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NONLINEAR CAUCHY PROBLEMS WITH SMALL ANALYTIC DATA AND THE LIFESPAN OF THEIR SOLUTIONS

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Abstract. We study the lifespan of solutions to fully nonlinear second-order Cauchy problems with small real- or complex-analytic data. The nonlinear term is an analytic function in u, $\bar{u}$ and their derivatives. We give an outline of the proof based on the method of majorants and the fixed point technique.

1. Introduction. Let us consider the Cauchy problem of a very simple ordinary differential equation with a small parameter ε:

$$\frac{d^2}{dt^2}u = 6u^2, \quad u(0) = \varepsilon^2, \quad u'(0) = 2\varepsilon^3.$$

Its solution is $u(t) = \varepsilon^2/(1-\varepsilon t)^2$ and its denominator vanishes at $t = 1/\varepsilon$. The solution exists as long as $|t| < 1/\varepsilon$ and the smaller ε is, the longer the lifespan $1/\varepsilon$ is. In the present paper, we want to prove analogous results for partial differential equations. Smallness of the initial data is measured in terms of Cauchy type estimates.

Cauchy problems for semilinear wave equations with small data have been studied by many authors in the $\mathcal{C}^\infty$-category (see for instance [H] and [G]). On the other hand, some results have been obtained in the real-analytic category: weakly hyperbolic first order systems were dealt with in [DS] and [K] and the Kirchhoff equation was solved in [GM]. In the Gevrey class, m-th order equations have been solved in [GM2]. The latter two employ Banach algebras defined by means of majorants, rather than a scale of Banach spaces as in [Ni]. They are main tools of the present paper.

In the present paper, we consider fully nonlinear problems in the real- or complex-analytic category in the spirit of the Cauchy–Kowalevsky theorem, namely *without hyper-*

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bolicity assumption. In the real domain, our result is of Nagumo type ([Na]): we consider nonlinear terms which are continuous in the time variable t and are analytic in the space variable. In the complex domain, we deal with functions holomorphic in all the variables. Complete proofs are in [Y2].

2. Statement of the results. Let Ω be an open set of $\mathbb{R}^n_x$, $x = (x_1, \dots, x_n)$. A $\mathcal{C}^\infty$-function $\varphi(x)$ on Ω is said to be *uniformly analytic* on Ω if there exists a positive constant C such that

$$|\partial^\alpha \varphi(x)| \le C^{|\alpha|+1}|\alpha|! \quad (\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$$

holds for any $x \in \Omega$ and any $\alpha \in \mathbb{N}^n = \{0, 1, 2, \dots\}^n$.

This kind of estimate holds on compact subsets of Ω for any analytic function. What should be noted about a uniformly analytic function is that C is uniform on the whole open set Ω and there is no need to introduce compact subsets. We denote by $A(\Omega)$ the totality of uniformly analytic functions on Ω.

For $T > 0$, let I_T be the open interval $]-T, T[\, \subset \mathbb{R}_t$ and set $\Omega_T = I_T \times \Omega \subset \mathbb{R}_t \times \mathbb{R}^n_x$. For $k \in \mathbb{N}$, a continuous function $u(t, x)$ on Ω_T is said to belong to $\mathcal{C}^k(T; A(\Omega))$ if

(i) $\partial_t^j \partial^\alpha u \in \mathcal{C}(\Omega_T)$ for any $j \in \{0, \dots, k\}$ and any $\alpha \in \mathbb{N}^n$,
(ii) for any $T' \in]0, T[$, there exists a positive constant $C = C_{T'}$ such that

$$\sup_{|t| \le T', x \in \Omega} |\partial_t^j \partial^\alpha u(t, x)| \le C^{|\alpha|+1}|\alpha|!$$

for any $j \in \{0, \dots, k\}$ and any $\alpha \in \mathbb{N}^n$. (Notice that this estimate is uniform in $x \in \Omega$ but is only locally uniform in $t \in]-T, T[$.)

