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A NOTION OF BOUNDEDNESS FOR HYPERFUNCTIONS AND MASSERA TYPE THEOREMS

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Abstract. For some classes of periodic linear ordinary differential equations and functional equations, it is known that the existence of a bounded solution in the future implies the existence of a periodic solution. In order to think on such phenomena for hyperfunction solutions to linear functional equations, we introduced a notion of bounded hyperfunctions, and translated the problems into the problems on analytic solutions to some equations in complex domains.

In this article, after recalling the terminology, we briefly review our approach to the equations with finite delay, and announce our recent result on some classes of equations with infinite delay.

1. Introduction

1.1. Hyperfunctions. The notion of hyperfunction was introduced in the late 50's by M. Sato (refer to [S1], [S2], [S3]), and plays important roles in the study of analytic ordinary and partial differential equations. The following properties are well-known. Hyperfunctions form a flabby sheaf $\mathscr{B}$ on $\mathbb{R}^n$, or on a real-analytic manifold, and they admit boundary value representations by holomorphic defining functions on wedge domains. A linear differential operator $P \in \mathscr{D}$ with analytic coefficients acts on hyperfunctions, and its actions on hyperfunctions are directly given from the actions on defining functions. On the other hand, there is no good topology of $\mathscr{B}(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$. Moreover, there are no inequality nor boundedness for $f \in \mathscr{B}(\Omega)$.

Hyperfunctions on $\mathbb{R}^n$ are defined in terms of local cohomology groups of the sheaf of holomorphic functions on $\mathbb{C}^n$. Here, let us first recall the notion of hyperfunctions in one-dimensional case.

We denote by $\mathscr{O}$ the sheaf of holomorphic functions on $\mathbb{C}$. For $\Omega \subset \mathbb{R}$, a complex neighborhood of Ω is an open set in $\mathbb{C}$ including Ω as a closed subset.

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Definition 1.1. The space $\mathscr{B}(\Omega)$ of hyperfunctions on an open set $\Omega \subset \mathbb{R}$ is defined by

$$\mathscr{B}(\Omega) := \varinjlim_{U} \frac{\mathscr{O}(U \setminus \Omega)}{\mathscr{O}(U)} . \tag{1.1}$$

Here, U runs through complex neighborhoods of Ω.

As we already mentioned, the correspondence $\Omega \mapsto \mathscr{B}(\Omega)$ becomes a flabby sheaf on $\mathbb{R}$, and differential operators with analytic coefficients act on $\mathscr{B}(\Omega)$ via their actions on $\mathscr{O}(U \setminus \Omega)$.

1.2. Classical Massera type theorems. In [M], Massera studied the existence of a periodic solution to a periodic ordinary differential equation of normal form, and in the linear case, he gave the following result.

Theorem 1.2 (Massera, linear case). *Consider a linear ordinary differential equation*

$$\frac{dx}{dt} = A(t)x + f(t),$$

where $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ and $f : \mathbb{R} \to \mathbb{R}^m$ are 1-periodic and continuous. Then the existence of a bounded solution in the future (i.e., defined and bounded on $\{t > t_0\}$ with some t_0) implies the existence of a 1-periodic solution.

Note that the inverse implication of Theorem 1.2 follows from the fact that periodic C^1-functions are bounded on $\mathbb{R}$. Therefore, we have the equivalence between the existence of a bounded solution in the future and the existence of a 1-periodic solution.

After Massera, many generalizations by many authors have appeared also on linear functional equations. Let ω be a positive constant (which represents the period of a periodic equation). We are interested in the following

Problem. Consider an ω-periodic linear functional equation. Does "the existence of a bounded solution in the future" imply "the existence of an ω-periodic solution"?

We give here a few references on this problem.

Chow–Hale [CH] studied functional differential equations with retarded type. An example is

$$\frac{dx}{dt} = A(t)x + \int_0^r B(t,s)x(t-s)\,ds + f(t). \tag{1.2}$$

Here A, B (resp. f) are square matrices (resp. a vector) of size m, whose entries are continuous and ω-periodic in t, and $r > 0$ is a constant representing the "delay".

Hino–Murakami [HM1] studied similar equations with infinite delay,

$$\frac{dx}{dt} = A(t)x + \int_0^\infty B(t,s)x(t-s)\,ds + f(t). \tag{1.3}$$

Here we assume the continuity and ω-periodicity in t for A, B and f, and we also pose some integrability assumption on B. Moreover, solutions x must belong to some restricted class concerning the behavior near $-\infty$, so that the integral becomes well-defined.

Zubelevich [Z] studied discrete dynamical systems in reflexive Banach spaces and those in sequentially complete Hausdorff locally convex spaces with the Montel property.

Their results gave positive answers to the problem and we wonder whether such phenomena appear commonly in periodic linear equations.

