The Theory of Systems With Internal degrees of Freedom*

Lecture Notes

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A. INTRODUCTION

In Newtonian dynamics extended test bodies such as planets are usually represented by point masses or rigid bodies, i.e. in both cases by systems with a finite number of degrees of freedom. In general relativity a similar simplified representation of extended bodies is offered by multipole particles introduced by Mathisson [1]. Equations of motion of such particles are derived from the conservation law of energy and momentum. In the case of a pole-dipole particle there are 10 parameters to be determined and only 7 equations of motion. Thus the world line associated with a pole-dipole particle remains undetermined. This apparent difficulty is removed by noting that the world line is completely arbitrary unless it is related to the center of mass of the particle by suitable conditions. The resulting equations of motion in flat space

$$m\frac{\delta\dot{\xi}^{\mu}}{\mathrm{d}s}=0, \qquad \frac{\delta S^{\mu\nu}}{\mathrm{d}s}=0$$

state that the world line of the center of mass is straight and the internal angular momentum is constant.

The pole-dipole particle is frequently considered a classical model of the electron: Hönl and Papapetrou [2], Weissenhoff [3]. The Dirac theory or the electron leads to a strange phenomenon known as Zitterbewegung. Some authors [3] have used the freedom in the choice of the world line associated with a pole-dipole particle for a classical representation of this phenomenon. In deriving the Zitterbewegung, the Dirac equation is transformed into a non-relativistic Schrödinger form and the Hamiltonian obtained is used to find the Heisenberg equations of motion. The physical content of the Dirac equation is by no means clear, however, and therefore this method is not fully justified.

In this series of lectures we study the theory of systems with internal degrees of freedom on both the quantum and the classical level. The theory of spin $\frac{1}{2}$, mass m particles is reformulated. It is still equivalent to the Dirac theory in most respects. The Zitterbewegung is eliminated, however. On the classical level a system with spin $\frac{1}{2}$ and mass m is found to correspond to a pole-dipole particle with a straight world line.

Chapter B contains a mathematical introduction. Most of the material presented there can be found in standard texts on modern algebra, in particular in Pontrjagin [4], Weyl [5] and Boerner [6].

In Chapter C we formulate quantum mechanics in an operator form based on group algebra. This formulation is applied in particular to the theory of angular momentum. Theories of particles with mass m and spin 0 or $\frac{1}{2}$ are also formulated.

Finally in Chapter D the classical limit is discussed and compared with known results for multipole particles.

B. MATHEMATICAL PRINCIPLES

I. Preliminaries

1. Groups. A set G of elements $r, s, t, \ldots$ is called a *group* if there is an operation defined in G, called multiplication, such that

a) multiplication is associative: $r(st) = (rs)t = rst$,
b) there exists an identity, that is, an element 1 defined by $r1 = 1r = r$,
c) for each $r \in G$ there exists a unique inverse element r^{-1} such that $rr^{-1} = r^{-1}r = 1$.

The multiplication is in general non-commutative. If it is commutative the group is said to be *Abelian*. Sometimes additive notation is used for Abelian groups with the sum $r + s$ replacing the product rs, the negative $-r$ replacing the inverse r^{-1}, and with 0 instead of 1 for the neutral element,

A group can be *finite* or *infinite*. If it is finite then the number of its elements is called the *order* of the group.

The concept of an abstract group is a generalization of the concept of a group of transformations. Given a set M we can consider all possible one-to-one mappings of the set onto itself. This set of mappings obviously forms a group under composition of transformations, with identical mapping for the identity and inverse mapping as the inverse element. A transformation group need not contain all one-to-one mappings.

Given a group of transformations of a set M onto itself we can introduce a relation $\sim$ between elements of M: $p \sim q$ if there exists a transformation $s \in G$ such that $s(p) = q$. This relation is *reflexive*: $p \sim p$, *symmetric*: $p \sim q$ implies $q \sim p$, and *transitive*: $p \sim q$ and $q \sim n$ imply $p \sim n$. Any relation with these properties is called an *equivalence*. Under an equivalence relation the set M decomposes into mutually exclusive subsets called *equivalence classes*. Every group of transformations or an equivalence relation determines a *classification* of the elements of M. Examples of classifications by means of groups of transformations are known from geometry.