Let $P(\partial_t, \partial_x) = \sum_{j=1}^n p_j \partial_t \partial_j + \sum_{k=1}^n \sum_{j=1}^k p_{jk} \partial_j \partial_k$ ($\partial_j = \partial/\partial x_j$ and $p_j, p_{jk} \in \mathbb{C}$) be a second-order linear partial differential operator with constant complex coefficients. We consider the following fully nonlinear Cauchy problem:

$$\text{(CP)} \qquad \begin{cases} (\partial_t^2 - P(\partial_t, \partial_x))u = f(t; u;\ \partial_t u, \nabla u;\ \nabla \partial_t u, \nabla^2 u), \\ u(0, x) = \varphi(x),\ \partial_t u(0, x) = \psi(x), \end{cases}$$

where $\partial_t = \partial/\partial t$, $\nabla u = (\partial_j u)_{1 \le j \le n}$ and $\nabla^2 u = (\partial_j \partial_k u)_{1 \le j \le k \le n}$. Here $\varphi(x)$ and $\psi(x)$ are uniformly analytic on an open subset Ω of $\mathbb{R}^n$. A typical example of the operator $\partial_t^2 - P(\partial_t, \partial_x)$ is $\partial_t^2 \pm \Delta_x$ and we do not assume hyperbolicity. We assume that $f(t; X; Y; Z)$ is continuous and bounded on $\mathbb{R}_t \times \mathcal{U}$, where $\mathcal{U}$ is an open neighborhood of $(X, Y, Z) = 0 \in \mathbb{C} \times \mathbb{C}^{n+1} \times \mathbb{C}^N$, $N = n(n+3)/2$. Moreover we assume that it is complex-analytic in $\mathcal{U}$ for each fixed $t \in \mathbb{R}$ and is a sum of sufficiently large powers of X, Y, Z. Precisely speaking, we suppose that its Taylor expansion is of the form

$$f(t; X; Y; Z) = \sum_{L \ge 4} a_{\alpha\beta\gamma}(t) X^\alpha Y^\beta Z^\gamma, \quad L = \alpha + 2|\beta| + 3|\gamma|. \tag{1}$$

We want to show that the lifespan of a solution is large when the data are small in some sense. As will be explained later, their smallness implies that $X^\alpha Y^\beta Z^\gamma$ is small for $X = u$, $Y = (\partial_t u, \nabla u)$, $Z = (\nabla \partial_t u, \nabla^2 u)$ if $L = \alpha + 2|\beta| + 3|\gamma|$ is large.

THEOREM 1. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for any ε with* $0 < \varepsilon \le \varepsilon_0$:

If $\sup_{x\in\Omega} |\partial^\alpha \varphi(x)| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *and* $\sup_{x\in\Omega} |\partial^\alpha \psi(x)| \le \varepsilon^{|\alpha|+2}|\alpha|!$ *for all* $\alpha \in \mathbb{N}^n$, *then* (CP) *has a solution* $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ *for* $T = \delta/\varepsilon$.

REMARK 1. The order ε^{-1} of T is as good as in the linear case and is the best possible.

REMARK 2. In [Y] and [GM2], the nonlinear terms do not involve $u, \partial_t u$ and $\nabla\partial_t u$.

REMARK 3. As will be clear in Section 4, $\partial_t^2 u$ is unique in a Banach algebra $\mathcal{G}_{T,2e^2\varepsilon}(\Omega)$, which is a subspace of $\mathcal{C}^0(T; A(\Omega))$. See also Remark 4.

Let us formulate the complex version of (CP), which is referred to as (CPc). We assume that $\varphi(x)$ and $\psi(x)$ are complex-analytic on an open set $U \subset \mathbb{C}^n_x$. Let f be independent of t. (We assume it to be entire and bounded as in the real case. Then it is independent of t by Liouville's theorem.) For $T > 0$, we introduce the open ball $B_T = \{t \in \mathbb{C}; |t| < T\}$ instead of an open interval.

