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BOREL SUMMABILITY FOR A FORMAL SOLUTION OF $\frac{\partial}{\partial t}u(t,x) = \left(\frac{\partial}{\partial x}\right)^2 u(t,x) + t\left(t\,\frac{\partial}{\partial t}\right)^3 u(t,x)$

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Abstract. In this paper we study the Borel summability of a certain divergent formal power series solution for an initial value problem. We show the Borel summability under the condition that an initial value function $\phi(x)$ is an entire function of exponential order at most 2.

1. Introduction and statement of the main result. Let $t, x, \xi \in \mathbb{C}$. Let us introduce $D_R := \{x \in \mathbb{C} : \ |x| < R\}$ and $S_{d,\theta} := \{\xi \in \mathbb{C} \setminus \{0\} : |\arg \xi - d| < \theta\}$. Let $\mathcal{O}(D_R)$ (resp. $\mathcal{O}(S_{d,\theta} \times D_R)$) be the set of all holomorphic functions on D_R (resp. $S_{d,\theta} \times D_R$), $\mathcal{O}(D_R)[[t]]$ the set of all formal power series $\sum_{i=0}^{\infty} u_i(x)t^i$, where the coefficients $u_i(x)$ are in $\mathcal{O}(D_R)$. We denote by $[a]$ the integer part of $a \in \mathbb{R}$.

We consider the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = a\Big(\dfrac{\partial}{\partial x}\Big)^2 u(t,x) + bt\Big(t\,\dfrac{\partial}{\partial t}\Big)^3 u(t,x) \\ u(0,x) = \phi(x), \end{cases} \tag{1}$$

where the numbers a and b are any complex numbers.

Let us recall some known results. If $b = 0$, then equation (1) is the heat equation. Then we have the following two results.

i) Assume that the initial value function $\phi(x)$ is an entire function and satisfies with some positive constants C and K,

$$|\phi(x)| \le Ce^{K|x|^2} \quad \text{for} \quad x \in \mathbb{C}.$$

Then the formal power series solution $\hat{v}(t,x)$ of (1) is holomorphic in a neighborhood of $t = 0$. This is a classical result, (see [Kow]).

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ii) The following result is that of Lutz–Miyake–Schäfke in [L-M-S]. The following two statements a) and b) are equivalent:

a) The initial value function $\phi(x)$ is analytic on $\Omega = S_{d/2,\theta} \cup S_{d/2+\pi,\theta} \cup D_r$ and satisfies with some positive constants C and K,

$$|\phi(x)| \le Ce^{K|x|^2} \quad \text{on} \quad \Omega.$$

b) The formal power series solution $\hat{v}(t,x)$ of (1) is Borel summable in a direction d.

The case $a = 0$ is covered by Ōuchi [Ou1] and [Ou2], where he obtained some results on the multi-summability of some linear/nonlinear partial differential equations.

The purpose of this paper is to show the Borel summability for a formal solution of (1) in the case $ab \neq 0$.

At first we study the Borel summability for a formal power series solution $\widehat{v}(t,x)$. We refer the reader for details to Lutz–Miyake–Schäfke [L-M-S].

Let $\widehat{v}(t,x) = \sum_{i=0}^{\infty} v_i(x)t^i \in \mathcal{O}(D_R)[[t]]$ be a formal power series with coefficients holomorphic in D_R. By $\mathcal{O}(D_R)[[t]]_1$ we denote the subset of $\mathcal{O}(D_R)[[t]]$ whose coefficients satisfy with some positive constants A, B and $0 < r < R$,

$$\sup_{|x|\le r} |v_i(x)| \le AB^i\Gamma(i+1) \quad \text{for} \quad i = 0, 1, \dots,$$

The elements of $\mathcal{O}(D_R)[[t]]_1$ are called *formal series of Gevrey class one.*

We define $\mathcal{O}[[t]]_1$ by

$$\mathcal{O}[[t]]_1 := \bigcup_{R>0} \mathcal{O}(D_R)[[t]]_1.$$

Set $S^t_{d,\theta} := \{t \in \mathbb{C} \setminus \{0\} : |\arg \xi - d| < \theta\}$ and $S^t_{d,\theta}(T) = S^t_{d,\theta} \cup \{t : |t| < T\}$.