1.3. Massera phenomena in hyperfunctions. Now our interest is

QUESTION. Is there any counterpart to this phenomenon in the framework of hyperfunctions?

In order to think about Massera phenomena in hyperfunctions, we must at least think about a notion of

$$\text{“boundedness (in a neighborhood of } +\infty)\text{”}$$

for univariate hyperfunctions. But, since there is no notion of inequality for hyperfunctions, the usual notion of boundedness does not make sense. Moreover, it seems impossible to introduce “boundedness” for usual hyperfunctions defined on an open set $]t_0, +\infty[$ (that is, sections in $\mathscr{B}(]t_0, +\infty[)$).

To overcome this difficulty, we introduce a new class of “bounded hyperfunctions”.

2. Boundedness for hyperfunctions. In this section, we recall the sheaf $\mathscr{B}_{L^\infty}$ of bounded hyperfunctions at infinity in one variable, which we introduced in [O] in a similar manner as Sato defined the sheaf $\mathscr{B}$ of hyperfunctions and the sheaf $\mathscr{Q}$ of Fourier hyperfunctions in one variable. In fact, we define the sheaf $\mathscr{B}_{L^\infty}$ on a compactification $\mathbb{D}^1 := [-\infty, +\infty] = \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$, using the sheaf $\mathscr{O}_{L^\infty}$ of bounded holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$.

On the other hand, there is a notion of bounded hyperfunctions (the space $\mathcal{B}_{L^\infty}$) given by duality due to Chung–Kim–Lee [CKL]. The space of the global sections of $\mathscr{B}_{L^\infty}$ can be identified with $\mathcal{B}_{L^\infty}$ (in 1-dimensional case). Refer also to [O] for this identification.

2.1. Bounded hyperfunctions at infinity. As a preparation, we define a compactification $\mathbb{D}^1 := [-\infty, +\infty] = \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$ and consider the diagram of the topological spaces

$$\begin{array}{ccc} \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} \\ \cup & & \cup \\ \mathbb{R} =]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] \end{array}$$

Here, a set $\Omega \subset \mathbb{D}^1$ is a neighborhood of $+\infty$ (resp. $-\infty$) if it includes a set $]a, +\infty]$ (resp. $[-\infty, a[$) with some $a \in \mathbb{R}$. We take coordinates $t \in \mathbb{R}$ and $w \in \mathbb{C}$ ($\operatorname{Re} w = t$) and denote by $\mathscr{O}$ the sheaf of holomorphic functions on $\mathbb{C}$.

First we define the sheaf of bounded holomorphic functions.

DEFINITION 2.1. We define the sheaf $\mathscr{O}_{L^\infty}$ of *bounded holomorphic functions* on $\mathbb{D}^1 + i\mathbb{R}$, as the sheaf associated with the presheaf given by the correspondence:

$$\mathbb{D}^1 + i\mathbb{R} \overset{\text{open}}{\supset} U \mapsto \mathscr{O}(U \cap \mathbb{C}) \cap L^\infty(U \cap \mathbb{C}).$$

We have $\mathscr{O}_{L^\infty}(U) = \{f \in \mathscr{O}(U \cap \mathbb{C}) : \forall K \Subset U \ \|f\|_K < +\infty\}$, where $\|\cdot\|_K$ is given by

$$\|f\|_K := \sup_{w \in K \cap \mathbb{C}} |f(w)| \tag{2.1}$$

for $f \in \mathscr{O}(U \cap \mathbb{C})$ and $K \Subset U$. Note that $\mathscr{O}_{L^\infty}(U)$ is endowed with a Fréchet topology by the system of seminorms $\|\cdot\|_K$ for $K \Subset U$, and that $\mathscr{O}_{L^\infty}|_{\mathbb{C}} = \mathscr{O}$, i.e., $\mathscr{O}_{L^\infty}$ is an extension of $\mathscr{O}$ to $\mathbb{D}^1 + i\mathbb{R}$.

Next we introduce bounded hyperfunctions at infinity.

Definition 2.2 (sheaf $\mathscr{B}_{L^\infty}$ of bounded hyperfunctions at infinity). We define the sheaf $\mathscr{B}_{L^\infty}$ of *bounded hyperfunctions at infinity* on $\mathbb{D}^1$, as the sheaf associated with the presheaf given by correspondence:

$$\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{\mathscr{O}_{L^\infty}(U \setminus \Omega)}{\mathscr{O}_{L^\infty}(U)}\,. \tag{2.2}$$

Here U runs through complex neighborhoods of Ω, that is, open sets $U \subset \mathbb{D}^1 + i\mathbb{R}$ including Ω as a closed subset.