THEOREM 2. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with* $0 < \varepsilon \le \varepsilon_0$:

If $\sup_{x\in U} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *and* $\sup_{x\in U} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2}|\alpha|!$ *for all* $\alpha \in \mathbb{N}^n$, *then* (CPc) *has a unique solution $u(t,x)$ that is complex-analytic on $B_T \times U$ for $T = \delta/\varepsilon$ and satisfies the following estimate*: *for any T' with $0 < T' < T = \delta/\varepsilon$, there exists $C = C_{T'} > 0$ such that*

$$\sup_{|t|\le T', x\in U} |\partial^\alpha u(t,x)| \le C^{|\alpha|+1}|\alpha|!$$

for any $\alpha \in \mathbb{N}^n$.

REMARK 4. Some uniqueness result in the situation of Theorem 1 can be derived from Theorem 2. The functions φ and ψ in Theorem 1 extend to the $1/(4\varepsilon)$-neighborhood of Ω in $\mathbb{C}^n$ and satisfy $|\varphi^{(\alpha)}(x)| \le 2^n\varepsilon^{|\alpha|+1}|\alpha|!$, $|\psi^{(\alpha)}(x)| \le 2^n\varepsilon^{|\alpha|+2}|\alpha|!$ there. If f_1 in (CP1) is independent of t, Theorem 2 holds for a larger value of ε and a smaller value of $|t|$. It gives a *unique* real-analytic (in t, x) solution u to (CP1) for $|t| < \delta/(2^n\varepsilon)$, $x \in \Omega$.

We can deal with nonlinearities involving the complex conjugates of the derivatives of the unknown function. Let us consider

$$\text{(CPconj)}\qquad \begin{cases} \big(\partial_t^2 - P(\partial_x)\big)u = g(t; u, \bar{u};\, \partial_t u, \nabla u, \partial_t \bar{u}, \nabla\bar{u};\, \nabla\partial_t u, \nabla^2 u, \nabla\partial_t\bar{u}, \nabla^2\bar{u}), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \end{cases}$$

where $g(t; \tilde{X}; \tilde{Y}; \tilde{Z})$ is continuous and bounded on $\mathbb{R}_t \times \widetilde{\mathcal{V}}$, $\widetilde{\mathcal{V}}$ is an open neighborhood of $(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0 \in \mathbb{C}^2 \times \mathbb{C}^{2(n+1)} \times \mathbb{C}^{n(n+3)}$. Moreover, we assume that g is complex-analytic in $\widetilde{\mathcal{V}}$ for each fixed $t \in \mathbb{R}$ and has an expansion of the form

$$g(t; \tilde{X}; \tilde{Y}; \tilde{Z}) = \sum_{\tilde{L}\ge 4} a_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(t)\tilde{X}^{\tilde{\alpha}}\tilde{Y}^{\tilde{\beta}}\tilde{Z}^{\tilde{\gamma}},$$

$$\tilde{L} = |\tilde{\alpha}| + 2|\tilde{\beta}| + 3|\tilde{\gamma}|.$$

THEOREM 3. *There exist $\delta > 0$ and $\varepsilon_0 > 0$ such that the following holds for all ε with* $0 < \varepsilon \le \varepsilon_0$:

If $\sup_{x\in\Omega} |\partial^\alpha \varphi| \le \varepsilon^{|\alpha|+1}|\alpha|!$ *and* $\sup_{x\in\Omega} |\partial^\alpha \psi| \le \varepsilon^{|\alpha|+2}|\alpha|!$ *for all* $\alpha \in \mathbb{N}^n$, *then* (CPconj) *has a solution* $u(t,x) \in \mathcal{C}^2(T; A(\Omega))$ *for* $T = \delta/\varepsilon$.

EXAMPLE 1. The theorem above can be applied to the nonlinear wave equations

$$(\partial_t^2 - \Delta)u = |\nabla u|^2 = \sum_{j=1}^n \partial_j u\, \partial_j \bar{u},$$

$$(\partial_t^2 - \Delta)u = u|\nabla u|^2 = \sum_{j=1}^n u\partial_j u\, \partial_j \bar{u}.$$

3. Banach algebra. We will work in a Banach algebra with parameters T and ζ. It is a subspace of $\mathcal{C}^0(T; A(\Omega))$ and (CP) will be reduced to a 0-th order integro-differential equation in it. It can be solved with a suitable choice of (T, ζ) by using the contraction principle.