Every classification defines an equivalence relation but not necessarily a group of transformations. If all equivalence classes contain one element, then the corresponding relation is the equality. If under a group of transformations the whole set M forms one single class, then M is said to be *homogeneous* under this group with every point being equivalent to any other.

A one-to-one mapping φ of a group G onto another group G' is called an *isomorphism* if it preserves multiplication: $r' = \varphi(r)$ and $s' = \varphi(s)$ imply $r's' = \varphi(rs)$. The isomorphism between G and G' labels these groups as *isomorphic*. Two isomorphic groups are indistinguishable when treated as abstract groups. They may, however, differ as groups of transformations. If a mapping of G on G' preserves multiplication but is not a one-to-one correspondence, it is called a *homomorphism*.

An isomorphic mapping of a group onto itself is called an *automorphism*. An important class of automorphisms, called *inner automorphisms*, is obtained by associating with every

element $s \in G$ the mapping $r \mapsto \varphi_s(r) = srs^{-1}$ of G onto itself. Inner automorphisms form a group. So do all automorphisms.

The group of inner automorphisms gives rise to an equivalence relation called *conjugation*, where two elements r and s are conjugate if there exists an inner automorphism φ_t transforming r into s: $s = \varphi_t(r) = trt^{-1}$. The corresponding equivalence classes are called *conjugacy classes* and play an important part in investigating the structure of a group. The identity forms a class by itself. If the group is Abelian then conjugation reduces to equality and all classes contain one element each.

A homomorphism of an abstract group G onto a group of transformations is called a *representation*. An isomorphic representation is said to be *faithful*. An example of a representation is given by the correspondence between group elements and inner automorphisms: $s \mapsto \varphi_s$. Similar examples are provided by the correspondences $s \mapsto f_s$ and $s \mapsto g_s$ with f_s and g_s denoting mappings of the group G on itself defined by $r \mapsto f_s(r) = sr$ and $r \mapsto g_s(r) = rs$. In these last two examples the group G appears in two roles as a set of transformations and as a set in which the transformations operate. E.g. in the formula $r \mapsto f_s(r) = sr$, s is the transformation and r is the transformed variable. The group G appearing in the second role is called a *group space*. The group multiplication in a group space is of secondary importance; it is not preserved by the transformations f_s and g_s and is used only to give meaning to the formulae $f_s(r) = sr$ and $g_s(r) = rs$, where it appears as multiplication between elements of group space and the group itself,

The representation by inner automorphisms is not faithful except in special cases. The two other representations which are called *regular representations* are always faithful and the group space is homogeneous under both regular representations.

A subset H of a group G is called a *subgroup* of G if it forms a group under the same law of multiplication which operates in G. It means that r^{-1} belongs to H if r does and for any two elements r and s of H, their product rs belongs to H. Denoting by AB the set of elements of the form rs, $r \in A$, $s \in B$, and by A^{-1} the set of elements r^{-1}, $r \in A$ with A and B subsets of G, we can express conditions for H being a subgroup in the form $H^{-1} = H$ and $H^2 = HH = H$.

Every subgroup H of G defines an equivalence relation in G under which two elements r and s are equivalent if $rs \in H$. The equivalence classes of this relation are called *left cosets* of the subgroup H relative to G. Similarly the relation which holds between r and s when $rs \in H$ defines *right cosets* of H relative to G. Sets of the form rH are left cosets of H and every left coset is of this form. Here rH denotes the set of all elements of the form rs with $s \in H$. Further $r \in rH$ and $s \in rH$ if and only if $r^{-1}s \in H$. Similarly right cosets are of the form Hr. Of special importance are subgroups invariant under inner automorphisms said to be *normal* or *invariant* subgroups. A subgroup N of G is an invariant subgroup if $rNr^{-1} = N$ for all $r \in G$ or, equivalently, $rN = Nr$. The last relation shows that left cosets of N are at the same time right cosets. Every group has two trivial normal subgroups: the whole group and the subgroup containing the identity only. If G has no other normal subgroups it is said to be *simple*.