Let $v(t,x)$ be analytic on $S^t_{d,\theta}(T)$ for some $T > 0$. Then $\widehat{v}(t,x) \in \mathcal{O}[[t]]_1$ is called *a Gevrey asymptotic expansion* of $v(t,x)$ as $t \to 0$ in $S^t_{d,\theta}$, written as

$$v(t,x) \cong_1 \widehat{v}(t,x) \quad \text{in} \quad S^t_{d,\theta},$$

if for any proper subset $S' \Subset S^t_{d,\theta}(T)$ there exist positive constants A, B and $0 < r < R$ such that $\widehat{v}(t,x) \in \mathcal{O}(D_R)[[t]]_1$ and

$$\sup_{|x|\le r} \left| v(t,x) - \sum_{i=0}^{N-1} v_i(x)t^i \right| \le AB^N\Gamma(N+1)|t|^N \quad \text{for} \quad t \in S' \quad \text{and} \quad N = 1, 2, \dots.$$

Definition 1.1. We say that $\widehat{v}(t,x) \in \mathcal{O}[[t]]_1$ is *Borel summable in a direction* $d \in \mathbb{R}$ if there exist a sector $S^t_{d,\theta}$ with $\theta > \pi/2$ and a function $v(t,x)$ analytic on $S^t_{d,\theta} \times D_r$ such that $v(t,x) \cong_1 \widehat{v}(t,x)$ in $S^t_{d,\theta}$.

Remark 1.2. Let us remark that the function $v(t,x)$ is unique if it exists, in that case $v(t,x)$ is called *the Borel sum* of $\widehat{v}(t,x)$.

Definition 1.3. Let $\widehat{v}(t,x) = \sum_{i=0}^{\infty} v_i(x)t^i \in \mathcal{O}(D_R)[[t]]$. Then *the formal Borel transform* $(\widehat{\mathcal{B}}\widehat{v})(\xi,x)$ is defined by

$$(\widehat{\mathcal{B}}\widehat{v})(\xi,x) = v_0(x)\delta(\xi) + \sum_{i=1}^{\infty} \frac{v_i(x)}{\Gamma(i)}\,\xi^{i-1},$$

where $\delta(\xi)$ means the delta function with support at $\xi = 0$.

The Borel summability of $\widehat{v}(t,x) \in \mathcal{O}[[t]]_1$ can be characterized by

PROPOSITION 1.4 ([L-M-S]). *The formal power series $\widehat{v}(t,x) \in \mathcal{O}(D_R)[[t]]_1$ is Borel summable in a direction d if one can find some $r < R$ so that the following two properties hold:*

1. *The power series $V(\xi,x) = (\widehat{\mathcal{B}}\widehat{v})(\xi,x) - v_0(x)\delta(\xi)$ converges for $|\xi| < R$ and $x \in D_r$.*
2. *There exists a $\theta > 0$ such that for any $x \in \overline{D_r}$ the function $V(\xi,x)$ can be continued with respect to ξ into the sector $S_{d,\theta}$. Moreover, for any $\theta_1 < \theta$ there exist constants C, $K > 0$ such that*

$$\sup_{|x|\le r} |V(\xi,x)| \le Ce^{K|\xi|} \quad \textit{for} \quad \xi \in S_{d,\theta_1}.$$

Then $v_0(x) + (\mathcal{L}_d V)(t,x)$ is called the Borel summation in a direction d of $\widehat{v}(t,x)$, where $\mathcal{L}_d$ is the Laplace transform that is defined by

$$(\mathcal{L}_d\phi)(t,x) := \int_0^{\infty e^{id}} \exp\{-(\tfrac{\xi}{t})\}\phi(\xi,x)\,d\xi.$$

Note that by changing variables $s = b^{1/2}t$ and $y = a^{-1/2}b^{1/4}x$ in (1) we can assume $a = 1$ and $b = 1$, which we shall do from now on.

For the equation (1) set

$$A_0(\xi) = 1 - \xi^2. \tag{2}$$

DEFINITION 1.5. Set $Z = \{\xi : A_0(\xi) = 0\}$. A *singular direction* is an argument of an element of Z. We denote by Ξ the totality of singular directions, i.e. $\Xi = \{d \in \mathbb{R} : d = 0 \bmod (\pi)\}$.

Now we are ready to state the main result.