Some facts in this section have already appeared in [W], [GM], [GM2] and [Y], but we nevertheless present them for the readers' convenience. The proofs given here are just sketches and their conciseness makes essential ideas clear. Complete proofs can be found in [Y] and [Y2].

Let $f(X) = \sum_{k=0}^\infty a_k X^k$ and $g(X) = \sum_{k=0}^\infty b_k X^k$ be two formal series with $a_k \in \mathbb{R}$, $b_k \in \mathbb{R}_+$. We write $f(X) \ll g(X)$ if $|a_k| \le b_k$ for all $k \ge 0$.

For a formal power series $f(X) = \sum_{k=0}^\infty a_k X^k$, set

$$Df(X) = \sum_{k=1}^\infty k a_k X^{k-1} = \sum_{k=0}^\infty (k+1) a_{k+1} X^k,$$

$$D^{-1} f(X) = \sum_{k=0}^\infty \frac{a_k}{k+1} X^{k+1} = \sum_{k=1}^\infty \frac{a_{k-1}}{k} X^k.$$

We have $DD^{-1}f(X) = f(X)$ but $D^{-1}Df(X) = \sum_{k=1}^\infty a_k X^k \ne f(X)$.

Set

$$\theta(X) = K^{-1} \sum_{k=0}^\infty \frac{X^k}{(k+1)^2}, \quad K = \frac{4\pi^2}{3}.$$

It is a series due to Lax. It is useful in nonlinear analysis because $\theta^2(X) \ll \theta(X)$; roughly speaking, *multiplication is bounded.* On the other hand, we have $D^{-j}\theta(X) \ll (9/2)^j \theta(X)$ if $j \ge 0$ by Lemma 2.5 of [W]; *integration is bounded.*

If $\zeta > 0$, then a continuous function $u(t,x)$ on Ω_T is said to be an element of $\mathcal{G}_{T,\zeta}(\Omega)$ if it is infinitely differentiable in x and there exists a constant $C > 0$ such that

$$\sup_{x\in\Omega} |\partial^\alpha u(t,x)| \le C\zeta^{|\alpha|} D^{|\alpha|}\theta(|t|/T) \tag{2}$$

for any $\alpha \in \mathbb{N}^n$ and any $t \in I_T$. We define the norm $\|u\|$ to be the infimum of such C's.

PROPOSITION 1. *The space $\mathcal{G}_{T,\zeta}(\Omega)$ is a Banach algebra. Moreover, it is a subalgebra of* $\mathcal{C}^0(T; A(\Omega))$.

Proof. The boundedness of multiplication ($\|uv\| \le \|u\|\|v\|$) follows from $\theta^2(X) \ll \theta(X)$.

We can show that $\mathcal{G}_{T,\zeta}(\Omega)$ is a Banach space by injecting it into the Fréchet space $\mathcal{C}^{0,\infty}(\Omega_T)$ of continuous functions on Ω_T that are infinitely differentiable in x. A Cauchy sequence (u_k) in the former space is one in the latter space. It converges in $\mathcal{C}^{0,\infty}(\Omega_T)$. Let the limit be $u \in \mathcal{C}^{0,\infty}(\Omega_T)$. It can be proved that $u \in \mathcal{G}_{T,\zeta}(\Omega)$ and that $(u_k) \to u$ in $\mathcal{G}_{T,\zeta}(\Omega)$. ■

PROPOSITION 2. *Let the integral operator ∂_t^{-1} be defined by $\partial_t^{-1} w(t,x) = \int_0^t w(s,x)\,ds$. Then for any $(k,\alpha) \in (-\mathbb{N}) \times \mathbb{N}^n$ with $k + |\alpha| \le 0$, there exists a constant $C_{k,|\alpha|} > 0$ independent of T and ζ such that $\partial_t^k \partial^\alpha$ is an endomorphism of $\mathcal{G}_{T,\zeta}(\Omega)$ and its norm is not larger than $C_{k,|\alpha|} T^{-k} \zeta^{|\alpha|}$.*