The cosets of an invariant subgroup N have another important property. If Nr and Ns are cosets, then $NrNs = Nrs$ is another coset. It is easy to see that the set of all

cosets of an invariant subgroup N forms a group under multiplication of cosets with $N1 = N$ as the identity and Nr^{-1} as the element inverse to Nr. This group is called the *factor group* of G by the normal subgroup N and is denoted by G/N.

The correspondence $r \mapsto f(r) = Nr$ is a homomorphism of G onto G/N. If φ is a homomorphism of G onto a group G^*, then the subset of G that goes to the identity of G^* is a normal subgroup N of G and is called the *kernel* of φ. To each coset of N there corresponds an element of G^* to which the coset goes under φ. This correspondence is an isomorphism of G^* with G/N. It follows that φ can be composed of the homomorphism $r \mapsto f(r) = Nr$ of G onto G/N, where N is the kernel of φ, followed by the isomorphism of G/N onto G defined in the above sentence.

2. Linear spaces and operators. A set S of *vectors* $u, v, x, \ldots$ is called a *linear space* over the field $\mathbb{C}$ of complex numbers $\alpha, \beta, \gamma, \ldots$ if in S there are defined operations of addition of vectors and multiplication of vectors by complex numbers satisfying the following conditions:

a) S is an Abelian group under addition of vectors,
b) $\lambda(\mu u) = (\lambda\mu)u$,
c) $\lambda(u + v) = \lambda u + \lambda v$,
d) $(\lambda + \mu)u = \lambda u + \mu u$,
e) $1u = u$, where 1 is the complex number 1.

Vectors $u_1, u_2, \ldots, u_k$ are said to be *linearly independent* if $\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_k u_k = 0$ implies $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$. They are *dependent* if there exists a linear relation $\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_k u_k = 0$ with not all coefficients vanishing.

If the number of linearly independent vectors of a space S is bounded (does not exceed a certain number N) then S is said to be *finite dimensional*. Otherwise S is *infinite dimensional*.

A system of linearly independent vectors $u_1, u_2, \ldots, u_n$ of a finite dimensional space S is called a *basis* of S if it is maximal in the sense that adjoining to the system an additional vector $x \in S$ produces a dependent system $x, u_1, u_2, \ldots, u_n$. Take the relation

$$x\xi + u_1\xi_1 + u_2\xi_2 + \ldots + u_n\xi_n = 0$$

with not all coefficients vanishing. The coefficient ξ must certainly be different from 0, otherwise all coefficients would vanish. We can put $\xi = -1$ without any loss of generality. The result is

$$x = u_1\xi_1 + u_2\xi_2 + \ldots + u_n\xi_n$$

and we see that given a basis we can express all vectors as linear combinations of basis vectors. Each vector is completely characterized by coefficients of these combinations which are called *components*.

Given a second basis $u_{1'}, u_{2'}, \ldots, u_{n'}$ we can express the vectors $u_{i'}$ as combinations of u_i and vice versa:

$$u_{i'} = u_1\varepsilon_{1i'} + u_2\varepsilon_{2i'} + \ldots + u_n\varepsilon_{ni'} = u_j\varepsilon_{ji'},$$

$$u_i = u_{1'}\varepsilon_{1'i} + u_{2'}\varepsilon_{2'i} + \ldots + u_{n'}\varepsilon_{n'i} = u_{j'}\varepsilon_{j'i}.$$

By substituting one combination into the other we get $u_{i'} = u_{j'}\varepsilon_{j'k}\varepsilon_{ki'}$ and $u_i = u_j\varepsilon_{jk'}\varepsilon_{k'i}$ which implies

$$\varepsilon_{j'k}\varepsilon_{ki'} = \begin{cases} 1, & \text{for } j' = i' \\ 0, & \text{for } j' \neq i' \end{cases}$$

and

$$\varepsilon_{jk'}\varepsilon_{k'i} = \begin{cases} 1, & \text{for } j = i \\ 0, & \text{for } j \neq i. \end{cases}$$

These relations are possible only if $n = n'$ which means that the number of vectors of a maximal independent set is characteristic for the space being the same for all such sets. This number is called the *dimension* of S. Components of a vector x in different bases are related by the transformation $\xi_{i'} = \varepsilon_{i'j}\xi_j$.