Refer to a similar formula (1.1), and also to the case of the sheaf $\mathscr{Q}$ of Fourier hyperfunctions on $\mathbb{D}^1$ (in one variable): $\mathscr{Q}$ is defined as the sheaf associated with the presheaf $\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_U \frac{\tilde{\mathscr{O}}(U\setminus\Omega)}{\tilde{\mathscr{O}}(U)}$, where $\tilde{\mathscr{O}}$ is the sheaf on $\mathbb{D}^1 + i\mathbb{R}$ of holomorphic functions with infra-exponential growth on the real direction. See Kawai [K] for the theory of Fourier hyperfunctions.

We also consider vector-valued variants of these sheaves. Let E be a sequentially complete Hausdorff locally convex space over $\mathbb{C}$. We denote by ${}^E\mathscr{O}$ the sheaf of E-valued holomorphic functions on $\mathbb{C}$. See, for example, [BS] for the notion of vector-valued holomorphic functions. Then, the sheaves ${}^E\mathscr{O}_{L^\infty}$ and ${}^E\mathscr{B}_{L^\infty}$, the E-valued variants of $\mathscr{O}_{L^\infty}$ and $\mathscr{B}_{L^\infty}$, can be defined in a similar manner.

Definition 2.3. We define the sheaf ${}^E\mathscr{O}_{L^\infty}$ of *E-valued bounded holomorphic functions* on $\mathbb{D}^1 + i\mathbb{R}$, as the sheaf associated with the presheaf:

$$U \mapsto \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}) : f \text{ is bounded}\},$$

and the sheaf ${}^E\mathscr{B}_{L^\infty}$ of *E-valued bounded hyperfunctions at infinity* on $\mathbb{D}^1$, as that associated with the presheaf:

$$\Omega \mapsto \varinjlim_{U} \frac{{}^E\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathscr{O}_{L^\infty}(U)}\,.$$

The sheaf ${}^E\mathscr{O}_{L^\infty}$ is also an extension of ${}^E\mathscr{O}$ to $\mathbb{D}^1 + i\mathbb{R}$, that is, ${}^E\mathscr{O}_{L^\infty}(U) = {}^E\mathscr{O}(U)$ if $U \subset \mathbb{C}$. Denoting by $\mathcal{N}(E)$ the system of continuous seminorms of E, we define the seminorms $\|\cdot\|_{K,p}$ ($K \Subset U$, $p \in \mathcal{N}(E)$) of ${}^E\mathscr{O}_{L^\infty}(U)$ for $U \subset \mathbb{D}^1 + i\mathbb{R}$, similarly as (2.1), by

$$\|f\|_{K,p} := \sup_{w \in K \cap \mathbb{C}} p(f(w)). \tag{2.3}$$

The space ${}^E\mathscr{O}_{L^\infty}(U)$ is endowed with the locally convex topology by those seminorms.

Note that Definition 2.3 extends Definitions 2.1 and 2.2. In fact, when $E = \mathbb{C}$, ${}^E\mathscr{O}_{L^\infty}$ and ${}^E\mathscr{B}_{L^\infty}$ coincide with their scalar valued variants $\mathscr{O}_{L^\infty}$ and $\mathscr{B}_{L^\infty}$ respectively.

Sheaves of bounded hyperfunctions are extensions of those of usual hyperfunctions. In fact, in the scalar case, $\mathscr{B}_{L^\infty}$ is an extension of $\mathscr{B}$ to $\mathbb{D}^1$, i.e.,

$$\mathscr{B}_{L^\infty}|_{\mathbb{R}} = \mathscr{B}.$$

When E is a Fréchet space, ${}^E\mathscr{B}_{L^\infty}$ is an extension of the sheaf ${}^E\mathscr{B}$ of E-valued hyperfunctions due to Ion–Kawai [IK]. On the other hand, in the general case, we denote the restriction ${}^E\mathscr{B}_{L^\infty}|_{\mathbb{R}}$ by ${}^E\mathscr{B}$.

We are interested in the following

EXAMPLE 2.4. $E = \mathscr{O}(V)$ for an open $V \subset \mathbb{C}^n_z$, is a Fréchet space, while $E = \mathscr{A}(V)$ for an open $V \subset \mathbb{R}^n_x$, the space of real-analytic functions on V, is not a Fréchet space.

Bounded hyperfunctions can be represented by defining functions.

FACT. *A section u on a compact set $[a, +\infty]$ admits a boundary value representation in the interior*:

$$u(t) = [v(w)] = v(t + i0) - v(t - i0) \quad \textit{on }]a, +\infty],$$

where $v \in {}^E\mathscr{O}(\{t + is \in \mathbb{C} : t > a, 0 < |s| < d\})$ is bounded on $\{t > a + \delta, \delta < |s| < d - \delta\}$ for any $\delta > 0$. We call v a defining function of u.