Proof. The norm is measured in terms of $\theta(X)$. The boundedness of $\partial_t^k \partial^\alpha$ follows from $D^{-j}\theta(X) \ll (9/2)^j \theta(X)$ $(j \ge 0)$. ■

PROPOSITION 3. *Assume that $\varphi(x)$ and $\psi(x)$ satisfy $\sup_{x\in\Omega} |\partial^\alpha \varphi(x)| \le \varepsilon^{|\alpha|+1}|\alpha|!$, $\sup_{x\in\Omega} |\partial^\alpha \psi(x)| \le \varepsilon^{|\alpha|+2}|\alpha|!$ for all $\alpha \in \mathbb{N}^n$ and a constant $\varepsilon > 0$. Then for $\zeta = 2e^2\varepsilon$ and any $T > 0$, we have $\varphi, \psi \in \mathcal{G}_{T,\zeta}(\Omega)$. Moreover, they satisfy the following estimates: $\|\varphi\| \le K\varepsilon$, $\|\psi\| \le K\varepsilon^2$, $\|\partial_j \varphi\| \le K\varepsilon^2$, $\|\partial_j \psi\| \le K\varepsilon^3$, $\|\partial_j\partial_k \varphi\| \le 3K\varepsilon^3$, $\|\partial_j\partial_k\psi\| \le 3K\varepsilon^4$ for $j,k \in \{1,2,\dots,n\}$.*

Proof. By using $|\alpha|! = K(|\alpha|+1)^2 D^{|\alpha|}\theta(0) \le K(|\alpha|+1)^2 D^{|\alpha|}\theta(|t|/T)$, we can derive estimates of the form (2) for $\varphi(x)$ and $\psi(x)$ (independent of t) and their derivatives. ■

REMARK 5. Set $\zeta = 2e^2\varepsilon$ as in Proposition 3. It is a small quantity. If T is not too large, then the product $C_{k,|\alpha|}T^{-k}\zeta^{|\alpha|}$ in Proposition 2 is relatively small. In particular, it is equal to const.$\varepsilon^{k+|\alpha|}$ if $T = \delta/\varepsilon$. In some cases $k + |\alpha|$ is negative, but it causes no problem when $\partial_t^k\partial^\alpha$ acts on a small function. For example, if $k + |\alpha| = -1$ and the norm of $w \in \mathcal{G}_{T,\zeta}(\Omega)$ is of order ε^3, then $\|\partial_t^k\partial^\alpha w\|$ is of order ε^2. Notice that its powers are smaller ($\|(\partial_t^k\partial^\alpha w)^j\|$ is of order ε^{2j}). The assumption $L = |\alpha| + 2|\beta| + 3|\gamma| \ge 4$ guarantees sufficient smallness by this effect.

PROPOSITION 4. *Assume that $\varphi(x)$ and $\psi(x)$ satisfy the assumption in Proposition 3. Set $\zeta = 2e^2\varepsilon$, $T = \delta/\varepsilon$ for $0 < \delta < 1$. Then there exist positive constants C_1 and C_2 independent of ε and δ such that*

$$
\begin{aligned}
&\|P\partial_t^{-2} w\| \le C_1\delta\|w\|, \quad \|P(\varphi + t\psi)\| \le C_2\varepsilon^3,\\
&\|\partial_t^{-2} w + \varphi + t\psi\| \le C_{-2,0}\varepsilon^{-2}\|w\| + 2K\varepsilon,\\
&\|\partial_t(\partial_t^{-2} w + \varphi + t\psi)\| \le C_{-1,0}\varepsilon^{-1}\|w\| + K\varepsilon^2,\\
&\|\partial_j(\partial_t^{-2} w + \varphi + t\psi)\| \le 2e^2 C_{-2,1}\varepsilon^{-1}\|w\| + 2K\varepsilon^2,\\
&\|\partial_t\partial_j(\partial_t^{-2} w + \varphi + t\psi)\| \le 2e^2 C_{-1,1}\|w\| + K\varepsilon^3,\\
&\|\partial_j\partial_k(\partial_t^{-2} w + \varphi + t\psi)\| \le (2e^2)^2 C_{-2,2}\|w\| + 6K\varepsilon^3
\end{aligned}
$$

for any $w \in \mathcal{G}_{T,\zeta}(\Omega)$.

