York, 1972, 197–217.

[13] M. Kawagishi, T. Yamanaka, *On the Cauchy problem for non linear PDEs in the Gevrey class with shrinkings*, J. Math. Soc. Japan 54 (2002), 649–677.

[14] M. Kawagishi, T. Yamanaka, *The heat equation and the shrinking*, Electron. J. Differential Equations 2003, no. 97, 1–14.

[15] A. Lastra, S. Malek, J. Sanz, *On q-asymptotics for linear q-difference-differential equations with Fuchsian and irregular singularities*, J. Differential Equations 252 (2012), 5185–5216.

[16] S. Malek, *On functional linear partial differential equations in Gevrey spaces of holomorphic functions*. Ann. Fac. Sci. Toulouse Math. (6) 16 (2007), 285–302.

[17] S. Malek, *On singularly perturbed partial integro-differential equations with irregular singularity*, J. Dyn. Control Syst. 13 (2007), 419–449.

[18] S. Malek, *On Gevrey functional solutions of partial differential equations with Fuchsian and irregular singularities*, J. Dyn. Control Syst. 15 (2009), 277–305.

[19] S. Malek, *On complex singularity analysis for linear q-difference-differential equations*. J. Dyn. Control Syst. 15 (2009), 83–98.

[20] B. Malgrange, *Sommation des séries divergentes*, Exposition. Math. 13 (1995), 163–222.

[21] T. Mandai, *Existence and nonexistence of null-solutions for some non-Fuchsian partial differential operators with* $T$*-dependent coefficients*, Nagoya Math. J. 122 (1991), 115–137.

[22] T. Mandai, *The method of Frobenius to Fuchsian partial differential equations*, J. Math. Soc. Japan 52 (2000), 645–672.

[23] S. Ōuchi, *Multisummability of formal solutions of some linear partial differential equations*, J. Differential Equations 185 (2002), 513–549.

[24] D. W. Pravica, M. J. Spurr, *Analytic continuation into the future*, in: Dynamical Systems and Differential Equations (Wilmington, 2002), Discrete Contin. Dyn. Syst. 2003, suppl., 709–716.

[25] D. W. Pravica, M. J. Spurr, *Unique summing of formal power series solutions to advanced and delayed differential equations*, Discrete Contin. Dyn. Syst. 2005, suppl., 730–737.

[26] J.-P. Ramis, *Dévissage Gevrey*, Astérisque, 59–60 (1978), 173–204.

[27] J.-P. Ramis, *About the growth of entire functions solutions of linear algebraic q-difference equations*, Ann. Fac. Sci. Toulouse Math. (6) 1 (1992), 53–94.

[28] J.-P. Ramis, J. Sauloy, C. Zhang, *Local analytic classification of q-difference equations*, preprint `arXiv:0903.0853`.

[29] J.-P. Ramis, C. Zhang, *Développement asymptotique q-Gevrey et fonction thêta de Jacobi*, C. R. Math. Acad. Sci. Paris 335 (2002), 899–902.

[30] T. Yamanaka, M. Kawagishi, *A Cauchy-Kovalevskaja type theory in the Gevrey class for PDEs with shrinkings*, Nonlinear Anal. 64 (2006), 1860–1884.

[31] H. Yamane, *Global Fuchsian Cauchy problem*, J. Math. Sci. Univ. Tokyo 7 (2000), 147–162.

[32] C. Zhang, *Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques: théorie générale et exemples*, in: Differential Equations and the Stokes Phenomenon, World Sci. Publ., River Edge, NJ, 2002, 309–329.

[33] O. Zubelevich, *An elliptic equation with perturbed* $p$*-Laplace operator*, J. Math. Anal. Appl. 328 (2007), 1188–1195.