A subset S_1 of S is called a *subspace* of S if it is a linear space under linear operations defined in S, that is, if for any u and v in S_1 their linear combination $\lambda u + \mu v$ belongs to S_1. Starting with any set of k independent vectors $v_1, \ldots, v_k$ in S and taking their linear combinations $y = v_\mu \eta_\mu$, $\mu = 1, 2, \ldots, k$ we obtain a k-dimensional subspace of S with $v_1, \ldots, v_k$ as basis vectors.

Two subspaces S_1 and S_2 of S are said to be *independent* if any non-zero vectors $u \in S_1$ and $v \in S_2$ are independent. It follows that S_1 and S_2 have 0 as the only common element. The space S is said to *decompose* into a *direct sum* of its linear subspaces S_1 and S_2 if an arbitrary vector x of S can be expressed uniquely as a sum of vectors x_1 and x_2 from S_1 and S_2 respectively. This definition is equivalent to

$$S = S_1 + S_2, \quad S_1 \cap S_2 = \{0\},$$

where $\{0\}$ denotes the set containing 0 only. By putting together basis vectors of S_1 and S_2 we obtain a basis of S which is called adapted to the decomposition. It follows that the sum of the dimensions of S_1 and S_2 is equal to the dimension of S.

A mapping $x \mapsto ax$ of S onto itself is called *linear* if

$$\lambda x + \mu y \mapsto a(\lambda x + \mu y) = \lambda ax + \mu ay.$$

In particular

$$ax = au_i\xi_i = u_j\alpha_{ji}\xi_i,$$

where α_{ji} are determined from $au_i = u_j\alpha_{ji}$. We see that given a basis we can represent a linear mapping, called also a *linear operator*, by a matrix. On a change of basis the matrix representing a transforms according to $\alpha_{i'j'} = \varepsilon_{i'i}\alpha_{ij}\varepsilon_{jj'}$. Linear operators can be multiplied, with the product ab defined by $(ab)x = a(bx) = abx$. The matrix representing ab in an arbitrary basis is equal to the product $\alpha_{ik}\beta_{kj}$ of matrices α_{ik} and β_{kj} representing a and b respectively. Linear operators can also be added and multiplied by complex numbers, definitions of these operations being given by $(a + b)x = ax + bx$ and $(\lambda a)x = \lambda(ax) = \lambda ax$. In matrix representation this corresponds to adding and multiplying matrices by complex numbers.

A linear space S is said to be *unitary* if to each pair of vectors x and y in S there corresponds a complex number (x, y), called the *scalar product*, which satisfies:

a) $(x, y) = (y, x)^*$ ($*$ – complex conjugation),
b) $(x, x) > 0$ if $x \neq 0$,

c) $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$.

It follows immediately that $(\lambda x + \mu y, z) = \lambda^*(x, z) + \mu^*(y, z)$.

The length of a vector x is given by $\|x\| = \sqrt{(x, x)}$. Two vectors x and y are said to be *orthogonal* if their scalar product vanishes: $(x, y) = 0$. A system of vectors $u_1, u_2, \ldots, u_n$ forms an *orthonormal basis* if all vectors u_i are of unit length and are orthogonal to each other: $(u_i, u_k) = \varepsilon_{ik}$. The scalar product of two vectors x and y can be expressed in terms of their components in an orthonormal basis in a particularly simple form: $(x, y) = (u_i, u_k)\xi^*{}_i\eta_k = \xi^*{}_i\eta_i$. The components of a vector x in an orthonormal basis can be calculated from the formula

$$(u_i, x) = (u_i, u_j)\xi_j = \varepsilon_{ij}\xi_j = \xi_i.$$

An operator s is said to be *unitary* if it preserves scalar products: $(sx, sy) = (x, y)$. The operator $a^\dagger$ defined by $(ax, y) = (x, a^\dagger y)$ for arbitrary x and y is called the *Hermitian conjugate* of a. For a unitary operator s we have $(sx, sy) = (x, s^\dagger sy) = (x, y)$ which implies that $s^\dagger s = e$ (the identity operator) or $s^\dagger = s^{-1}$ (inverse operator). An operator a is said to be *Hermitian* if $a^\dagger = a$. The matrix $\alpha^\dagger{}_{ik}$ representing $a^\dagger$ satisfies $\alpha^\dagger{}_{ik} = (\alpha_{ki})^*$, where α_{ik} represents a. Hermitian operators are represented by Hermitian matrices $(\alpha_{ki})^* = \alpha_{ik}$ and unitary operators are represented by unitary matrices $(\sigma_{ki})^*\sigma_{jk} = \varepsilon_{ij}$.

