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MAILLET TYPE THEOREM AND GEVREY REGULARITY IN TIME OF SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We will consider the nonlinear partial differential equation

$$t^{\gamma}(\partial/\partial t)^{m}u = F(t,x,\{(\partial/\partial t)^{j}(\partial/\partial x)^{\alpha}u\}_{j+|\alpha|\le L,j<m}) \tag{E}$$

(with $\gamma \ge 0$ and $1 \le m \le L$) and show the following two results: (1) (Maillet type theorem) if (E) has a formal solution it is in some formal Gevrey class, and (2) (Gevrey regularity in time) if (E) has a solution $u(t,x) \in C^{\infty}([0,T],\mathcal{E}^{\{\sigma\}}(V))$ it is in some Gevrey class also with respect to the time variable t. It will be explained that the mechanism of these two results are quite similar, but still there appears some difference between them which is very interesting to the author.

1. Introduction. We denote by t the time variable in $\mathbb{R}_t$, and by $x = (x_1,\dots,x_n)$ the space variable in $\mathbb{R}^n_x$. We use the notation: $\mathbb{N} = \{0,1,2,\dots\}$, $\mathbb{N}^* = \{1,2,\dots\}$, $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1+\ldots+\alpha_n$, $\partial_t = \partial/\partial t$, $\partial_x = (\partial_{x_1},\dots,\partial_{x_n})$ with $\partial_{x_i} = \partial/\partial x_i$ $(i=1,\dots,n)$ and $\partial_x^{\alpha} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$.

For $\sigma \ge 1$ and an open subset V of $\mathbb{R}^n_x$ we denote by $\mathcal{E}^{\{\sigma\}}(V)$ the set of all functions $f(x) \in C^{\infty}(V)$ satisfying the following: for any compact subset K of V there are $C>0$ and $h>0$ such that

$$\max_{x\in K}|\partial_x^{\alpha}f(x)| \le Ch^{|\alpha|}|\alpha|!^{\sigma} \quad \forall\alpha\in\mathbb{N}^n.$$

A function in the class $\mathcal{E}^{\{\sigma\}}(V)$ is called *a function of the Gevrey class of order* σ.

The class $\mathcal{E}^{\{1\}}(V)$ is nothing but the set of all analytic functions on V and usually is denoted by $\mathcal{A}(V)$. For convenience, we set $\mathcal{E}^{\{\infty\}}(V) = C^{\infty}(V)$. If $1<\sigma_1<\sigma_2<\infty$

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we have

$$\mathcal{A}(V) \subset \mathcal{E}^{\{\sigma_1\}}(V) \subset \mathcal{E}^{\{\sigma_2\}}(V) \subset C^\infty(V).$$

Thus, functions in the class $\mathcal{E}^{\{\sigma_1\}}(V)$ are closer to analytic functions than those in $\mathcal{E}^{\{\sigma_2\}}(V)$; in this sense, we can say that functions in $\mathcal{E}^{\{\sigma_1\}}(V)$ are more regular than those in $\mathcal{E}^{\{\sigma_2\}}(V)$.

For an interval $[0,T] = \{t \in \mathbb{R} : 0 \le t \le T\}$ we denote by $C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ the set of all infinitely differentiable functions $u(t,x)$ in $t \in [0,T]$ with values in $\mathcal{E}^{\{\sigma\}}(V)$ equipped with the usual local convex topology.

Similarly, for $s \ge 1$ and $\sigma \ge 1$ we denote by $\mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ the set of all functions $u(t,x) \in C^\infty([0,T] \times V)$ satisfying the following: for any compact subset K of V there are $C > 0$ and $h > 0$ such that

$$\max_{(t,x)\in[0,T]\times K} |\partial_t^k \partial_x^\alpha u(t,x)| \le C h^{k+|\alpha|} k!^s |\alpha|!^\sigma \quad \forall (k,\alpha) \in \mathbb{N} \times \mathbb{N}^n.$$

Obviously, we have

$$\mathcal{E}^{\{s,\sigma\}}([0,T] \times V) \subset C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V)).$$

In the case $s = \sigma$ we write $\mathcal{E}^{\{\sigma\}}([0,T] \times V)$ instead of $\mathcal{E}^{\{\sigma,\sigma\}}([0,T] \times V)$.

In this paper, we will consider the nonlinear partial differential equation

$$t^\gamma \partial_t^m u = F\big(t, x, \{\partial_t^j \partial_x^\alpha u\}_{j+|\alpha|\le L, j<m}\big) \tag{1.1}$$

where $\gamma \ge 0$ and $L \ge m \ge 1$ are integers, and $F(t, x, \{z_{j,\alpha}\}_{j+|\alpha|\le L, j<m})$ is a suitable function in a Gevrey class (for the precise assumptions, see Section 2). And, we will consider the following problem on Gevrey regularity in time:

PROBLEM 1.1. Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (1.1); can we have the result $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ for a suitable $s \ge 1$? If this is true, determine the precise bound of the index s of the time regularity.

Some particular cases are studied by Hannah–Himonas–Petronilho [HHP] (for KdV equation), Łysik [L1] (for KdV equation), Gramchev–Łysik [GL1, GL2] (for semilinear heat equation), Tahara [T1] (for linear Fuchsian equation) and Kinoshita–Taglialatela [KT] (for linear hyperbolic equation). In this paper, we will show a general result on the equation (1.1).

2. Maillet type theorem. Let $\gamma \in \mathbb{N}$, $m \in \mathbb{N}^*$, $L \in \mathbb{N}^*$, Λ be a subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n : j + |\alpha| \le L,\ j < m\}$ and $d = \#\Lambda$ (the cardinal of Λ). We will consider the following nonlinear partial differential equation

$$t^\gamma \partial_t^m u = F\big(t, x, \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}\big). \tag{2.1}$$

For simplicity we write

$$Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda} :$$

we denote the corresponding variable by $z = \{z_{j,\alpha}\}_{(j,\alpha)\in\Lambda} \in \mathbb{R}^d$. Let Ω be an open subset of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$, and let $F(t,x,z)$ be a C^∞ function on Ω. Let $s_1 \ge 1$, $\sigma \ge 1$ and $s_2 \ge 1$, let $T > 0$, and let V be an open subset of $\mathbb{R}^n$. The main assumptions are as follows.

a$_1$) $\gamma \geq 0$, $L \geq m \geq 1$, $s_1 \geq 1$ and $\sigma \geq s_2 \geq 1$.
a$_2$) Λ is a subset of $\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n : j + |\alpha| \leq L,\ j < m\}$.
a$_3$) $F(t,x,z) \in \mathcal{E}^{\{s_1,\sigma,s_2\}}(\Omega)$.
a$_4$) $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ is a solution of (2.1) on $[0,T] \times V$; this involves the property: $(t,x) \in [0,T] \times V \Longrightarrow (t,x,Du(t,x)) \in \Omega$.

Before considering Problem 1.1, we analyze a formal power series on this solution. Let

$$\hat{u}(t,x) = \sum_{k=0}^{\infty} u_k(x) t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]] \tag{2.2}$$

be the formal Taylor expansion at $t=0$ of the solution $u(t,x)$ in a$_4$), and let us look for Gevrey type estimates of the coefficients $u_k(x)$ $(k = 1,2,\dots)$.

In the case $\sigma = 1$ (that is, in the analytic case), we have many results on Gevrey type estimates of the coefficients of a formal power series solution (see Ōuchi [O], Gérard–Tahara [GT], Shirai [S], Lysik [L2], and their references), and such results are called *Maillet type theorem.*

In order to state our result in the general case, we define

Definition 2.1. For a formal power series $\hat{f}(t,x) = \sum_{k\geq 0} a_k(x) t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]]$ we define *the valuation of* $\hat{f}(t,x)$ *in* t (which we denote by $\mathrm{val}_t(\hat{f})$) by

$$\mathrm{val}_t(\hat{f}) = \min\{k \in \mathbb{N} : a_k(x) \not\equiv 0 \text{ on } V\}$$

(if $a_k(x) \equiv 0$ on V for all $k \in \mathbb{N}$, we set $\mathrm{val}_t(\hat{f}) = \infty$). If $f(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ we define $\mathrm{val}_t(f)$ by using the formal Taylor expansion of $f(t,x)$ at $t = 0$.