Usual bounded functions can be regarded as bounded hyperfunctions. In fact, in the scalar case, there exists a natural embedding

$$L^\infty(]a, +\infty[) \hookrightarrow \mathscr{B}_{L^\infty}(]a, +\infty]),$$

while in the general case, bounded continuous maps $f :]a, +\infty[\to E$ are regarded as sections in ${}^E\mathscr{B}_{L^\infty}(]a, +\infty])$. Moreover, these two maps are compatible with the standard morphism ${}^E\mathscr{O}_{L^\infty}|_{\mathbb{D}^1} \hookrightarrow {}^E\mathscr{B}_{L^\infty}$.

In the scalar case, $\mathscr{B}_{L^\infty}$ admits the following properties.

- The sheaf $\mathscr{B}_{L^\infty}$ is flabby. Therefore, the restriction $\mathscr{B}_{L^\infty}(]a, +\infty]) \to \mathscr{B}(]a, +\infty[)$ is surjective. But it is not injective.
- The standard morphism $\mathscr{B}_{L^\infty} \to \mathscr{Q}$ is injective.

2.2. Duality. Chung–Kim–Lee [CKL] introduced the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions on the n-dimensional Euclidean space $\mathbb{R}^n$. Their space is defined as the dual space of the space of test functions:

$$\mathcal{B}_{L^\infty} = (\mathcal{A}_{L^1})'.$$

Here, $\mathcal{A}_{L^1} := \varinjlim_{h>0} \mathcal{A}_{L^1,h}$ and

$$\mathcal{A}_{L^1,h} := \Big\{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{1,h} := \sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^1(\mathbb{R}^n)}}{h^{|\alpha|}\alpha!} < +\infty\Big\}.$$

We have

THEOREM 2.5. *$\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ can be identified with $\mathcal{B}_{L^\infty}$ of the case $n = 1$.*

3. Operators and periodicity. In this section, we recall classes of operators for $\mathscr{B}_{L^\infty}$ given by in [O], and also introduce new classes of operators in §3.2. In examples of equations with retarded type, the former classes correspond to equations with finite delay (see (1.2) as an example), and the latter classes to equations with infinite delay (see (1.3)).

3.1. Operators of type K with a closed interval $K \subset \mathbb{R}$. Let $K = [a, b] \subset \mathbb{R}$ be a closed interval, and $U \subset \mathbb{D}^1 + i\mathbb{R}$ an open set. We consider also the case $a = b$, that is, $K = \{a\}$. Typically U is a strip neighborhood $U = \mathbb{D}^1 + i]-d, d[$ of $\mathbb{D}^1$ with some $d > 0$.

DEFINITION 3.1 (operators of type K). Let $P = \{P_V : {}^E\mathscr{O}_{L^\infty}(V + K) \to {}^E\mathscr{O}_{L^\infty}(V)\}_{V \subset U}$ be a family of linear continuous maps, where V runs through open subsets in U. P is said to be *an operator of type* K for ${}^E\mathscr{O}_{L^\infty}$ on U, if the diagram below commutes for any pair $V_1 \supset V_2$.

$$\begin{array}{ccc} {}^E\mathscr{O}_{L^\infty}(V_1 + K) & \xrightarrow{P_{V_1}} & {}^E\mathscr{O}_{L^\infty}(V_1) \\ \downarrow & & \downarrow \\ {}^E\mathscr{O}_{L^\infty}(V_2 + K) & \xrightarrow{P_{V_2}} & {}^E\mathscr{O}_{L^\infty}(V_2). \end{array} \tag{3.1}$$

Here the vertical arrows are the restriction maps.

Note that $V + K$ denotes the vectorial addition $\{w + t : w \in V, t \in K\}$, which is also defined in case $V \not\subset \mathbb{C}$, using the convention $w + t := w$ if $w \in \{\pm\infty\} + i\mathbb{R}$ and $t \in \mathbb{R}$. See the beginning of §3.2 for a brief discussion.

Let P be an operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U. Then, P induces a family $\{P_\Omega : {}^E\mathscr{B}_{L^\infty}(\Omega + K) \to {}^E\mathscr{B}_{L^\infty}(\Omega)\}_{\Omega \subset \mathbb{D}^1 \cap U}$ with Ω open in $\mathbb{D}^1 \cap U$, which makes the following diagram commute for any $\Omega_1 \supset \Omega_2$.

$$\begin{array}{ccc} {}^E\mathscr{B}_{L^\infty}(\Omega_1 + K) & \xrightarrow{P_{\Omega_1}} & {}^E\mathscr{B}_{L^\infty}(\Omega_1) \\ \downarrow & & \downarrow \\ {}^E\mathscr{B}_{L^\infty}(\Omega_2 + K) & \xrightarrow{P_{\Omega_2}} & {}^E\mathscr{B}_{L^\infty}(\Omega_2). \end{array} \tag{3.2}$$

An operator P of type $\{0\}$ for ${}^E\mathscr{O}_{L^\infty}$ on U (case $K = \{0\}$) is nothing but a sheaf endomorphism of ${}^E\mathscr{O}_{L^\infty}|_U$ consisting of continuous maps. Then P induces a sheaf morphism of ${}^E\mathscr{B}_{L^\infty}|_{\mathbb{D}^1 \cap U}$, and when $U \ni +\infty$, P acts on the stalk $({}^E\mathscr{B}_{L^\infty})_{+\infty}$. We denote its action by $P_{+\infty}$. Therefore we can consider $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solutions to $Pu = f$ for $f \in ({}^E\mathscr{B}_{L^\infty})_{+\infty}$, as a germ $u \in ({}^E\mathscr{B}_{L^\infty})_{+\infty}$ satisfying $P_{+\infty}u = f$.