Two subspaces S_1 and S_2 of a unitary space S are called *orthogonal* if any two vectors $x \in S_1$ and $y \in S_2$ are orthogonal. Orthogonal subspaces are independent. Two orthogonal subspaces of S of joint dimension n determine an *orthogonal decomposition* of S. By putting together orthonormal bases of S_1 and S_2 an *adapted orthonormal basis* of S is obtained. All vectors orthogonal to a subspace S_1 of S form a linear subspace $S_2 = S_1^*$ of S orthogonal to S_1. S^* is called the *orthogonal complement* of S_1. S decomposes into an orthogonal sum of S_1 and S_2.

A subspace $S_1 \subset S$ is said to be *invariant* under a set M of linear operators if $ax \in S_1$ for arbitrary $x \in S_1$ and $a \in M$. An invariant subspace S_1 is called *irreducible* if it does not contain invariant subspaces different from S_1 and $\{0\}$. The set M is called *irreducible* if S is irreducible. The operators of M induce linear transformations of an invariant subspace S_1 onto itself. If S_1 is irreducible, then the induced transformations form an irreducible set.

A homomorphism of an abstract group G onto a group of linear operators is called a *linear representation*. A representation is called *unitary* if it is a homomorphism onto a group of unitary operators. A linear representation is called *irreducible* if it is a homomorphism onto an irreducible set of operators.

3. Abstract algebras. The set A of all linear operators in a linear space S is closed under multiplication and linear operations and satisfies the following conditions:

a) A is a linear space over the field $\mathbb{C}$ of complex numbers,
b) A contains a unit element e,
c) multiplication is associative,
d) $a(\lambda b + \mu c) = \lambda ab + \mu ac$.

We take conditions a) to d) to define an abstract *associative algebra* A, with *identity* e, over the field $\mathbb{C}$ of complex numbers. The dimension of A as a linear space is called the *order* of A.

A subset M of A is called an *ideal* if it is a linear subspace of A and if for $a \in M$, $b \in A$, and $c \in A$, bac belongs to M. Every algebra A has two *improper ideals*: the whole of A and the set $\{0\}$ containing 0 only. Other ideals are said to be *proper*. An ideal M is said to be *minimal* if it does not contain smaller proper ideals of A. A is said to be *simple* if it does not contain proper ideals. A is said to be *semi-simple* if it decomposes into a direct sum of its minimal ideals:

$$A = M_1 + M_2 + \cdots + M_n.$$

A subset I of A is called a *left ideal* if it is a linear space of A and if for $a \in I$ and $b \in A$, $ba \in I$. A *right ideal* J is defined in a similar way. A left (right) ideal is said to be *minimal* if it does not contain smaller left (right) proper ideals of A.

If M_1 and M_2 are ideals of A then $M_1 + M_2$ and $M_1 \cap M_2$ are ideals. If M_1 and M_2 are independent as linear subspaces of A: $M_1 \cap M_2 = \{0\}$, then $M_1 M_2 = M_2 M_1 = \{0\}$. Similarly if I_1 and I_2 are left ideals of A then $I_1 + I_2$ and $I_1 \cap I_2$ are left ideals and the same is true of right ideals.

A (one-to-one) mapping of an abstract algebra A onto an algebra A' of linear operators in a linear space S is called a (faithful) linear representation if it preserves multiplication and linear operations. Multiplication of elements of A by a fixed element $a \in A$ from the left induces a linear mapping of A, treated as a linear space, into itself. Associating with every element $a \in A$ the linear mapping defined above gives a linear representation of A called the *left regular representation*. The *right regular representation* is defined in a similar manner. Regular representations of an algebra with identity are always faithful. It is easy to see that left (right) ideals form invariant subspaces under the left (right) regular representation. Minimal ideals are irreducible. An element of A which commutes with all elements of A is said to be *central*. The set of all central elements of A is called the *centre* of A.