MAIN THEOREM 1.6. *Assume that the initial value function $\phi(x)$ is an entire function and satisfies with some positive constants C and K,*

$$|\phi(x)| \le Ce^{K|x|^2} \quad \textit{on} \quad \mathbb{C}.$$

Then the equation (1) has a formal power series solution $\widehat{v}(t,x)$ which is Borel summable in a direction d with $\overline{S_{d,\theta}} \cap \Xi = \emptyset$ for a sufficiently small $\theta > 0$.

REMARK 1.7. In the case $\phi(x) = e^{x^2}$, the formal solution $\widehat{v}(t,x)$ of (1) satisfies for $x \in \mathbb{R}$,

$$\begin{aligned} u_i(x) &\ge 2^{i-3}\,\frac{((i-2)/2)!^3}{(i/2)!}\Big(\frac{\partial}{\partial x}\Big)^4\phi(x) && \text{for } i \ge 2,\ i \text{ even},\\ u_i(x) &\ge \frac{1}{2^{i-4}}\,\frac{((i-1)/2)!}{i!}\,\frac{(i-2)!^3}{((i-3)/2)!^3}\Big(\frac{\partial}{\partial x}\Big)^2\phi(x) && \text{for } i \ge 3,\ i \text{ odd}. \end{aligned} \tag{3}$$

It is not trivial to prove using this estimate that $\widehat{v}(t,x)$ is Borel summable however the initial value function $\phi(x)$ is an entire function of exponential order 2.

2. Formal solution. In this section we construct a formal power series solution of (1) and give an estimation of its coefficients.

First of all note that a formal power series solution $\widehat{v}(t,x) = \sum_{i=0}^{\infty} u_i(x)t^i$ of (1) is unique and satisfies the recurrence relations

$$\begin{cases} u_0(x) = \phi(x), \quad u_1(x) = \Big(\dfrac{\partial}{\partial x}\Big)^2 \phi(x), \\ iu_i(x) = \Big(\dfrac{\partial}{\partial x}\Big)^2 u_{i-1}(x) + (i-2)^3 u_{i-2}(x) \quad \text{for} \quad i \geq 2. \end{cases} \tag{4}$$

We have

LEMMA 2.1. *Assume that the initial value function $\phi(x) \in \mathcal{O}(D_R)$. Then the coefficients $u_i(x)$ of $\widehat{v}(t,x)$ are holomorphic in D_R and there exist positive constants A, B such that*

$$|u_i(x)| \leq AB^i\Gamma(i+1) \quad \text{on} \quad D_r \quad \text{for} \quad i \in \mathbb{N}_0 \quad \text{and} \quad 0 < r < R. \tag{5}$$

Proof. For serii $f(x) = \sum_{j=0}^{\infty} f_j x^j$ and $g(x) = \sum_{j=0}^{\infty} g_j x^j$ with $g_j \geq 0$ write $f(x) \ll g(x)$ if $|f_j| \leq g_j$ for $j \in \mathbb{N}_0$.

For $A > 0$ and $R > 0$ set $\theta_R(x) = \frac{A}{1-x/R}$ and $\theta_R^{(n)}(x) = (\frac{\partial}{\partial x})^n \theta_R(x) = \frac{An!}{R^n(1-x/R)^{n+1}}$ for $n \geq 0$. For the function $\theta_R(x)$ we get

$$\theta_R^{(n)}(x) \ll \frac{R}{n+1}\,\theta_R^{(n+1)}(x) \quad \text{for} \quad n \geq 0. \tag{6}$$

We will show that the coefficients $u_i(x)$ in (4) satisfy with some some $A > 0$ and $R > 0$,

$$u_i(x) \ll \frac{C_R^i}{i!}\,\theta_R^{(2i)}(x) \quad \text{for} \quad i \geq 0, \tag{7}$$

where $C_R = 1 + R^4$.