In the case $K \neq \{0\}$, an operator P type K for ${}^E\mathscr{O}_{L^\infty}$ on U does not induce a sheaf endomorphism of ${}^E\mathscr{B}_{L^\infty}$, but when $U \ni +\infty$, P still acts on $({}^E\mathscr{B}_{L^\infty})_{+\infty}$. Again we denote the action by $P_{+\infty}$, and we consider $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solutions to $Pu = f$ for $f \in ({}^E\mathscr{B}_{L^\infty})_{+\infty}$, in the same way. On the other hand, for a fixed open set $\Omega \subset \mathbb{D}^1$, u is said to be an ${}^E\mathscr{B}_{L^\infty}(\Omega)$-solution to the equation $Pu = f$, if u is a section of ${}^E\mathscr{B}_{L^\infty}$ on $\Omega + K$ (not on Ω), and if $P_\Omega u = f$ holds in ${}^E\mathscr{B}_{L^\infty}(\Omega)$.

EXAMPLE 3.2. We define the sheaf $\mathscr{D}_{L^\infty}$ of ordinary differential operators with $\mathscr{O}_{L^\infty}$ coefficients on $\mathbb{D}^1 + i\mathbb{R}$ as the sheaf associated with the presheaf

$$U \mapsto \left\{ P(w, \partial_w) := \sum_{j=0}^{m} a_j(w)\partial_w^j : m \in \mathbb{N},\, a_j \in \mathscr{O}_{L^\infty}(U) \right\}.$$

Sections of $\mathscr{D}_{L^\infty}$ are operators of type $\{0\}$.

Let $k(t) \in \mathscr{B}(\mathbb{R})$ be a hyperfunction with compact support, and $K \subset \mathbb{R}$ a convex hull of $-\operatorname{supp} k$. The convolution $k*$ with kernel k becomes an operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$ for any E. In particular, we give

EXAMPLE 3.3. For a fixed $\omega > 0$, the ω-translation operator $T_\omega : u(t) \mapsto u(t+\omega)$ is an operator of type $\{\omega\}$, and the ω-difference operator $T_\omega - 1 : u(t) \mapsto u(t+\omega) - u(t)$ is an operator of type $[0, \omega]$.

3.2. Operators of type K with $K = [-\infty, b]$. We extend the notion of operators of type K into the case $K = [-\infty, b]$.

We start by a remark on vectorial additions $U + K$ and $\Omega + K$ of open sets $U \subset \mathbb{D}^1 + i\mathbb{R}$, $\Omega \subset \mathbb{D}^1$, and a closed interval $K \subset \mathbb{D}^1$. We have in mind the case $K = [-\infty, b]$ for $b \in \mathbb{R}$. Note that we do not admit $K = \{-\infty\}$ nor $K = \{+\infty\}$, while we admitted $K = \{a\}$ for $a \in \mathbb{R}$ in §3.1.

Recall that $A_1 + A_2$ of two subsets $A_i \subset \mathbb{C}$ $(i = 1, 2)$ is defined via the vectorial addition in $\mathbb{C}$, i.e., $A_1 + A_2 := \{w_1 + w_2 : w_i \in A_i \ (i = 1, 2)\}$, and it is extended to the case of $A_1 \subset \mathbb{D}^1 + i\mathbb{R}$ and $A_2 \subset \mathbb{C}$ (see §3.1 for the case $A_2 \subset \mathbb{R}$), and also to the case A_1, $A_2 \subset [-\infty, +\infty[+ i\mathbb{R}$ and so on in a similar way. In fact, we use the convention

$$(t_1 + is_1) + (t_2 + is_2) = t_1 + i(s_1 + s_2) \quad \text{for } t_1 = \pm\infty \text{ and } t_2 \in \mathbb{R} \sqcup \{t_1\}. \tag{3.3}$$

The convention (3.3) cannot be extended to the case $(\pm\infty + is_1) + (\mp\infty + is_2)$. Instead, we define $A_1 + A_2$, first in the case where each A_1 and A_2 contains exactly one point, by

$$\{t_1 + is_1\} + \{t_2 + is_2\} := \begin{cases} \{t + i(s_1 + s_2) : t \in \mathbb{D}^1\}, & t_1 = \pm\infty,\ t_2 = \mp\infty, \\ \{(t_1 + is_1) + (t_2 + is_2)\}, & \text{otherwise}, \end{cases} \tag{3.4}$$

and then, next for general A_1, $A_2 \subset \mathbb{D}^1 + i\mathbb{R}$, by

$$A_1 + A_2 := \bigcup_{w_1 \in A_1, w_2 \in A_2} \{w_1\} + \{w_2\}.$$

Note that we used the convention (3.3) again in the second case in (3.4).