A subset A_1 of A is called a subalgebra if it is an algebra under operations defined in A. This means that it is a linear subspace and that for arbitrary $a \in A_1$ and $b \in A_1$, $ab \in A_1$. An algebra A is called a *division algebra* if every element of A distinct from 0 has an inverse.

Let a be an arbitrary non-zero element of a division algebra A of order h. Elements $e, a, a^2, \ldots, a^h$ must be dependent, which means that they satisfy a linear relation

$$W(a) = a^m + \lambda_1 a^{m-l} + \lambda_2 a^{m-2} + \cdots + \lambda_m e = 0$$

with $m \leqslant h$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the roots of the polynomial $W(\alpha)$. Then

$$W(a) = (a - \alpha_1 e)(a - \alpha_2 e) \cdots (a - \alpha_m e) = 0.$$

Since A is a division algebra, the last equation holds only if for some i, $a = \alpha_i e$. It follows that all elements of A are multiples of the identity e, which means that A is one-dimensional and isomorphic to the field $\mathbb{C}$.

II. Group algebra of a finite group

1. Definition and basic properties. A finite group G of order h can be used to generate a *group algebra* A_G of the same order by a formal introduction of linear operations, that is, addition and multiplication by complex numbers, under which group elements are assumed to be independent. Multiplication is extended from group elements to their linear combinations, that is, elements of A_G, by a natural relation

$$r(\lambda s + \mu t) = \lambda rs + \mu rt.$$

Group elements form by definition a natural linear basis for A_G. Elements of A_G are linear combinations of group elements

$$a = \sum_{r\in G} \alpha(r) r,$$

and are represented by coefficients $\alpha(r)$ of these combinations, which can also be interpreted as complex-valued functions on G. Components in the natural basis of a combination $\lambda a+\mu b$ of elements $a = \sum_{r\in G} \alpha(r)r$ and $b = \sum_{r\in G} \beta(r)r$ are obviously combinations of components of a and b:

$$\lambda a + \mu b = \sum_{r\in G} (\lambda\alpha(r) + \mu\beta(r)).$$

Components of a product are obtained from

$$ab = \sum_{r,s} \alpha(r)\beta(s)rs = \sum_{s,t} \alpha(ts^{-1})\beta(s)t = \sum_{r,t} \alpha(r)\beta(r^{-1}t)t.$$

In addition to being an algebra, A_G is a unitary space with the scalar product

$$\mathrm{tr}(a^\dagger b),$$

where the *trace* $\mathrm{tr}(a)$ and the *Hermitian conjugate* $a^\dagger$ are defined below.

First we define the *complex conjugate* a^* of a by $s^* = s$ and $(\lambda a+\mu b)^* = \lambda^* a^* + \mu^* b^*$. This gives for $a = \sum_{r\in G} \alpha(r)r$:

$$a^* = \sum_{r\in G} \alpha^*(r)r = \sum_{r\in G} \alpha(r)^* r,$$

or

$$\alpha^*(r) = \alpha(r)^*.$$

The components in the natural basis of the complex conjugate of a are obtained by taking complex conjugates of components of a. For a product ab we have

$$(ab)^* = \sum_{r,s} (\alpha(r)\beta(s))^* rs = \sum_{r,s} \alpha(r)^*\beta(s)^* rs = a^* b^*.$$

Next we introduce the transpose a^T of a by $s^\mathsf{T} = s^{-1}$ and $(\lambda a + \mu b)^\mathsf{T} = \lambda a^\mathsf{T} + \mu b^\mathsf{T}$. This gives for $a = \sum_{r\in G} \alpha(r)r$:

$$a^\mathsf{T} = \sum_{r\in G} \alpha^\mathsf{T}(r)r = \sum_{r\in G} \alpha(r)r^{-1} = \sum_{r\in G} \alpha(r^{-1})r,$$

or

$$\alpha^\mathsf{T}(r) = \alpha(r^{-1}).$$

For a product ab we have

$$(ab)^{\mathsf{T}} = \sum_{r,s} \alpha(r)\beta(s)(rs)^{-1} = \sum_{r,s} \alpha(r)\beta(s)s^{-1}r^{-1} = b^{\mathsf{T}}a^{\mathsf{T}}.$$