Under the conditions a$_1$)–a$_4$) we set

$$k_{j,\alpha} = \mathrm{val}_t\Big(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,Du(t,x)), V\Big), \quad (j,\alpha) \in \Lambda$$

and we suppose:

$$\begin{cases} k_{j,\alpha} \geq \gamma - m + j, & \text{if } (j,\alpha) \in \Lambda \text{ and } |\alpha| = 0, \\ k_{j,\alpha} \geq \gamma - m + j + 1, & \text{if } (j,\alpha) \in \Lambda \text{ and } |\alpha| > 0. \end{cases} \tag{2.3}$$

Then, if we use the norm

$$|||f|||_{K,\rho} = \sum_{|\alpha|\geq 0} \frac{\|\partial_x^\alpha f\|_K}{|\alpha|!^\sigma} \rho^{|\alpha|} \quad \text{for } f(x) \in \mathcal{E}^{\{\sigma\}}(V) \text{ and } K \Subset V \tag{2.4}$$

(where $\|\partial_x^\alpha f\|_K$ denotes the maximum norm on K, and ρ is a parameter) we can apply the same arguments as in references quoted above and we have

Theorem 2.2 (Maillet type theorem). *Suppose the conditions* a$_1$)–a$_4$) *and* (2.3). *Then the coefficients* $u_k(x)$ $(k = 1,2,\dots)$ *of the formal Taylor expansion* (2.2) *satisfy the estimate*: *for any compact subset* K *of* V *there are* $\rho > 0$, $C > 0$ *and* $h > 0$ *such that*

$$|||u_k|||_{K,\rho} \leq C h^k k!^{s-1} \quad \forall k \in \mathbb{N} \tag{2.5}$$

for any $s \geq \max\{s_0^*, s_1, s_2\}$ *with*

$$s_0^* = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j + \sigma|\alpha| - m}{k_{j,\alpha} - \gamma + m - j}\Big)\Big]. \tag{2.6}$$

From now, we use the following notation: for $\hat{u}(t,x) = \sum_{k=0}^{\infty} u_k(x)t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]]$ we write $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\},V)$ if it satisfies the estimate (2.5). In this case we say that $\hat{u}(t,x)$ is in a formal Gevrey class of order s in t.

Similarly, for $U(t,x,z) = \sum_{k+|\nu|\ge 0} u_{k,\nu}(x)t^k z^\nu \in \mathcal{E}^{\{\sigma\}}(K)[[t,z]]$ we write $U(t,x,z) \in \mathcal{E}^{\{s,\sigma\}}(\{t,z\},K)$ if there are $\rho > 0$, $C > 0$ and $h > 0$ such that $\|u_{k,\nu}\|_{K,\rho} \le Ch^{k+|\nu|} \times (k+|\nu|)!^{s-1}$ for all $(k,\nu) \in \mathbb{N} \times \mathbb{N}^d$.

Sketch of the proof of Theorem 2.2. Let $\hat{u}(t,x)$ be as in (2.2), and set $s = \max\{s_0^*, s_1, s_2\}$. Take any compact subset K of V. Our purpose is to show the estimate (2.5) on K. We will give a sketch of the proof only in the case $s > 1$.

Step 1. *Reduction.* We take $q \in \mathbb{N}$ sufficiently large and we divide our formal solution $\hat{u}(t,x)$ into the form

$$\hat{u}(t,x) = \varphi(t,x) + t^q \hat{w}(t,x) \quad \text{with } \varphi(t,x) = \sum_{k=0}^{q} u_k(x)t^k. \tag{2.7}$$

Then we have $\hat{w}(t,x) \in \mathcal{E}^{\{\sigma\}}(K)[[t]]$, $\hat{w}(0,x) \equiv 0$, and we see that $\hat{w}(t,x)$ satisfies the formal equation of the form

$$\begin{aligned} C\big(t\partial_t, x\big)\hat{w} = \hat{f}(t,x) &+ \sum_{(j,\alpha)\in\Lambda} \hat{a}_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha \hat{w} \\ &+ \sum_{|\nu|\ge 2} \hat{b}_\nu(t,x) \prod_{(j,\alpha)\in\Lambda} \Big[(t\partial_t)^j \partial_x^\alpha \hat{w}\Big]^{\nu_{j,\alpha}} \end{aligned} \tag{2.8}$$

where $C(\lambda,x) = \lambda^m + c_1(x)\lambda^{m-1} + \ldots + c_m(x) \in \mathcal{E}^{\{\sigma\}}(K)[\lambda]$, $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in\Lambda} \in \mathbb{N}^d$ and $|\nu| = \sum_{(j,\alpha)\in\Lambda} \nu_{j,\alpha}$. Moreover, we have the following conditions:

1) $C(x,k) \ne 0$ on K for all $k = 1, 2, \ldots$,
2) $\hat{f}(t,x), \hat{a}_{j,\alpha}(t,x), \hat{b}_\nu(t,x) \in \mathcal{E}^{\{s^*,\sigma\}}(\{t\},K)$ for $s^* = \max\{s_1, s_2\}$,
3) $\mathrm{val}_t(\hat{f}) \ge 1$, $\mathrm{val}_t(\hat{a}_{j,\alpha}) \ge 1$ and $\mathrm{val}_t(\hat{b}_\nu) \ge (q-m+1)|\nu| - (q+\gamma-m)$,
4) $\sum_{|\nu|\ge 2} \hat{b}_\nu(t,x)z^\nu \in \mathcal{E}^{\{s^*,\sigma\}}(\{t,z\},K)$, and
5) if we set $q_{j,\alpha} = \mathrm{val}_t(\hat{a}_{j,\alpha})$ we have

$$s_0^* = 1 + \max\left[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\right]. \tag{2.9}$$

We set $p_\nu = \mathrm{val}_t(\hat{b}_\nu)$, and for $|\nu| \ge 2$ we set $L_\nu = \max\{j + \sigma|\alpha| : \nu_{j,\alpha} > 0\}$. Since we are considering the case $s > 1$ and q is sufficiently large, we may suppose the condition

$$s - 1 > \frac{L_\nu - m}{p_\nu + |\nu| - 1} \quad \text{for any } |\nu| \ge 2. \tag{2.10}$$

Thus, to prove Theorem 2.2 it is sufficient to show that

$$\hat{w}(t,x) = \sum_{k=1}^{\infty} w_k(x)t^k \in \mathcal{E}^{\{s,\sigma\}}(\{t\},K). \tag{2.11}$$

Step 2. *Basic lemmas.* We present two lemmas which are needed in the proof of (2.11). In the discussion below we regard $\|f\|_{K,\rho}$ in (2.4) as a formal power series in ρ. We write $\sum_{k\ge 0} a_k\rho^k \ll \sum_{k\ge 0} b_k\rho^k$ if $|a_k| \le b_k$ holds for all $k \ge 0$. We take $c_0 > 0$ so that $|C(k,x)| \ge c_0 k^m$ holds on K for any $k = 1, 2, \ldots$, and also we take $C_0 > 0$ and $R_0 > 0$

so that

$$\|C(k)\|_{K,\rho} \ll \frac{C_0 k^m}{(1-\rho/R_0)^n}\,, \quad k = 1, 2, \ldots .$$

LEMMA 2.3. *Let $w(x), g(x) \in \mathcal{E}^{\{\sigma\}}(K)$ satisfy the equation $C(k,x)w(x) = g(x)$ on K. If $\|g\|_{K,\rho} \ll A/(1-\rho/R)^a$ for some $A > 0$, $0 < R < R_0$ and $a \geq 1$, and if R satisfies*

$$\frac{C_0}{c_0}\Big[\frac{1}{(1-R/R_0)^n} - 1\Big] \leq 1/2, \tag{2.12}$$

then we have

$$\|w\|_{K,\rho} \ll \frac{(2/c_0)A}{k^m(1-\rho/R)^a}\,.$$

LEMMA 2.4 (Nagumo's type lemma, [T2, Proposition 4.5]). *If $\|f\|_{K,\rho} \ll C/(1-\rho/R)^a$ for some $C > 0$, $a \geq 1$ and $R > 0$, we have*

$$\|\partial_{x_i} f\|_{K,\rho} \ll \frac{Ce^{\sigma}(a+\sigma)^{\sigma}/R}{(1-\rho/R)^{a+\sigma}} \quad \textit{for } i = 1, \ldots, n.$$