Then, if $U \subset \mathbb{D}^1 + i\mathbb{R}$ is open, then $U + K \subset \mathbb{D}^1 + i\mathbb{R}$ is open. And a similar conclusion holds for an open set $\Omega \subset \mathbb{D}^1$. If U is a complex neighborhood of Ω satisfying $U \cap \mathbb{D}^1 = \Omega$, then $U + K$ is a complex neighborhood of $\Omega + K$ satisfying $(U + K) \cap \mathbb{D}^1 = \Omega + K$. Moreover, if K_1 and K_2 are closed intervals in $\mathbb{D}^1$, then we have $(U + K_1) + K_2 = U + (K_1 + K_2)$.

After this preparation, we can give the following definition. Let $U \subset \mathbb{D}^1 + i\mathbb{R}$ be an open set and $b \in \mathbb{R}$ a constant.

DEFINITION 3.4 (operators of type $[-\infty, b]$). Let $P = \{P_V : {}^E\mathscr{O}_{L^\infty}(V + [-\infty, b]) \to {}^E\mathscr{O}_{L^\infty}(V)\}_{V \subset U}$ be a family of linear continuous maps, where V runs through open subsets in U. P is said to be *an operator of type* $[-\infty, b]$ for ${}^E\mathscr{O}_{L^\infty}$ on U, if the diagram (3.1) commutes for any pair $V_1 \supset V_2$, under the notation $K = [-\infty, b]$.

Such operator P also induces a family $\{P_\Omega : {}^E\mathscr{B}_{L^\infty}(\Omega + [-\infty, b]) \to {}^E\mathscr{B}_{L^\infty}(\Omega)\}_{\Omega \subset \mathbb{D}^1 \cap U}$ with Ω open in $\mathbb{D}^1 \cap U$, which makes the diagram (3.2) commute for any $\Omega_1 \supset \Omega_2$, again under the notation $K = [-\infty, b]$.

Remark 3.5. Let P be an operator of type $K = [-\infty, b]$ ($b \in \mathbb{R}$).

If $V \subset \mathbb{D}^1 + i\mathbb{R}$ is non-empty then $V + K$ always contains points at $-\infty$, that is, there exists a point $-\infty + is$ in $V + K$ for some $s \in \mathbb{R}$. Moreover if $\Omega \subset \mathbb{D}^1$ is open and non-empty, then $\Omega + K$ is equal either to $\mathbb{D}^1$ or to $[-\infty, c[$ with some $c \in \mathbb{R} \sqcup \{+\infty\}$, depending on $\Omega \ni +\infty$ or not. Therefore, any operand of P_Ω ($\Omega \neq \emptyset$) must be defined at least in a neighborhood of $-\infty$.

Unlike the case in §3.1, P does not act on the stalk $({}^E\mathscr{B}_{L^\infty})_{+\infty}$ any more. For any neighborhood Ω in $\mathbb{D}^1$ of $+\infty$, $\Omega + K$ coincides with $\mathbb{D}^1$, and the subfamily $\{P_\Omega\}$ for all the neighborhoods Ω of $+\infty$ induces the map

$$P_{+\infty} : {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1) \to ({}^E\mathscr{B}_{L^\infty})_{+\infty}.$$

Therefore, an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution u to $Pu = f$ for a germ $f \in ({}^E\mathscr{B}_{L^\infty})_{+\infty}$ is a global section $u \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ satisfying $P_{+\infty}u = f$. In other words, u is an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution to $Pu = f$ if there exists a neighborhood $\Omega \subset \mathbb{D}^1$ of $+\infty$ such that f belongs to ${}^E\mathscr{B}_{L^\infty}(\Omega)$ and that $P_\Omega u = f$.

3.3. Periodicity of bounded hyperfunctions and operators. Let ω be a positive constant. $u \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ (resp. ${}^E\mathscr{B}(\mathbb{R})$) is called ω-periodic if it satisfies $(T_\omega - 1)u = 0$.