Since the function $\phi(x)$ is holomorphic in a neighborhood of the origin, for some $A > 0$ and $R > 0$ we have

$$u_0(x) = \phi(x) \ll \theta_R^{(0)}(x).$$

By the relation (4) we get

$$u_1(x) \ll \theta_R^{(2)}(x) \ll \frac{C_R}{1!}\,\theta_R^{(2)}(x), \quad u_2(x) \ll \frac{C_R}{2!}\,\theta_R^{(4)}(x) \ll \frac{C_R^2}{2!}\,\theta_R^{(4)}(x).$$

For $i \geq 3$ let us show the estimate (7) by induction. By the inductive assumption, we have

$$u_j(x) \ll \frac{C_R^j}{j!}\,\theta_R^{(2j)}(x) \quad \text{for} \quad 0 \leq j < i. \tag{8}$$

Next by the estimates (6) and (8) we get

$$\begin{aligned} \Big(\frac{\partial}{\partial x}\Big)^2 u_{i-1}(x) &\ll \frac{C_R^{i-1}}{(i-1)!}\,\theta_R^{(2i)}(x) \\ (i-2)^3 u_{i-2}(x) &\ll \frac{(i-1)(i-2)^3 C_R^{i-2} R^4}{(i-1)!(2i-3)(2i-2)(2i-1)(2i)}\,\theta_R^{(2i)}(x) \\ &\ll \frac{C_R^{i-2}R^4}{(i-1)!}\,\theta_R^{(2i)}(x) \ll \frac{C_R^{i-1}R^4}{(i-1)!}\,\theta_R^{(2i)}(x). \end{aligned} \tag{9}$$

Hence the relation (4) and the estimate (9) imply that the estimate (7) holds for $i \geq 0$. ∎

3. Preparatory lemmas. In this section we recall two lemmas that we need to prove Theorem 1.6. These lemmas are in [Ou1], so we omit their proofs.

DEFINITION 3.1. Let $\phi_i(\xi,x) \in \mathcal{O}(S_{d,\theta} \times D_R)$, $i = 1,2$, satisfy $|\phi_i(\xi,x)| \le C|\xi|^{\epsilon-1}$ for $\epsilon > 0$. Then the *convolution* of $\phi_1(\xi,x)$ and $\phi_2(\xi,x)$ is defined by

$$(\phi_1 * \phi_2)(\xi,x) = \int_0^\xi \phi_1(\xi-\eta,x)\phi_2(\eta,x)\,d\eta.$$

Then we have the following lemma.

LEMMA 3.2 ([Ou1, Lemma 1.4, p. 516]). *Assume that the functions $\phi_i(\xi,x)$, $i = 1,2$, belonging to $\mathcal{O}(S_{d,\theta} \times D_R)$ satisfy*

$$|\phi_i(\xi)| \le C_i \frac{|\xi|^{s_i-1}}{\Gamma(s_i)} \quad \text{on} \quad S_{d,\theta} \times D_R$$

for $i = 1,2$. Then the convolution $(\phi_1 * \phi_2)(\xi,x)$ satisfies

$$|(\phi_1 * \phi_2)(\xi,x)| \le C_1C_2 \frac{|\xi|^{s_1+s_2-1}}{\Gamma(s_1+s_2)} \quad \text{on} \quad S_{d,\theta} \times D_R.$$

LEMMA 3.3 ([Ou1, Lemma 3.2, p. 526]). *For a series $\widehat{v}(t,x) = \sum_{n=1}^\infty v_n(x)t^n$ set $(\widehat{\mathcal{B}}\widehat{v})(\xi,x) = V(\xi,x)$. Then for $1 \le k \le \delta$ we have*

$$\widehat{\mathcal{B}}\Big(t^\delta\Big(t\frac{\partial}{\partial t}\Big)^k \widehat{v}\Big)(\xi,x) = \sum_{s=1}^k C_{k,s}\frac{\xi^{\delta-(s+1)}}{\Gamma(\delta-s)} * (\xi^s V(\xi,x)),$$

where the constants $C_{k,s}$ satisfy

$$C_{1,1} = 1, \quad C_{k,s} = -sC_{k-1,s} + C_{k-1,s-1}. \tag{10}$$

4. Proof of the Main Theorem. In this section we will prove the Main Theorem by analyzing a convolution equation that is constructed from the equation (1). Firstly, let us construct the convolution equation. To this end set $u(x,t) = \phi(x) + v(t,x)$. Substituting $u(t,x)$ into (1) we get

$$\frac{\partial}{\partial t}v(t,x) = \Big(\frac{\partial}{\partial x}\Big)^2\phi(x) + \Big(\frac{\partial}{\partial x}\Big)^2 v(t,x) + t\Big(t\frac{\partial}{\partial t}\Big)^3 v(t,x). \tag{11}$$