Step 3. *Proof of* (2.11). Let $\hat{f}(t,x)$, $\hat{a}_{j,\alpha}(t,x)$ and $\hat{b}_\nu(t,x)$ be as

$$\hat{f}(t,x) = \sum_{k\geq 1} f_k(x)t^k, \quad \hat{a}_{j,\alpha}(t,x) = \sum_{k\geq q_{j,\alpha}} a_{j,\alpha,k}(x)t^k, \quad \hat{b}_\nu(t,x) = \sum_{k \geq p_\nu} b_{\nu,k}(x)t^k.$$

Since $\hat{w}(t,x)$ is a formal solution of the equation (2.8), the coefficients $w_k(x)$ ($k = 1, 2, \ldots$) satisfy the recurrent formulas

$$C(1,x)w_1 = f_1(x) \tag{2.13}$$

and for $k \geq 2$

$$\begin{aligned} C(k,x)w_k = f_k(x) &+ \sum_{(j,\alpha)\in\Lambda} \sum_{q_{j,\alpha}\leq h\leq k-1} a_{j,\alpha,k}(x)(k-h)^j \partial_x^\alpha w_{k-h} \\ &+ \sum_{|\nu|\geq 2} \sum_{p_\nu\leq h\leq k-2} b_{\nu,h}(x) \sum_{|k^*|+h=k} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} \Big[k_{j,\alpha}(i)^j \partial_x^\alpha w_{k_{j,\alpha}(i)}\Big] \end{aligned} \tag{2.14}$$

where $|k^*| = \sum_{(j,\alpha)\in\Lambda}(k_{j,\alpha}(1)+\ldots+k_{j,\alpha}(\nu_{j,\alpha}))$. Since only the terms $w_1(x), \ldots, w_{k-1}(x)$ and their derivatives appear in the right-hand side of (2.14), in the estimation of $w_k(x)$ ($k = 1, 2, \ldots$) we can use the induction argument on k.

Let us take $F_k \geq 0$, $A_{j,\alpha,k} \geq 0$ and $B_{\nu,k} \geq 0$ so that

$$\|f_k\|_{K,\rho} \ll \frac{F_k}{(1-\rho/R_0)^n}\,, \quad \|a_{j,\alpha,k}\|_{K,\rho} \ll \frac{A_{j,\alpha,k}}{(1-\rho/R_0)^n}\,, \quad \|b_{\nu,k}\|_{K,\rho} \ll \frac{B_{\nu,k}}{(1-\rho/R_0)^n}$$

and that the series

$$\sum_{k\geq 1} \frac{F_k}{k!^{s-1}}\,t^k, \quad \sum_{k\geq q_{j,\alpha}} \frac{A_{j,\alpha,k}}{k!^{s-1}}\,t^k, \quad \sum_{|\nu|\geq 2, k\geq p_\nu} \frac{B_{\nu,k}}{(k+|\nu|)!^{s-1}}\,t^k z^\nu$$

are convergent in a neighborhood of $t = 0$ or $(t,z) = (0,0)$.

Since $w_1(x) \in \mathcal{E}^{\{\sigma\}}(K)$ is known, we can choose $A > 0$ so that

$$\|\partial_x^\alpha w_1\|_{K,\rho} \ll \frac{A}{(1-\rho/R_0)^n}\,, \quad (j,\alpha) \in \Lambda. \tag{2.15}$$

We choose $\mu \in \mathbb{N}$ so that $\mu \geq \max\{j + \sigma|\alpha| : (j,\alpha) \in \Lambda\}$, $\mu > m$ and $\mu \geq n$ hold. Take $N \in \mathbb{N}^*$ sufficiently large so that $s-1 \geq (\mu-m)/(N-1)$, and set $F_k^* = k^{\mu-m}F_k/(k-1)!^{s-1}$

and

$$A^*_{j,\alpha,k} = \begin{cases} A_{j,\alpha,k}, & \text{if } k+1 \le N, \\ \dfrac{A_{j,\alpha,k}}{(k+1-N)!^{s-1}}, & \text{if } k+1 > N, \end{cases}$$

$$B^*_{\nu,k} = \begin{cases} B_{\nu,k}, & \text{if } k+|\nu| \le N, \\ \dfrac{B_{\nu,k}}{(k+|\nu|-N)!^{s-1}}, & \text{if } k+|\nu| > N. \end{cases}$$

Then the series

$$\sum_{k\ge 1} F^*_k t^k, \quad \sum_{k\ge q_{j,\alpha}} A^*_{j,\alpha,k} t^k \quad \text{and} \quad \sum_{|\nu|\ge 2, k\ge p_\nu} B^*_{\nu,k} t^k z^\nu$$

are convergent in a neighborhood of $t=0$ or $(t,z)=(0,0)$.

Now, let us consider the following functional equation with respect to (Y,t):

$$\begin{aligned} (2.16)\quad Y = \frac{At}{(1-\rho/R)^\mu} + \frac{2/c_0}{(1-\rho/R)^\mu} \Bigg[& \sum_{k\ge 2} \frac{F^*_k}{(1-\rho/R)^{\mu(2k-2)}} t^k \\ & + H \sum_{(j,\alpha)\in\Lambda} \sum_{k\ge q_{j,\alpha}} \frac{A^*_{j,\alpha,k}(k+1)^\mu}{(1-\rho/R)^{\mu(2k-1)}} t^k (\beta Y) \\ & + H \sum_{|\nu|\ge 2} \sum_{k\ge p_\nu} \frac{B^*_{\nu,k}(k+|\nu|)^\mu}{(1-\rho/R)^{\mu(2k+|\nu|-2)}} t^k (\beta Y)^{|\nu|} \Bigg], \end{aligned}$$

where $R>0$ is the constant in (2.12), ρ is regarded as a parameter with $0<\rho<R$, $H = e^{N(s-1)}$ and $\beta = (2\mu e/R)^\mu$. Since this is an analytic functional equation, the implicit function theorem tells us that for any $0<\rho<R$ equation (2.16) has a unique holomorphic solution Y of the form

$$Y = \sum_{k\ge 1} Y_k(\rho) t^k,$$

and the coefficients $Y_k = Y_k(\rho)$ $(k=1,2,\dots)$ are determined by the recurrent formulas:

$$(2.17)\qquad Y_1 = \frac{A}{(1-\rho/R)^\mu}$$

and for $k\ge 2$

$$\begin{aligned} (2.18)\quad Y_k = \frac{2/c_0}{(1-\rho/R)^\mu} \Bigg[& \frac{F^*_k}{(1-\rho/R)^{\mu(2k-2)}} \\ & + H \sum_{(j,\alpha)\in\Lambda} \sum_{q_{j,\alpha}\le h\le k-1} \frac{A^*_{j,\alpha,h}(h+1)^\mu}{(1-\rho/R)^{\mu(2h-1)}} (\beta Y_{k-h}) \\ & + H \sum_{|\nu|\ge 2} \sum_{p_\nu\le h\le k-2} \frac{B^*_{\nu,h}(h+|\nu|)^\mu}{(1-\rho/R)^{\mu(2h+|\nu|-2)}} \prod_{(j,\alpha)\in\Lambda} \prod_{i=1}^{\nu_{j,\alpha}} [\beta Y_{k_{j,\alpha}(i)}] \Bigg]. \end{aligned}$$

Moreover, by induction on k we see that the coefficients $Y_k(\rho)$ have the form

$$Y_k(\rho) = \frac{C_k}{(1-\rho/R)^{\mu(2k-1)}}, \quad k=1,2,\dots$$

where $C_1 = A$ and $C_k \ge 0$ $(k\ge 2)$ are constants independent of the parameter ρ.