Theorem 3.6. *The restriction map* ${}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1) \to {}^E\mathscr{B}_{L^\infty}(\mathbb{R})$ *induces an isomorphism*

$$\{u \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1) : (T_\omega - 1)u = 0\} \xrightarrow{\sim} \{u \in {}^E\mathscr{B}(\mathbb{R}) : (T_\omega - 1)u = 0\}. \tag{3.5}$$

Therefore, every ω-periodic hyperfunction $u \in {}^E\mathscr{B}(\mathbb{R})$ has the unique ω-periodic extension $\hat{u} \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$. In other words, there is no non-trivial periodic bounded hyperfunction supported in $\{\pm\infty\}$.

Moreover, we have

Theorem 3.7. *Every ω-periodic bounded hyperfunction $f \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ admits an ω-periodic boundary value representation. That is, there exists a section*

$$g \in {}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\{s : 0 < |s| < d\})$$

such that $f = [g]$ and that $(T_\omega - 1)g = 0$.

Let $U = \mathbb{D}^1 + iI$ be a strip domain in $\mathbb{D}^1 + i\mathbb{R}$ with an open interval $I \subset \mathbb{R}$, and P an operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U, where $K = [a, b] \subset \mathbb{R}$ or $K = [-\infty, b]$ ($b \in \mathbb{R}$).

Definition 3.8. P is said to be *ω-periodic* if it commutes with the ω-translation operator, that is, the following diagram commutes for any $V \subset U$.

$$\begin{array}{ccc}
{}^E\mathscr{O}_{L^\infty}(V + \omega + K) & \xrightarrow{P_{V+\omega}} & {}^E\mathscr{O}_{L^\infty}(V + \omega) \\
\big\downarrow T_\omega & & \big\downarrow T_\omega \\
{}^E\mathscr{O}_{L^\infty}(V + K) & \xrightarrow{P_V} & {}^E\mathscr{O}_{L^\infty}(V)
\end{array}$$

Here we abbreviated "$+\{\omega\}$" to "$+\omega$".

It follows directly from the definition that ω-periodic operators preserve the ω-periodicity of their operands.

4. Massera type theorems in $\mathscr{B}_{L^\infty}$. In this section, we recall the results in the case $K \subset \mathbb{R}$ studied in [O], and also a recent result in the case $K = [-\infty, b]$, under an additional assumption, which we call the fading memory condition.

4.1. Massera type theorem in the case $K \subset \mathbb{R}$. We prepare a notion of (sequential) Montel property for locally convex spaces, and a weak variant of Montel type lemma.

DEFINITION 4.1 (Montel property). Let E be a sequentially complete Hausdorff locally convex space. We say that E admits *the Montel property*, if

(M) any bounded sequence in E has a convergent subsequence.

LEMMA 4.2. *Assume that E admits the Montel property. Then for any bounded sequence $(f_j)_j$ in ${}^E\mathscr{O}_{L^\infty}(U)$, there exists a subsequence $(f_{j_k})_k$ and a section $f \in {}^E\mathscr{O}_{L^\infty}(U)$ such that $f_{j_k} \to f$ in the topology of ${}^E\mathscr{O}(U \cap \mathbb{C})$.*

Recall that ${}^E\mathscr{O}_{L^\infty}(U)$ is a subspace of ${}^E\mathscr{O}(U \cap \mathbb{C}) = {}^E\mathscr{O}_{L^\infty}(U \cap \mathbb{C})$, and that they are endowed with locally convex topologies given by the seminorms $\|\cdot\|_{K,p}$ (see (2.3)), where K runs through compact sets in U and in $U \cap \mathbb{C}$ respectively, and p runs through continuous seminorms of E. Note that the topology of ${}^E\mathscr{O}_{L^\infty}(U)$ is stronger than that induced from the ${}^E\mathscr{O}(U \cap \mathbb{C})$, and that $(f_j)_j$ does not necessarily have a subsequence convergent in ${}^E\mathscr{O}_{L^\infty}(U)$.

Let $K \subset \mathbb{R}$ be a closed interval, ω a positive constant, $U = \mathbb{D}^1 + i]-d, d[$ a strip neighborhood of $\mathbb{D}^1$, and $f \in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ an ω-periodic E-valued bounded hyperfunction on $\mathbb{D}^1$. Now we give,

THEOREM 4.3 (see [O, Theorem 4.3]). *Let P be an ω-periodic operator of type K on U. Assume that E admits* (M). *Then $Pu = f$ has an ω-periodic ${}^E\mathscr{B}(\mathbb{R})$-solution if and only if it has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution.*

4.2. Massera type theorems in the case $K = [-\infty, b]$. In Section 3, we defined the notion of operators of type $K = [-\infty, b]$, using the continuity and the commutativity with restrictions. Under the commutativity with restrictions, the continuity reads as follows.