Now we multiply each term of (11) by t^2 and apply the formal Borel transformation. Then by Lemma 3.3 we get the convolution equation

$$\xi V(\xi,x) = \frac{\xi^{2-1}}{\Gamma(2)}\Big(\frac{\partial}{\partial x}\Big)^2\phi(x) + \frac{\xi^{2-1}}{\Gamma(2)} * \Big(\frac{\partial}{\partial x}\Big)^2 V(\xi,x) + \sum_{s=1}^3 C_{3,s}\frac{\xi^{3-(s+1)}}{\Gamma(3-s)} * \big(\xi^s V(\xi,x)\big). \tag{12}$$

For the formal solution $\widehat{v}(t,x)$ in Lemma 2.1 set $\widehat{v}_0(t,x) := \widehat{v}(t,x) - \phi(x)$ and

$$V(\xi,x) = (\widehat{\mathcal{B}}\widehat{v}_0)(\xi,x). \tag{13}$$

Then by Lemma 2.1, $V(\xi,x)$ is a holomorphic function on $|\xi| < \tau$ for some $\tau > 0$ and satisfies (12). We will show that $V(\xi,x)$ is analytic on $S_{d,\theta}$ for some directions d in ξ.

By

$$C_{3,3}\delta(\xi) * (\xi^3 V(\xi,x)) = \xi^3 V(\xi,x)$$

we rewrite (12) as

$$(\xi-\xi^3)V(\xi,x)=\frac{\xi^{2-1}}{\Gamma(2)}\Big(\frac{\partial}{\partial x}\Big)^2\phi(x) + \frac{\xi^{2-1}}{\Gamma(2)}*\Big(\frac{\partial}{\partial x}\Big)^2V(\xi,x)+\sum_{s=1}^{2}C_{3,s}\frac{\xi^{3-(s+1)}}{\Gamma(3-s)}*\big(\xi^sV(\xi,x)\big). \tag{14}$$

Note that $\xi-\xi^3=\xi A_0(\xi)$, where $A_0(\xi)$ is given by (2). Let us construct a formal solution $V(\xi,x)=\sum_{i=0}^{\infty}V_i(\xi,x)$ of (14) with

$$\xi A_0(\xi)V_0(\xi,x)=\frac{\xi^{2-1}}{\Gamma(2)}\Big(\frac{\partial}{\partial x}\Big)^2\phi(x) \tag{15}$$

$$\xi A_0(\xi)V_i(\xi,x)=\frac{\xi^{2-1}}{\Gamma(2)}*\Big(\frac{\partial}{\partial x}\Big)^2V_{i-1}(\xi,x)+\sum_{s=1}^{2}C_{3,s}\frac{\xi^{3-(s+1)}}{\Gamma(3-s)}*\big(\xi^sV_{i-1}\big) \tag{16}$$

for $i\ge 1$, where $V_{-1}(\xi,x)\equiv 0$.

Let $0<\tau<1$. Set $\Omega=S_{d,\theta}\cup\{\xi\in\mathbb{C}:|\xi|<\tau\}$ with $\overline{S_{d,\theta}}\cap\Xi=\emptyset$. Now we estimate functions $V_i(\xi,x)$ on Ω. For $A_0(\xi)$ we have

$$|\{A_0(\xi)\}^{-1}|\le C_0(|\xi|^2+1)^{-1}\quad\text{on}\quad\Omega. \tag{17}$$

To estimate functions $V_i(\xi,x)$ we need the following lemma, which can be found in [Kow] and [Pic].

Lemma 4.1. *The following two statements are equivalent*:

(i) *A function $\phi(x)$ is an entire function and satisfies with some positive constants C,K,*

$$|\phi(x)|\le Ce^{K|x|^2}\quad on\quad \mathbb{C}.$$

(ii) *For any $R>0$ there exist positive constants A and B depending on R such that*

$$\Big\|\Big(\frac{\partial}{\partial x}\Big)^i\phi\Big\|_R\le AB^i\Gamma\big(\tfrac{i}{2}+1\big)$$

for all $i=0,1,\dots,$ where $\|\phi\|_R=\sup_{x\in D_R}|\phi(x)|$.