The following lemma says that $Y(t)$ is a majorant series of the formal solution $\hat{w}(t,x)$:

LEMMA 2.5. *For any* $k=1,2,\dots$ *we have*

$$k^j \lVert\!|\partial_x^\alpha w_k|\!\rVert_{K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-j-\sigma|\alpha|}}\,\beta Y_k(\rho) \quad \text{for any } (j,\alpha)\in\Lambda. \tag{2.19}$$

Proof. The case $k=1$ follows from (2.15) and (2.17). The general case is proved by induction on k. To do so, we apply the induction hypothesis to (2.14), we note that $(k-\ell)!/(k-1)! \le e^\ell/k^{\ell-1}$ holds for any $k>\ell>1$, and we use formula (2.18) and Lemma 2.3. Then we have

$$\lVert\!|w_k|\!\rVert_{K,\rho} \ll \frac{(k-1)!^{s-1}}{k^\mu}(1-\rho/R)^\mu Y_k(\rho) = \frac{(k-1)!^{s-1}}{k^\mu}\,\frac{C_k}{(1-\rho/R)^{\mu(2k-2)}}\,.$$

Thus, by using Lemma 2.4 we have the estimate (2.19). Since the argument is similar to the argument in [GT, Chapter 6], we may omit the details. ∎

Since if we fix $\rho>0$ the series $Y=\sum_{k\ge1}Y_k(\rho)t^k$ is convergent in a neighborhood of $t=0$, we have $Y_k(\rho)\le Ch^k$ $(k=1,2,\dots)$ for some $C>0$ and $h>0$. By applying these estimates to (2.19) we have the result (2.11). ∎

3. Gevrey regularity in time. Now, let us return to Problem 1.1. In order to treat time regularity problem, we will use the norm

$$\lVert\!|f|\!\rVert_{[0,T]\times K,\rho} = \sum_{|\alpha|\ge0} \frac{\lVert\partial_x^\alpha f\rVert_{[0,T]\times K}}{|\alpha|!^\sigma}\,\rho^{|\alpha|} \tag{3.1}$$

(where $f(t,x)\in C^\infty([0,T],\mathcal{E}^{\{\sigma\}}(V))$, $K\Subset V$, $\lVert\partial_x^\alpha f\rVert_{[0,T]\times K}$ denotes the maximum norm on $[0,T]\times K$, and ρ is a parameter). It is clear that $u(t,x)\in\mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$, if and only if for any compact subset K of V there are $\rho>0$, $C>0$ and $h>0$ such that

$$\lVert\!|\partial_t^k u/k!|\!\rVert_{[0,T]\times K,\rho} \le Ch^k k!^{s-1} \quad \forall k\in\mathbb{N}.$$

If we use Faà di Bruno's formula (see Johnson [J]) instead of the recurrent formulas (2.14) which appear in the calculation of formal power series solution, we can apply the same argument to Problem 1.1 as in Maillet type theorem in Section 2.

Therefore, by Theorem 2.2 it will be expected that the solution $u(t,x)$ satisfies $u(t,x)\in\mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ for any $s\ge\max\{s_0^*,s_1,s_2\}$, that is, the Gevrey order of the regularity in time for actual solution will be the same as the Gevrey order for formal solution in Maillet type theorem. But, in fact, it seems not true in the general case as is seen in the theorem given below.

Instead of the valuation, we define

DEFINITION 3.1. For $f(t,x)\in C^\infty([0,T]\times V)$ we define *the order of the zero of* $f(t,x)$ *on* V *at* $t=0$ (which we denote by $\mathrm{ord}_t(f,V)$) by

$$\mathrm{ord}_t(f,V)=\min\{k\in\mathbb{N} : (\partial_t^k f)(0,x)\not\equiv 0 \text{ on } V\}$$

(if $(\partial_t^k f)(0,x)\equiv0$ on V for all $k\in\mathbb{N}$, we set $\mathrm{ord}_t(f,V)=\infty$).

Under the conditions a_1)–a_4) we set

$$k_{j,\alpha}=\mathrm{ord}_t\Big(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,Du(t,x)),V\Big), \quad (j,\alpha)\in\Lambda,$$

and we suppose

$$\begin{cases} k_{j,\alpha} \geq \gamma - m + j, & \text{if } (j,\alpha) \in \Lambda \text{ and } |\alpha| = 0, \\ k_{j,\alpha} \geq \gamma - m + j + 1, & \text{if } (j,\alpha) \in \Lambda \text{ and } |\alpha| > 0. \end{cases} \tag{3.2}$$

By Tahara [T2, Theorem 2.2] we have

THEOREM 3.2 (Gevrey regularity in time). *Suppose the conditions* a$_1$)–a$_4$) *and* (3.2). *Then we have* $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times V)$ *for any* $s \geq \max\{s_0, s_1, s_2\}$ *with*

$$s_0 = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha} - \gamma + m - j, m - j\}}\Big)\Big]. \tag{3.3}$$

We note that $s_0^* \leq s_0$ holds; in general, the time regularity in the case of formal solutions is better than the case of actual solutions.

REMARK 3.3.

(1) In the case $\gamma = 0$, the index s_0 is expressed as

$$s_0 = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j + \sigma|\alpha| - m}{m - j}\Big)\Big]. \tag{3.4}$$

(2) In the case $\gamma = m$, the index s_0 is expressed as

$$s_0 = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j + \sigma|\alpha| - m}{\min\{k_{j,\alpha} - j,\, m - j\}}\Big)\Big]. \tag{3.5}$$

EXAMPLE 3.4.

(1) Let us consider the periodic KdV equation:

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \quad u(0,x) = \varphi(x) \quad \text{on } \mathbb{T} \tag{3.6}$$

where $\varphi(x)$ is an analytic function on the torus $\mathbb{T}$. The following results are known: i) The problem (3.6) is well-posed in $H^s(\mathbb{T})$ for $s \gg 1$; ii) Gorsky–Himonas [GH] showed that $u(t,x) \in C^\infty((-\delta,\delta), \mathcal{E}^{\{1\}}(\mathbb{T}))$; and iii) Hannah–Himonas–Petronilho [HHP] showed that $u(t,x) \in \mathcal{E}^{\{3,1\}}((-\delta,\delta) \times \mathbb{T})$. Since our index s_0 is given by $s_0 = 3\sigma$, the result iii) just coincides with our result.

(2) Let $a > 0$, $k \in \mathbb{N}^*$ and let us consider

$$(t\partial_t + a)^2 u - t^k \partial_x^2 u = f(t,x). \tag{3.7}$$

The following results are known in [T1]: i) (3.7) is uniquely solvable in $C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(\mathbb{R}))$ for any $\sigma \geq 1$; ii) if $f(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R})$ we have the time regularity

$$\begin{cases} u(t,x) \in \mathcal{E}^{\{\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k \geq 2, \\ u(t,x) \in \mathcal{E}^{\{2\sigma-1,\sigma\}}([0,T] \times \mathbb{R}), & \text{if } k = 1. \end{cases} \tag{3.8}$$

Since our index s_0 is given by $s_0 = 1 + (2\sigma - 2)/\min\{k, 2\}$, the result (3.8) just coincides with our result.

Sketch of the proof of Theorem 3.2. Let $u(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(V))$ be a solution of (2.1) given in a$_4$), and set $s = \max\{s_0, s_1, s_2\}$. Take any compact subset K of V. Our purpose is to show that $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times K)$. We will give a sketch of the proof only in the case $s > 1$. The complete proof is given in [T2].