Proof. Apply Propositions 2 and 3. ■

4. Sketch of the proof of Theorem 1. Set $w(t,x) = \partial_t^2 u(t,x)$ (a new unknown function). Then we get $u = \partial_t^{-2} w + \varphi + t\psi$ and we have only to find $w \in \mathcal{G}_{T,\zeta}(\Omega) \subset \mathcal{C}^0(T; A(\Omega))$ with $T = \delta/\varepsilon$. From now on we will work in $\mathcal{G}_{T,\zeta}(\Omega)$. Let us introduce the following mappings Q and $\mathcal{L}$:

$$\begin{aligned} &Qu = (u; \partial_t u, \nabla u; \nabla\partial_t u, \nabla^2 u), \\ &\mathcal{L}(w) = P(\partial_t^{-2} w + \varphi + t\psi) + f\big(t; Q(\partial_t^{-2} w + \varphi + t\psi)\big). \end{aligned}$$

Then (CP) is reduced to the 0-th order integro-differential equation $w = \mathcal{L}(w)$. We shall find a fixed point w of $\mathcal{L}$ by showing that $\mathcal{L}$ is a contraction mapping on a closed ball centered at 0 of $\mathcal{G}_{T,\zeta}(\Omega)$, where

$$T = \delta/\varepsilon, \quad \zeta = 2e^2\varepsilon \quad (\varepsilon > 0,\ 0 < \delta < 1). \tag{3}$$

Assume that $\delta > 0$ is so small that $1 - 2C_1\delta > 0$. Let the radius r of the ball be defined by $r = 2C_2\varepsilon^3/(1 - 2C_1\delta)$. Notice that r is of order ε^3.

If w is in the above mentioned closed ball, then $\|w\|$ is of order $O(\varepsilon^3)$. By Proposition 4, $\|P\partial_t^{-2} w\|$ and other quantities are of orders shown in the following table:

$\|P\partial_t^{-2} w\|$	$\|P(\varphi + t\psi)\|$
$O(\varepsilon^3)$	$O(\varepsilon^3)$

(estimates concerning the linear part of $\mathcal{L}$)

$\|\partial_t^{-2} w + \varphi + t\psi\|$	$\|\partial_t(\partial_t^{-2} w + \varphi + t\psi)\|$	$\|\partial_j(\partial_t^{-2} w + \varphi + t\psi)\|$
$O(\varepsilon)$	$O(\varepsilon^2)$	$O(\varepsilon^2)$

(nonlinear)

$\|\partial_t\partial_j(\partial_t^{-2} w + \varphi + t\psi)\|$	$\|\partial_j\partial_k(\partial_t^{-2} w + \varphi + t\psi)\|$
$O(\varepsilon^3)$	$O(\varepsilon^3)$

(nonlinear)

Recall that the original unknown function u is given by $u = \partial_t^{-2} w + \varphi + t\psi$. The exponents 1, 2 and 3 concerning the nonlinear part in the above table corresponds to the coefficients in $L = \alpha + 2|\beta| + 3|\gamma|$. The assumption $L \geq 4$ in the main theorem implies that the nonlinear term of $\mathcal{L}$ is of order $O(\varepsilon^4)$. Since it is of higher order than r, we have

$$\|\text{the nonlinear part of } \mathcal{L}\| \leq r/2$$

if ε is sufficiently small. On the other hand, we can show that the linear part is small enough. Indeed, $\|w\| \leq r = 2C_2\varepsilon^3/(1 - 2C_1\delta)$ and Proposition 4 lead to

$$\|\text{the linear part of } \mathcal{L}\| = \|P(\partial_t^{-2} w + \varphi + t\psi)\| \leq C_1\delta r + C_2\varepsilon^3 = r/2.$$

Combining these estimates, we find that $\|w\| \leq r$ implies $\|\mathcal{L}(w)\| \leq r$; in other words, $\mathcal{L}$ is a mapping from the closed ball with radius r to itself. By a similar but more complicated calculation, we can show that $\mathcal{L}$ is a contraction and has a unique fixed point w.