Then for functions $V_i(\xi,x)$ we have

Proposition 4.2. *Set $\varphi(x)=(\frac{\partial}{\partial x})^2\phi(x)$. Assume that for some positive constants A,B,*

$$\|\Big(\frac{\partial}{\partial x}\Big)^i\varphi\|_R\le AB^i\Gamma\big(\tfrac{i}{2}+1\big). \tag{18}$$

Then for a sufficiently small $R>0$

$$\|\Big(\frac{\partial}{\partial x}\Big)^iV_k\|_R\le AB^{i+2k}K^k\Gamma\big(\tfrac{i}{2}+k+1\big)\frac{|\xi|^k}{(k+1)!k!}\quad for\quad i,k\in\mathbb{N}_0\quad and\quad \xi\in\Omega, \tag{19}$$

where $K=C_0\big(1+\sum_{s=1}^{2}|C_{3,s}|/B^2\big)$.

We will give a proof of Proposition 4.2 in the next section.

By Proposition 4.2 we have

$$\|V_k\|_R\le A(B^2K)^k\frac{|\xi|^k}{(k+1)!}\quad\text{for}\quad \xi\in\Omega. \tag{20}$$

Next by Lemma 4.1 and the estimate (20) we obtain the following proposition.

Proposition 4.3. *Assume that the initial value function $\phi(x)$ is an entire function and satisfies with some positive constants C and K,*

$$|\phi(x)| \le Ce^{K|x|^2} \quad on \quad \mathbb{C}.$$

Then for $(\xi, x) \in \Omega \times D_\rho$ with $\overline{S_{d,\theta}} \cap \Xi = \emptyset$,

$$|V(\xi, x)| \le C_1 e^{K_1|\xi|}$$

for $0 < \rho < R$.

Proof. By (20) we get

$$\|V\|_R \le \sum_{i=0}^{\infty} A(B^2K)^i \frac{|\xi|^i}{(i+1)!} \quad \text{for} \quad \xi \in \Omega.$$

Hence Proposition 4.3 follows by Lemma 4.1. ∎

Finally to end the proof of the Main Theorem note that by Proposition 4.3 the solution $V(\xi, x)$ satisfies the conditions of Proposition 1.4. ∎

5. Proof of Proposition 4.2. Firstly, let us estimate the function $V_0(\xi, x)$. By the relation (15) we have

$$\Big(\frac{\partial}{\partial x}\Big)^i V_0(\xi, x) = \{A_0(\xi)\}^{-1} \frac{\xi^{1-1}}{\Gamma(2)} \Big(\frac{\partial}{\partial x}\Big)^i \varphi(x).$$

Then by the estimates (17) and (18) we get

$$\Big\|\Big(\frac{\partial}{\partial x}\Big)^i V_0\Big\|_R \le C_0 \frac{|\xi|^{1-1}}{\Gamma(2)} AB^i \Gamma\big(\tfrac{i}{2} + 1\big)$$

and (19) follows for $k = 0$.

To show that the functions $V_k(\xi, x)$ satisfy (19) for $k \ge 1$ we use the induction. So assume that

$$\Big\|\Big(\tfrac{\partial}{\partial x}\Big)^i V_{k-1}\Big\|_R \le AB^{i+2(k-1)} K^{k-1} \Gamma\big(\tfrac{i}{2} + k - 1 + 1\big) \frac{|\xi|^{k-1}}{k!(k-1)!} \quad \text{for} \quad \xi \in \Omega. \tag{21}$$

Let us give an estimate for the right hand side of the relation (16). For the first term, by the inductive assumption (21) we have

$$\Big\|\Big(\frac{\partial}{\partial x}\Big)^{i+2} V_{k-1}\Big\|_R \le AB^{i+2k} K^{k-1} \Gamma\big(\tfrac{i}{2} + k + 1\big) \frac{|\xi|^{k-1}}{k!(k-1)!} \quad \text{for} \quad \xi \in \Omega.$$