Step 1. *Reduction.* We take $q \in \mathbb{N}$ sufficiently large and we divide our solution $u(t,x)$ into the form

$$u(t,x) = \varphi(t,x) + t^q w(t,x) \quad \text{with } \varphi(t,x) = \sum_{k=0}^{q-1} \frac{(\partial_t^k u)(0,x)}{k!}\, t^k. \tag{3.9}$$

Then we have $w(t,x) \in C^\infty([0,T], \mathcal{E}^{\{\sigma\}}(K))$, and we see that $w(t,x)$ satisfies an equation of the form

$$C\big(t\partial_t, x\big)w = G\big(t, x, \Theta w\big) \quad \text{with } \Theta w = \{(t\partial_t)^j \partial_x^\alpha w\}_{(j,\alpha)\in\Lambda}, \tag{3.10}$$

where $C(\lambda, x) = \lambda^m + c_1(x)\lambda^{m-1} + \ldots + c_m(x) \in \mathcal{E}^{\{\sigma\}}(K)[\lambda]$ and $G(t,x,z) \in \mathcal{E}^{\{s^*,\sigma,s_2\}}(\Omega_1)$ for $s^* = \max\{s_1, s_2\}$ and some open subset Ω_1 of $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_z^d$ satisfying the property: $(t,x) \in [0,T] \times K \Longrightarrow (t, x, \Theta w(t,x)) \in \Omega_1$. Moreover, we have the following conditions:

1) $\mathrm{ord}_t\Big(\dfrac{\partial G}{\partial z_{j,\alpha}}(t,x,\Theta w(t,x)), K\Big) \geq 1$,
2) $\mathrm{ord}_t\Big(\dfrac{\partial^{|\nu|} G}{\partial z^\nu}(t,x,\Theta w(t,x)), K\Big) \geq (q-m+1)|\nu| - (q+\gamma-m)$ for $|\nu| \geq 2$,
3) if we set $q_{j,\alpha} = \mathrm{ord}_t\Big(\dfrac{\partial G}{\partial z_{j,\alpha}}(t,x,\Theta w(t,x)), K\Big)$ we have

$$s_0 = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{\min\{q_{j,\alpha}, m-j\}}\Big)\Big]. \tag{3.11}$$

We set

$$q_{p,\nu}^* = \mathrm{ord}_t\Big(\frac{\partial^{p+|\nu|} G}{\partial t^p \partial z^\nu}(t,x,\Theta w(t,x)), K\Big), \quad p + |\nu| \geq 1, \ |\nu| \geq 1$$

(if $p=0$ and $\nu = e_{j,\alpha}$ we have $q_{0,e_{j,\alpha}}^* = q_{j,\alpha}$) and set $\Lambda_\nu = \{(j,\alpha) \in \Lambda : \nu_{j,\alpha} > 0\}$. Since q is taken sufficiently large we may suppose the condition

$$s_0 - 1 \geq \max_{(j,\alpha)\in\Lambda_\nu} \Big(\frac{j+\sigma|\alpha|-m}{p+|\nu|+\min\{q_{p,\nu}^*, m-j\}-1}\Big), \quad p+|\nu| \geq 2, \ |\nu| \geq 1.$$

In the case $|\nu| \geq 2$ this is verified in [T2, Proposition 6.2]. In the case $p \geq 1$ and $\nu = e_{j,\alpha}$ we have $q_{p,e_{j,\alpha}}^* \geq \max\{q_{j,\alpha} - p, 0\}$, and so this is verified by the following: if $q_{j,\alpha} - p \leq 0$ we have $p + |\nu| + \min\{q_{p,\nu}^*, m-j\} - 1 = p + \min\{q_{p,\nu}^*, m-j\} \geq p \geq q_{j,\alpha} \geq \min\{q_{j,\alpha}, m-j\}$, if $0 < q_{j,\alpha} - p \leq m-j$ we have $p + |\nu| + \min\{q_{p,\nu}^*, m-j\} - 1 \geq p + \min\{q_{j,\alpha} - p, m-j\} = p + (q_{j,\alpha} - p) = q_{j,\alpha} \geq \min\{q_{j,\alpha}, m-j\}$, and if $q_{j,\alpha} - p > m-j$ we have $p + |\nu| + \min\{q_{p,\nu}^*, m-j\} - 1 \geq p + \min\{q_{j,\alpha} - p, m-j\} = p + (m-j) \geq m-j \geq \min\{q_{j,\alpha}, m-j\}$.

Thus, to prove Theorem 3.2 it is sufficient to show that

$$w(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T] \times K). \tag{3.12}$$

Step 2. *Basic lemma.* We take $C > 0$ and $R_0 > 0$ so that $\|c_i\|_{K,\rho} \ll C/(1-\rho/R_0)^n$ $(i = 1, \ldots, m)$. Then, instead of Lemma 2.3, we have

Lemma 3.5 ([T2, Lemma 5.2.2]). *There are $k_0 > 0$ and $M_0 > 0$ such that if $w(t,x)$, $g(t,x) \in C^\infty([0,T] \times K)$ satisfy the equation $C(t\partial_t + k, x)w(t,x) = g(t,x)$ on $[0,T] \times K$ for some $k \geq k_0$, if $\|g\|_{[0,T]\times K,\rho} \ll A/(1-\rho/R)^a$ holds for some $A > 0$, $0 < R < R_0$ and $a \geq 1$, and if R satisfies*

$$M_0 C\Big[\frac{1}{(1-R/R_0)^n} - 1\Big] \leq 1/2, \tag{3.13}$$

then we have

$$\|(t\partial_t+k)^i w\|_{[0,T]\times K,\rho} \ll \frac{2M_0A}{k^{m-i}(1-\rho/R)^a}\,, \quad i=0,1,\dots,m.$$

Step 3. *Proof of* (3.12). We know that $w(t,x)$ is a solution of equation (3.10). By applying ∂_t^k to both sides of (3.10) and by using Faà di Bruno's formula (or Lemma 4.2 in [T2]) we have

$$\begin{aligned}(3.14)\quad & C(t\partial_t+k,x)\frac{1}{k!}\,\partial_t^k w \\ &= f_k(t,x)+\sum_{1\le p+|\nu|\le k,|\nu|\ge 1} a_{p,\nu}(t,x)\sum_{|k^*|=k-p}\prod_{(j,\alpha)\in\Lambda}\Bigg[\frac{1}{k_{j,\alpha}(1)!}\,\partial_t^{k_{j,\alpha}(1)}(t\partial_t)^j\partial_x^\alpha w \\ &\qquad\times\dots\times\frac{1}{k_{j,\alpha}(\nu_{j,\alpha})!}\,\partial_t^{k_{j,\alpha}(\nu_{j,\alpha})}(t\partial_t)^j\partial_x^\alpha w\Bigg],\end{aligned}$$

where $|k^*|=\sum_{(j,\alpha)\in\Lambda}(k_{j,\alpha}(1)+\dots+k_{j,\alpha}(\nu_{j,\alpha}))$,

$$\begin{aligned} f_k(t,x)&=\frac{1}{k!}\,\frac{\partial^k G}{\partial t^k}(t,x,\Theta w(t,x)) \quad (k\ge 1),\\ a_{p,\nu}(t,x)&=\frac{1}{p!\nu!}\,\frac{\partial^{p+|\nu|}G}{\partial t^p\partial z^\nu}(t,x,\Theta w(t,x)) \quad (p+|\nu|\ge 1,\ |\nu|\ge 1).\end{aligned}$$

By the definition, we have $q^*_{p,\nu}=\mathrm{ord}_t(a_{p,\nu}(t,x),K)$ ($p+|\nu|\ge 1$ and $|\nu|\ge 1$). Since $q_{j,\alpha}(=q^*_{0,e_{j,\alpha}})\ge 1$, we can express the right-hand side of (3.14) by a polynomial of the terms $(t\partial_t+h)^i\partial_x^\alpha(\partial_t^h w/h!)$ ($i=0,1,\dots,m$, $|\alpha|\le L$ and $h=1,\dots,k-1$). Therefore, in the estimation of $\partial_t^k w/k!$ we can use the induction argument on k. We note that formula (3.14) corresponds to the formula (2.14) in Section 2. Moreover, we have constants $F_k\ge 0$ and $A_{p,\nu}\ge 0$ such that

$$\|f_k\|_{[0,T]\times K,\rho}\ll\frac{F_k}{(1-\rho/R_0)^n}\,,\quad \|a_{p,\nu}\|_{[0,T]\times K,\rho}\ll\frac{A_{p,\nu}}{(1-\rho/R_0)^n}$$

and that the series

$$\sum_{k\ge 1}\frac{F_k}{k!^{s-1}}\,t^k \quad\text{and}\quad \sum_{p+|\nu|\ge 1,|\nu|\ge 1}\frac{A_{p,\nu}}{(p+|\nu|)!^{s-1}}\,t^p z^\nu$$

are convergent in a neighborhood of $t=0$ or $(t,z)=(0,0)$.