$P = \{P_V\}_{V \subset U}$ is an operator of type $K = [-\infty, b]$ for ${}^E\mathscr{O}_{L^\infty}$ on $U \subset \mathbb{D}^1 + i\mathbb{R}$, if and only if

$$\forall L \Subset \forall M \Subset \forall V \Subset U \quad \forall p \in \mathcal{N}(E) \quad \exists q = q_{L,M,p} \in \mathcal{N}(E) \quad \exists C = C_{L,M,p} > 0$$
$$\forall u \in {}^E\mathscr{O}_{L^\infty}(V + K) \quad \|P_V u\|_{L,p} \le C\|u\|_{M+K,q}.$$

Here q and C can be taken independent of V.

We pose a further assumption on P.

DEFINITION 4.4. Let $P = \{P_V\}_{V \subset U}$ be an operator of type $[-\infty, b]$ for ${}^E\mathscr{O}_{L^\infty}$ on $U \subset \mathbb{D}^1 + i\mathbb{R}$. P is said to satisfy *the fading memory condition*, or *the condition* (FM) for short, if

$$\forall L \Subset \forall M \Subset \forall V \subset U \quad \forall p \in \mathcal{N}(E) \quad \exists q = q_{L,M,p} \in \mathcal{N}(E)$$
$$\forall \varepsilon > 0 \quad \exists K^0 = K^0_{L,M,p,\varepsilon} \Subset K \cap \mathbb{R} \quad \exists C = C_{L,M,p,\varepsilon} > 0$$
$$\forall u \in {}^E\mathscr{O}_{L^\infty}(V + K) \quad \|P_V u\|_{L,p} \le C\|u\|_{M+K^0,q} + \varepsilon\|u\|_{M+K,q}.$$

Here we may assume that K^0 is a closed interval.

The condition (FM) seems to have relation with the notion of (uniform) fading memory spaces, studied in Hino–Murakami–Naito [HMN].

Consider, for example, the case $E=\mathbb{C}$ and $K=[-\infty,0]$. When P is an operator of type K, the value $(Pu)(w)$ at a point $w\in\mathbb{C}$ can be estimated by the values $u(w+s)$ where s runs through a fixed size neighborhood of the negative real axis. Roughly speaking, in general $(Pu)(w)$ may depend heavily on "$u(w+(-\infty))$", but under the condition (FM), we can make the dependence of $(Pu)(w)$ on "$u(w+(-\infty))$" as small as we please.

Now we come back to the situation that $K=[-\infty,b]$, w is a positive constant, $U=\mathbb{D}^1+i]-d,d[$ is a strip neighborhood of $\mathbb{D}^1$, and $f\in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ is an ω-periodic E-valued bounded hyperfunction on $\mathbb{D}^1$. Moreover let P be an ω-periodic operator of type K for ${}^E\mathscr{O}_{L^\infty}$ on U. In order to express the result in a similar form to Theorem 4.3, we explain the terminology: "an ω-periodic ${}^E\mathscr{B}(\mathbb{R})$-solution to the equation $Pu=f$".

By Theorem 3.6, we have the canonical isomorphism

$$\{u\in {}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1):\omega\text{-periodic}\}\xrightarrow{\sim}\{u\in {}^E\mathscr{B}(\mathbb{R}):\omega\text{-periodic}\},$$

which allows us to identify these spaces. Since P acts on the left hand side, we say, by abuse of terminology, that $u\in {}^E\mathscr{B}(\mathbb{R})$ is an ω-periodic ${}^E\mathscr{B}(\mathbb{R})$ solution to $Pu=f$, if u is ω-periodic and the canonical extension $\hat{u}$ of u satisfies the equation $P\hat{u}=f$.

Under these preparations, we give

THEOREM 4.5. *Assume that E admits* (M) *and that P satisfies* (FM)*. Then $Pu=f$ has an ω-periodic ${}^E\mathscr{B}(\mathbb{R})$-solution if and only if it has an $({}^E\mathscr{B}_{L^\infty})_{+\infty}$-solution.*

We give an example of an operator of type $K:=[-\infty,0]$ for $\mathscr{O}_{L^\infty}$ on $U:=\mathbb{D}^1+i]-d,d[$ satisfying the fading memory condition.

EXAMPLE 4.6 (integral operators of Volterra type). Consider a kernel function $k(w,s)\in C((U\cap\mathbb{C})\times(K\cap\mathbb{R}))$, holomorphic and ω-periodic in w, such that for any $L\Subset U\cap\mathbb{C}$, the function $s\mapsto \sup_{w\in L}|k(w,s)|$ on $K\cap\mathbb{R}=]-\infty,0]$ is integrable. We define linear maps $P_V:\mathscr{O}_{L^\infty}(V+K)\to\mathscr{O}_{L^\infty}(V)$ for $V\subset U$ by

$$P_Vu(w)=\int_{-\infty}^{0}k(w,s)u(w+s)\,ds,\quad u\in\mathscr{O}_{L^\infty}(V+K).$$

Then, $P=\{P_V\}_{V\subset U}$ is an operator of type K for $\mathscr{O}_{L^\infty}$ on U, satisfying (FM).

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