By Lemma 3.2 it follows that

$$\begin{aligned} \Big\|\frac{\xi^{2-1}}{\Gamma(2)} * \Big(\frac{\partial}{\partial x}\Big)^{i+2} V_{k-1}\Big\|_R &\le AB^{i+2k} K^{k-1} \Gamma\big(\tfrac{i}{2} + k + 1\big) \frac{|\xi|^{k+2-1}}{k!(k+1)!} \\ &= AB^{i+2k} K^{k-1} \Gamma\big(\tfrac{i}{2} + k + 1\big) \frac{|\xi|^{k+2-1}}{(k+1)!k!} \quad \text{for} \quad \xi \in \Omega. \end{aligned} \tag{22}$$

For the second term, by the inductive assumption (21) for $s = 1, 2$ we have

$$\begin{aligned} \Big\|\Big(\frac{\partial}{\partial x}\Big)^i (\xi^s V_{k-1})\Big\|_R &\le AB^{i+2(k-1)} K^{k-1} \Gamma\big(\tfrac{i}{2} + k - 1 + 1\big) \frac{|\xi|^{k+s-1}}{k!(k-1)!} \\ &= AB^{i+2(k-1)} K^{k-1} \Gamma\big(\tfrac{i}{2} + k - 1 + 1\big) \frac{\Gamma(k+s)}{k!(k-1)!} \frac{|\xi|^{k+s-1}}{\Gamma(k+s)} \end{aligned} \tag{23}$$

for $\xi \in \Omega$. So by Lemma 3.2 we derive

$$
\begin{aligned}
&\Big\|\Big(\frac{\partial}{\partial x}\Big)^i\Big\{\sum_{s=1}^{2} C_{3,s}\frac{\xi^{3-s-1}}{\Gamma(3-s)} * (\xi^s V_{k-1})\Big\}\Big\|_R \\
&\le \sum_{s=1}^{2} |C_{3,s}| AB^{i+2(k-1)}K^{k-1}\Gamma\big(\tfrac{i}{2}+k-1+1\big)\frac{\Gamma(k+s)}{k!(k-1)!}\frac{|\xi|^{k+3-1}}{\Gamma(k+3)} \qquad (24)\\
&\le \frac{\sum_{s=1}^{2}|C_{3,s}|}{B^2} AB^{i+2k}K^{k-1}\Gamma\big(\tfrac{i}{2}+k-1+1\big)\frac{k(k+1)(k+s-1)!}{(k+2)!}\frac{|\xi|^{k+3-1}}{(k+1)!k!}
\end{aligned}
$$

for $\xi \in \Omega$. Moreover, we have

$$\Gamma\big(\tfrac{i}{2}+k-1+1\big)k \le \Gamma\big(\tfrac{i}{2}+k+1\big)$$

and

$$\frac{(k+1)(k+s-1)!}{(k+2)!} \le 1$$

for $s = 1, 2$. Hence we have

$$
\begin{aligned}
&\Big\|\Big(\frac{\partial}{\partial x}\Big)^i\Big\{\sum_{s=1}^{2} C_{3,s}\frac{\xi^{3-s-1}}{\Gamma(3-s)} * (\xi^s V_{k-1})\Big\}\Big\|_R \\
&\qquad \le \frac{\sum_{s=1}^{2}|C_{3,s}|}{B^2} AB^{i+2k}K^{k-1}\Gamma\big(\tfrac{i}{2}+k+1\big)\frac{|\xi|^{k+3-1}}{(k+1)!k!} \qquad (25)
\end{aligned}
$$

for $\xi \in \Omega$. Finally, by (16), (17), (22) and (25), we obtain

$$
\begin{aligned}
\Big\|\Big(\frac{\partial}{\partial x}\Big)^i V_k\Big\|_R &\le C_0(1+|\xi|^2)^{-1}\Big\{AB^{i+2k}K^{k-1}\Gamma\big(\tfrac{i}{2}+k+1\big)\frac{|\xi|^{k+1-1}}{(k+1)!k!} \\
&\quad + \frac{\sum_{s=1}^{2}|C_{3,s}|}{B^2} AB^{i+2k}K^{k-1}\Gamma\big(\tfrac{i}{2}+k+1\big)\frac{|\xi|^{k+2-1}}{(k+1)!k!}\Big\} \qquad (26)\\
&\le AB^{i+2k}K^{k}\Gamma\big(\tfrac{i}{2}+k+1\big)\frac{|\xi|^{k+1-1}}{(k+1)!k!}
\end{aligned}
$$

with $K = C_0(1+\sum_{s=1}^{2}|C_{3,s}|/B^2)$. ∎

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