We choose $\mu\in\mathbb{N}$ so that $\mu\ge\max\{j+\sigma|\alpha| : (j,\alpha)\in\Lambda\}$, $\mu>m$ and $\mu\ge n$. Take $N\in\mathbb{N}^*$ sufficiently large so that $s-1\ge(\mu-m)/(N-1)$, $(m+1)N\ge k_0$ and $N\ge 2$, we take $A^*_{p,\nu}>0$ ($2\le p+|\nu|\le N$, $|\nu|\ge 1$) sufficiently large, and we set

$$\begin{aligned}F^*_k&=\frac{k^{\mu-m}F_k}{(k-1)!^{s-1}}\quad(k\ge 1),\\ A^*_{p,\nu}&=\frac{A_{p,\nu}}{(p+|\nu|-N)!^{s-1}}\quad(p+|\nu|\ge N,\ |\nu|\ge 1).\end{aligned}$$

We take $A^* > 0$ and $C_k^* > 0$ ($k = 1, 2, \ldots, (m+1)N$) so that

$$\|(t\partial_t)^j \partial_x^\alpha w\|_{[0,T]\times K,\rho} \ll \frac{A^*}{(1-\rho/R_0)^n}, \quad j = 0, 1, \ldots, m \text{ and } |\alpha| \le L, \tag{3.15}$$

$$\|(t\partial_t + k)^j \partial_x^\alpha (\partial_t^k w/k!)\|_{[0,T]\times K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-j-\sigma|\alpha|}} \frac{C_k^*}{(1-\rho/R_0)^n} \tag{3.16}$$

$$\text{for } j = 0, 1, \ldots, m,\ |\alpha| \le L \text{ and } k = 1, 2, \ldots, (m+1)N.$$

Now, let us consider the following functional equation with respect to (Y, t):

$$\begin{aligned} Y = {} & \frac{A^*}{(1-\rho/R)^\mu} t + \frac{2}{(1-\rho/R)^{2\mu}} tY + \sum_{1\le k\le (m+1)N} \frac{C_k^*}{(1-\rho/R)^{\mu(2k-1)}} t^k \\ & + \frac{2M_0}{(1-\rho/R)^\mu}\Bigg[\sum_{k>(m+1)N} \frac{F_k^*}{(1-\rho/R)^{\mu(2k-2)}} t^k + H \sum_{(j,\alpha)\in\Lambda} \frac{A_{0,e_{j,\alpha}}^* (m+1)^\mu}{(1-\rho/R)^\mu} t \times 2\beta Y \\ & + H \sum_{p+|\nu|\ge 2, |\nu|\ge 1} \frac{A_{p,\nu}^* (p + (m+1)|\nu|)^\mu}{(1-R/R_0)^n (1-\rho/R)^{\mu(2p+|\nu|-2)}} t^p (\beta Y)^{|\nu|}\Bigg] \end{aligned} \tag{3.17}$$

where $R > 0$ is the constant in (3.13), ρ is regarded as a parameter with $0 < \rho < R$, $H = e^{(s-1)(m+1)N}(m+1)^\mu$ and $\beta = (2\mu e/R)^\mu$. Since this is an analytic functional equation, the implicit function theorem tells us that for any $0 < \rho < R$ equation (3.17) has a unique holomorphic solution Y of the form

$$Y = \sum_{k\ge 1} Y_k(\rho) t^k.$$

Moreover, we see that the coefficients $Y_k(\rho)$ have the form

$$Y_k(\rho) = \frac{C_k}{(1-\rho/R)^{\mu(2k-1)}}, \quad k = 1, 2, \ldots$$

where $C_k \ge 0$ ($k \ge 1$) are constants independent of the parameter ρ. We can show

Lemma 3.6. *For any $k = 1, 2, \ldots$ we have*

$$\|(t\partial_t + k)^j \partial_x^\alpha (\partial_t^k w/k!)\|_{[0,T]\times K,\rho} \ll \frac{(k-1)!^{s-1}}{k^{\mu-j-\sigma|\alpha|}} \beta Y_k(\rho) \tag{3.18}$$

for $j = 0, 1, \ldots, m$ and $|\alpha| \le L$.

The cases $k = 1, 2, \ldots, (m+1)N$ follow from (3.16), and the general case is proved by induction on k. The complete proof is given in [T2, Section 7].

Since if we fix $\rho > 0$ the series $Y = \sum_{k\ge 1} Y_k(\rho) t^k$ is convergent in a neighborhood of $t = 0$, we have $Y_k(\rho) \le Ch^k$ ($k = 1, 2, \ldots$) for some $C > 0$ and $h > 0$. By applying these estimates to (3.18) we have the result (3.12). ∎

4. On the necessity of the condition. In this last section, we will derive a necessary condition for a solution $u(t,x)$ (or $\hat{u}(t,x)$) to belong to the class $\mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ (or $\mathcal{E}^{\{s,\sigma\}}(\{t\}, V)$).

4.1. Fuchsian case. We set $C(\lambda, x) = \lambda^m + c_1(x)\lambda^{m-1} + \ldots + c_0(x) \in \mathcal{E}^{\{\sigma\}}(V)[\lambda]$. If the equation is written in the form

$$C(t\partial_t, x)u = F\big(t, x, \Theta u\big) \quad \text{with } \Theta u = \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda} \tag{4.1}$$

our indices (2.6) and (3.3) are written as

$$s_0^* = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\Big], \tag{4.2}$$

$$s_0 = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{\min\{q_{j,\alpha}, m-j\}}\Big)\Big] \tag{4.3}$$

with

$$q_{j,\alpha} = \mathrm{ord}_t\Big(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u(t,x)), V\Big), \quad (j,\alpha)\in\Lambda. \tag{4.4}$$

In this case, let us estimate the index s of the Gevrey class $\mathcal{E}^{\{s,\sigma\}}$ from below such that $u(t,x)\in\mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ or $\hat{u}(t,x)\in\mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ holds.

Let $T>0$, V be an open neighborhood of $x=0\in\mathbb{R}^n$, and Ω be an open neighborhood of $(t,x,z)=(0,0,0)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^d$. For a function $f(t,x)\in C^\infty([0,T]\times V)$, we write $f(t,x)\gg 0$ (at $(t,x)=(0,0)$) if $(\partial_t^k\partial_x^\beta f)(0,0)\ge 0$ holds for all $(k,\beta)\in\mathbb{N}\times\mathbb{N}^n$. We assume:

b$_1$) $C(k,0)>0$ for any $k=1,2,\dots$;
b$_2$) $C(k,0)-C(k,x)\gg 0$ (at $x=0$) for any $k=1,2,\dots$;
b$_3$) $F(t,x,z)\gg 0$ (at $(t,x,z)=(0,0,0)$), and

$$\liminf_{|\beta|\to\infty}\Big(\frac{(\partial_t\partial_x^\beta F)(0,0,0)}{|\beta|!^\sigma}\Big)^{1/|\beta|} > 0; \tag{4.5}$$

b$_4$) $u(0,x)=0$ on V, and

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u)\Big|_{t=0} \equiv 0 \ \text{ on } V \text{ for any } (j,\alpha)\in\Lambda.$$

We note that by b$_4$) we have $(\Theta u)(0,x)=0$ and so by setting $t=0$ in (4.1) we have $F(0,x,0)=0$ on V. If we set $a(x)=(\partial_t F)(0,x,0)$ the condition (4.5) implies that there is an $h>0$ such that $(\partial_x^\beta a)(0)\ge h^{|\beta|}|\beta|!^\sigma$ holds for any sufficiently large $|\beta|$.

As before, we set $q_{j,\alpha}=\mathrm{ord}_t((\partial F/\partial z_{j,\alpha})(t,x,\Theta u(t,x)),V)$ $((j,\alpha)\in\Lambda)$. Then we have the expression

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,\Theta u(t,x)) = a_{j,\alpha}(x)t^{q_{j,\alpha}} + O(t^{q_{j,\alpha}+1}) \quad (\text{as } t\longrightarrow +0)$$

for some $a_{j,\alpha}(x)\in\mathcal{E}^{\{\sigma\}}(V)$ with $a_{j,\alpha}(x)\gg 0$ (at $x=0$). We set

$$\Lambda(+) = \{(j,\alpha)\in\Lambda : a_{j,\alpha}(0)>0, |\alpha|>0\}. \tag{4.6}$$

Then we have the following result.