5. Sketch of the proof of Theorem 2. The uniqueness is a consequence of the Cauchy–Kowalevsky theorem and analytic continuation. The existence is proved in the following way. A complex-analytic function on $B_T \times U$ is said to belong to $\mathcal{G}^{\mathbb{C}}_{T,\zeta}(U)$ if there exists a positive constant such that

$$\sup_{x \in U} |\partial^\alpha u(t,x)| \leq C\zeta^{|\alpha|} D^{|\alpha|}\theta(|t|/T)$$

for any $\alpha \in \mathbb{N}^n$ and any $t \in B_T$. We can show that $\mathcal{G}^{\mathbb{C}}_{T,\zeta}(U)$ is a Banach algebra and Theorem 2 can be proved in the same way as Theorem 1.

6. Sketch of the proof of Theorem 3. Let us solve the following system:

$$(\partial_t^2 - P(\partial_x))u_1 = g(t, Qu_1, Qu_2), \tag{4}$$

$$(\partial_t^2 - \overline{P}(\partial_x))u_2 = \overline{g}(t, Qu_2, Qu_1), \tag{5}$$

$$u_1(0,x) = \varphi(x),\ \partial_t u_1(0,x) = \psi(x), \tag{6}$$

$$u_2(0,x) = \overline{\varphi(x)},\ \partial_t u_2(0,x) = \overline{\psi(x)}, \tag{7}$$

where $g(t, Qu_1, Qu_2) = g(t; u_1, u_2; \partial_t u_1, \partial_t u_2; \nabla u_1, \nabla u_2; \dots)$ by abuse of notation. Moreover $\overline{P}$ and $\overline{g}$ are defined by

$$\overline{P}(\partial_x) = \sum_{k=1}^{n}\sum_{j=1}^{k} \overline{p_{jk}}\partial_j\partial_k,$$

$$\overline{g}(t; \tilde{X}; \tilde{Y}; \tilde{Z}) = \sum_{\tilde{L}\geq 4} \overline{a_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}(t)} \tilde{X}^{\tilde{\alpha}}\tilde{Y}^{\tilde{\beta}}\tilde{Z}^{\tilde{\gamma}}.$$

The system (4)–(7) can be uniquely solved in the same way as (CP) by introducing $\bigoplus^2 \mathcal{G}_{T,\zeta}(\Omega)$. By taking complex conjugates, we see that $(\overline{u_2}, \overline{u_1})$ is the unique solution of the following system:

$$(\partial_t^2 - P)\overline{u_2} = g(t, Q\overline{u_2}, Q\overline{u_1}), \tag{8}$$

$$(\partial_t^2 - \overline{P})\overline{u_1} = \overline{g}(t, Q\overline{u_1}, Q\overline{u_2}), \tag{9}$$

$$\overline{u_2}(0,x) = \varphi(x),\ \partial_t \overline{u_2}(0,x) = \psi(x), \tag{10}$$

$$\overline{u_1}(0,x) = \overline{\varphi(x)},\ \partial_t \overline{u_1}(0,x) = \overline{\psi(x)}. \tag{11}$$

It just means that $(\overline{u_2}, \overline{u_1})$ solves (4)–(7). By uniqueness we have $(\overline{u_2}, \overline{u_1}) = (u_1, u_2)$. We can set $u = u_1 = \overline{u_2}$. The solution $(u_1, u_2) = (u, \bar{u})$ of (4)–(7) gives the solution u to (CPconj).

7. Concluding remarks. Higher order equations are treated in [GM2]. It would be possible to extend the results in it to a wider class of nonlinear terms by introducing quantities like L.

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