THEOREM 4.1. *Let $u(t,x)\in C^\infty([0,T],\mathcal{E}^{\{\sigma\}}(V))$ be a solution of the equation* (4.1). *If $u(t,x)\in\mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ or $\hat{u}(t,x)\in\mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ for some $s\ge 1$, we have*

$$s \ge 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda(+)}\Big(\frac{j+\sigma|\alpha|-m}{q_{j,\alpha}}\Big)\Big]. \tag{4.7}$$

REMARK 4.2. Compare (4.7) with (4.2) or (4.3). In the case of formal solutions, our sufficient condition (4.2) is very close to the necessary condition (4.7); but in the case of actual solutions, there is a gap.

Sketch of the proof of Theorem 4.1. Let

$$\hat{u}(t,x) = \sum_{k=1}^{\infty} u_k(x) t^k \in \mathcal{E}^{\{\sigma\}}(V)[[t]] \tag{4.8}$$

be the formal Taylor expansion at $t = 0$ of the solution $u(t,x)$. By a formal calculation we can see that $u_k(x) \gg 0$ (at $x = 0$) for any $k = 1, 2, \ldots$.

Take any $(j, \alpha) \in \Lambda(+)$, then we have $q_{j,\alpha} \geq 1$ and $a_{j,\alpha}(0) > 0$. Our aim is to show the condition

$$s - 1 \geq \frac{j + \sigma|\alpha| - m}{q_{j,\alpha}}. \tag{4.9}$$

For simplicity we write $q = q_{j,\alpha}$ and $A = a_{j,\alpha}(0) > 0$. Since

$$\begin{aligned} C(t\partial_t + 1, x)\partial_t u &= (\partial_t F)(t, x, \Theta u) + \frac{\partial F}{\partial z_{j,\alpha}}(t, x, \Theta u) \times \partial_t (t\partial_t)^j \partial_x^\alpha u + \ldots \\ &\gg a(x) + At^q \times \partial_t (t\partial_t)^j \partial_x^\alpha u = a(x) + At^{q-1}(t\partial_t)^{j+1}\partial_x^\alpha u \end{aligned}$$

(where we used: $a(x) = (\partial_t F)(0, x, 0)$) and since $C(k, 0) - C(k, x) \gg 0$ for any $k = 1, 2, \ldots$, we have $C(1, 0)u_1 \gg a(x)$ and

$$C(q + \ell, 0)u_{q+\ell} \gg \frac{A\ell^{j+1}}{(q + \ell)} \partial_x^\alpha u_\ell, \quad \ell \geq 1.$$

Therefore

$$\begin{aligned} W_{kq+1}(x) \gg \frac{A^k (q+1)^{j+1} \cdots ((k-1)q + 1)^{j+1}}{C(1,0)C(q+1,0)\cdots C(kq+1,0)} \\ \times \frac{1}{(q+1)\cdots(kq+1)} \partial_x^{k\alpha}(a(x)) \quad \text{for } k = 1, 2, \ldots. \end{aligned} \tag{4.10}$$

Here we recall: by the assumption $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\}, V)$ we have $|u_p(0)| \leq BH^p p!^{s-1}$ ($p = 0, 1, 2, \ldots$) for some $B > 0$ and $H > 0$, and by the condition (4.5) we have an $h > 0$ such that $(\partial_x^\beta a)(0) \geq h^{|\beta|}|\beta|!^\sigma$ for any sufficiently large $|\beta|$. We note also that $C(k, 0) \leq ck^m$ (for $k = 1, 2, \ldots$) holds for some $c > 0$. By applying these conditions to (4.10), for any sufficiently large k we have

$$\begin{aligned} BH^{kq+1}(kq+1)!^{s-1} &\geq |u_{kq+1}(0)| = u_{kq+1}(0) \\ &\geq \frac{A^k (q+1)^{j+1} \cdots ((k-1)q+1)^{j+1}}{c^{k+1}(q+1)^{m+1} \cdots (kq+1)^{m+1}} h^{k|\alpha|}(k|\alpha|)!^\sigma \geq B_1 H_1^k k!^{j+\sigma|\alpha|-m} \end{aligned}$$

for some $B_1 > 0$ and $H_1 > 0$. This proves that $j + \sigma|\alpha| - m \leq q(s - 1)$. Thus, we have proved (4.9). The details are written in [T2, Section 8]. ∎

4.2. Non-singular case. Let us consider the initial value problem

$$\begin{cases} \partial_t^m u = F(t, x, Du) & \text{on } [0, T] \times V, \text{ where } Du = \{\partial_t^j \partial_x^\alpha u\}_{(j,\alpha)\in\Lambda}, \\ \partial_t^i u\big|_{t=0} = \varphi_i(x) & \text{on } V, \quad i = 0, 1, \ldots, m-1, \end{cases} \tag{4.11}$$

where $\varphi_i(x) \in \mathcal{E}^{\{\sigma\}}(V)$ $(0 \le i \le m-1)$ are supposed. In this case, our indices (2.6) and (3.3) are written as

$$s_0^* = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\Big)\Big], \tag{4.12}$$

$$s_0 = 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda, |\alpha|>0} \Big(\frac{j+\sigma|\alpha|-m}{m-j}\Big)\Big] \tag{4.13}$$

with

$$k_{j,\alpha} = \mathrm{ord}_t\Big(\frac{\partial F}{\partial z_{j,\alpha}}(t,x,Du(t,x)), V\Big), \quad (j,\alpha) \in \Lambda. \tag{4.14}$$

Let $T>0$, V be an open neighborhood of $x=0 \in \mathbb{R}^n$, and Ω be an open neighborhood of $(t,x,z) = (0,0,p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^d$ with $p = \{(\partial_x^\alpha \varphi_j)(0)\}_{(j,\alpha)\in\Lambda}$. We set $\varphi_m(x) = F(0,x,\{(\partial_x^\alpha\varphi_j)(x)\}_{(j,\alpha)\in\Lambda})$ and

$$a(x) = \frac{\partial F}{\partial t}\big(0,x,\{(\partial_x^\alpha\varphi_j)(x)\}_{(j,\alpha)\in\Lambda}\big) + \sum_{(j,\alpha)\in\Lambda} \frac{\partial F}{\partial z_{j,\alpha}}\big(0,x,\{(\partial_x^\alpha\varphi_j)(x)\}_{(j,\alpha)\in\Lambda}\big)(\partial_x^\alpha\varphi_{j+1})(x).$$

We assume:

c$_1$) $F(t,x,z) \gg 0$ (at $(t,x,z) = (0,0,p)$);
c$_2$) $\varphi_i(x) \gg 0$ (at $x=0$), $i = 0,1,\dots,m-1$;
c$_3$) $\liminf_{|\beta|\to\infty} \big((\partial_x^\beta a)(0)/|\beta|!^\sigma\big)^{1/|\beta|} > 0$.

As before, we set $k_{j,\alpha} = \mathrm{ord}_t\big((\partial F/\partial z_{j,\alpha})(t,x,Du(t,x)), V\big)$ $((j,\alpha)\in\Lambda)$. Then we have the expression

$$\frac{\partial F}{\partial z_{j,\alpha}}(t,x,Du(t,x)) = a_{j,\alpha}(x)t^{k_{j,\alpha}} + O(t^{k_{j,\alpha}+1}) \quad (\text{as } t \longrightarrow +0)$$

for some $a_{j,\alpha}(x) \in \mathcal{E}^{\{\sigma\}}(V)$ with $a_{j,\alpha}(x) \gg 0$ (at $x=0$). We set

$$\Lambda(+) = \{(j,\alpha)\in\Lambda : a_{j,\alpha}(0) > 0, |\alpha|>0\}. \tag{4.15}$$

Then we have the following result.

Theorem 4.3. *Let $u(t,x) \in C^\infty([0,T] \times \mathcal{E}^{\{\sigma\}}(V))$ be a solution of* (4.8). *If $u(t,x) \in \mathcal{E}^{\{s,\sigma\}}([0,T]\times V)$ or $\hat{u}(t,x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\};V)$ for some $s \ge 1$, we have*

$$s \ge 1 + \max\Big[0, \max_{(j,\alpha)\in\Lambda(+)} \Big(\frac{j+\sigma|\alpha|-m}{k_{j,\alpha}+m-j}\Big)\Big]. \tag{4.16}$$

Remark 4.4. Compare (4.16) with (4.12) or (4.13). In the case of formal solutions, our sufficient condition (4.12) is very close to the necessary condition (4.16); but in the case of actual solutions, there is a gap.

Proof of Theorem 4.3. By c$_1$) and c$_2$) we have $\varphi_m(x) \gg 0$. We set

$$u(t,x) = \sum_{i=0}^m \varphi_i(x)\frac{t^i}{i!} + t^m w(t,x).$$

Then we can reduce our equation (4.14) to an equation of type (4.1) with respect to $w(t,x)$, and we can apply Theorem 4.1. The condition (4.5) is verified by c$_3$). ■

4.3. A generalized KdV type equation. Let $k, \ell \in \{1, 2, 3, 4, 5, \dots\}$ and $m \in \{3, 4, 5, 6, \dots\}$, and let us consider

$$\partial_t u = \partial_x^m u + u^k \partial_x^\ell u, \ \ u(0, x) = \varphi(x), \tag{4.17}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ or $x \in \mathbb{T}$, and $\varphi(x)$ is an appropriate function in the Gevrey class $\mathcal{E}^{\{\sigma\}}$ for some $\sigma \geq 1$. This equation is discussed in Hannah–Himonas–Petronilho [HHP]. By Theorems 3.2 and 4.3 we have

THEOREM 4.5.

(1) *Let $I = (-\delta, \delta)$ and let V be an open subset of $\mathbb{R}$. If $u(t, x) \in C^\infty(I, \mathcal{E}^{\{\sigma\}}(V))$ is a solution of* (4.17), *we have $u(t, x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ (and so $\hat{u}(t, x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\}, V)$) for any $s \geq \max\{m\sigma, \ell\sigma\}$.*

(2) *Conversely, if $u(t, x) \in \mathcal{E}^{\{s,\sigma\}}(I \times V)$ is a solution of* (4.17) *and if $\varphi(x)$ satisfies $\varphi(0) > 0$, $\varphi(x) \gg 0$ (at $x = 0$) and*

$$\liminf_{\alpha \to \infty} \Big(\frac{(\partial_x^\alpha \varphi)(0)}{\alpha!^\sigma} \Big)^{1/\alpha} > 0, \tag{4.18}$$

we have $s \geq \max\{m\sigma, \ell\sigma\}$.

In the case $\sigma > 1$ there are many functions $\varphi(x) \in \mathcal{E}^{\{\sigma\}}(\mathbb{R})$ with compact support satisfying (4.18). In the case $\sigma = 1$, the necessity of the condition $s \geq \max\{m, \ell\}$ can be verified under the initial data

$$\varphi(x) = \begin{cases} \dfrac{i^{(m-\ell)/k} e^{ix}}{M - e^{ix}} \quad (M > 1) & \text{in the case } V = \mathbb{T} \\ \dfrac{1}{(i-x)^{(4p+m-\ell)/k}} \quad (p \in \mathbb{N}^*, \ k < 2m - 2\ell + 8p) & \text{in the case } V = \mathbb{R} \end{cases}$$

by a small modification of the argument in [HHP].

4.4. Heat equation. Let $k \in \{1, 2, \dots\}$ and let us consider

$$\partial_t u = t^k \partial_x^2 u, \quad u(0, x) = \varphi(x), \tag{4.19}$$

where $(t, x) \in (0, \infty) \times \mathbb{R}$. We know:

PROPOSITION 4.6. *If $\varphi(x)$ is a bounded continuous function on $\mathbb{R}$, the equation* (4.19) *has a unique solution $u(t, x) \in C^0([0, \infty) \times \mathbb{R}) \cap C^\infty((0, \infty) \times \mathbb{R})$ which is bounded on $[0, \infty) \times \mathbb{R}$; moreover, the unique solution is given by*

$$u(t, x) = \int_{-\infty}^{\infty} E\big(t^{k+1}/(k+1), x - y\big)\varphi(y)\, dy, \quad \textit{where } E(t, x) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}.$$

As to a solution in the Gevrey class, under the condition that $\varphi(x) \in \mathcal{B}^\infty(\mathbb{R})$ and

$$|\partial_x^m \varphi(x)| \leq AH^m m!^\sigma \quad \text{on } \mathbb{R}, \quad m = 0, 1, 2, \dots$$

for some $A > 0$ and $H > 0$, we have

THEOREM 4.7.

(1) *The solution $u(t, x)$ of* (4.19) *satisfies $u(t, x) \in \mathcal{E}^{\{2\sigma,\sigma\}}([0, \infty) \times \mathbb{R})$.*

(2) *Conversely, if $u(t, x) \in \mathcal{E}^{\{s,\sigma\}}([0, \infty) \times \mathbb{R})$ is a solution of* (4.19) *and if $\varphi(x)$ satisfies $\varphi(x) \gg 0$ (at $x = 0$) and*

$$\liminf_{\alpha \to \infty} \big((\partial_x^\alpha \varphi)(0)/\alpha!^\sigma\big)^{1/\alpha} > 0,$$

we have $s \geq (2\sigma + k)/(k + 1)$.

We note that if $k = 0$ we have $(2\sigma + k)/(k + 1) = 2\sigma$; but that if $k \geq 1$ we have

$$2\sigma = 1 + \frac{2\sigma - 1}{1} > 1 + \frac{2\sigma - 1}{k + 1} = \frac{2\sigma + k}{k + 1}\,.$$

In the case of formal solutions, we have $\hat{u}(t, x) \in \mathcal{E}^{\{s,\sigma\}}(\{t\}, \mathbb{R})$ for any $s \geq (2\sigma{+}k)/(k{+}1)$ which coincides with the necessity in (2).

[GL1] and [GL2] have discussed a similar problem for semilinear heat equations,

References

[GT] R. Gérard, H. Tahara, *Singular Nonlinear Partial Differential Equations*, Aspects Math., Friedr. Vieweg and Sohn, Braunschweig, 1996.

[GH] J. Gorsky, A. A. Himonas, *On analyticity in space variable of solutions to the KdV equation*, in: Geometric Analysis of PDE and Several Complex Variables, Contemp. Math. 368, Amer. Math. Soc., Providence RI, 2005, 233–247.

[GL1] T. Gramchev, G. Łysik, *Sharp analytic-Gevrey regularity estimates down to $t = 0$ for solutions to semilinear heat equations*, Differential Integral Equations 21 (2008), 771–799.

[GL2] T. Gramchev, G. Łysik, *Uniform analytic-Gevrey regularity of solutions for a semilinear heat equation*, in: Parabolic and Navier–Stokes Equations, Banach Center Publ. 81, Polish Acad. Sci. Inst. Math., Warsaw, 2008, 213–226.

[HHP] H. Hannah, A. A. Himonas, G. Petronilho, *Gevrey regularity in time for generalized KdV type equations*, in: Recent Progress on some Problems in Several Complex Variables and Partial Differential Equations, Contemp. Math. 400, Amer. Math. Soc., Providence RI, 2006, 117–127.

[J] W. P. Johnson, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly 109 (2002), 217–234.

[KT] T. Kinoshita, G. Taglialatela, *Time regularity of the solutions to second order hyperbolic equations*, Ark. Mat. 49 (2011), 109–127.

[L1] G. Łysik, *Non-analyticity in time of solutions to the KdV equation*, Z. Anal. Anwend. 23 (2004), 67–93.

[L2] G. Łysik, *Formal solutions of second order evolution equations*, Z. Anal. Anwend. 30 (2011), 95–104.

[O] S. Ōuchi, *Formal solutions with Gevrey type estimates of nonlinear partial differential equations*, J. Math. Sci. Univ. Tokyo 1 (1994), 205–237.

[S] A. Shirai, *Maillet type theorem for nonlinear partial differential equations and Newton polygons*, J. Math. Soc. Japan 53 (2001), 565–587.

[T1] H. Tahara, *Singular hyperbolic systems, VIII. On the well-posedness in Gevrey classes for Fuchsian hyperbolic equations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 39 (1992), 555–582.

[T2] H. Tahara, *Gevrey regularity in time of solutions to nonlinear partial differential equations*, J. Math. Sci. Univ. Tokyo 18 (2011), 67–137.

[T3] H. Tahara, *Time regularity in Gevrey classes of solutions to general nonlinear partial differential equations*, in: Recent development of micro-local analysis for the theory of asymptotic analysis. RIMS Kôkyûroku Bessatsu, Res. Inst. Math. Sci. (RIMS), Kyoto, to appear.