Hermitian conjugation is now defined as the composition of complex conjugation and transposition: $a^\dagger = (a^{\mathsf{T}})^* = (a^*)^{\mathsf{T}}$ or $s^\dagger = s^{-1}$ and $(\lambda a + \mu b)^\dagger = \lambda^* a^\dagger + \mu^* b^\dagger$, which gives for $a = \sum_{r\in G} \alpha(r)r$:

$$a^\dagger = \sum_{r\in G} \alpha^\dagger(r)r = \sum_{r\in G} \alpha(r)^* r^{-1} = \sum_{r\in G} \alpha(r^{-1})^* r,$$

or

$$\alpha^\dagger(r) = \alpha(r^{-1})^*.$$

For a product ab we have

$$(ab)^\dagger = \sum_{r,s} \alpha(r)^*\beta(s)^*(rs)^{-1} = \sum_{r,s} \alpha(r)^*\beta(s)^* s^{-1}r^{-1} = b^\dagger a^\dagger.$$

All three of the above defined operations are *involutions*:

$$a^{**} = a, \quad a^{\mathsf{TT}} = a, \quad a^{\dagger\dagger} = a.$$

The trace $\operatorname{tr}(a)$ of an element a is defined by

$$\operatorname{tr}(r) = \varepsilon(r) = \begin{cases} 1 & \text{for } r = 1, \\ 0 & \text{for } r \neq 1, \end{cases}$$

and $\operatorname{tr}(\lambda a + \mu b) = \lambda \operatorname{tr}(a) + \mu \operatorname{tr}(b)$. This gives

$$\operatorname{tr}(a) = \sum_{r\in G} \alpha(r) \operatorname{tr}(r) = \sum_{r\in G} \alpha(r)\varepsilon(r) = \alpha(1).$$

For the trace of a product we have

$$\begin{aligned} \operatorname{tr}(ab) &= \sum_{r,s} \alpha(r)\beta(s) \operatorname{tr}(rs) = \sum_{r,s} \alpha(r)\beta(s)\varepsilon(rs) = \sum_{r\in G} \alpha(r)\beta(r^{-1}) \\ &= \sum_{r,s} \alpha(r)\beta(s)\varepsilon(sr) = \operatorname{tr}(ba). \end{aligned}$$

We can now express the scalar product in terms of components in the natural basis

$$\operatorname{tr}(a^\dagger b) = \sum_{r\in G} \alpha(r)^*\beta(r).$$

It is worth noting that $a^\dagger$ is the Hermitian conjugate with respect to the scalar product:

$$\operatorname{tr}(b^\dagger a^\dagger c) = \operatorname{tr}((ab)^\dagger c).$$

The scalar product is preserved by the following transformations:

$$a \mapsto ra, \quad a \mapsto ar, \quad a \mapsto rar^{-1}, \quad a \mapsto a^{\mathsf{T}}, \quad a \mapsto e^{i\varphi}a,$$

with $r \in G$ and φ a real number. Under the transformations

$$a \mapsto a^* \quad \text{and} \quad a \mapsto a^\dagger$$

the scalar product goes into its complex conjugate. The transformations

$$a \mapsto rar^{-1} \quad \text{and} \quad a \mapsto a^*, \quad \lambda \mapsto \lambda^*$$

are automorphisms of the group algebra.

An element $a \in A_G$ is *central* if it commutes with all group elements $r \in G$:

$$ra = ar \quad \text{or} \quad rar^{-1} = a.$$

In the natural basis we have

$$\sum_{s\in G} \alpha(r^{-1}sr)s = \sum_{s\in G} \alpha(s)s$$

or

$$\alpha(r^{-1}sr) = \alpha(s),$$

which shows that components of a central element corresponding to group elements of the same class are equal.

In addition to the operations defined above we introduce a commutative product $a \times b$ which we call the *Kronecker product*. It is defined by

$$a \times b = \sum_{r\in G} \alpha(r)\beta(r)r.$$

2. Decomposition of A_G into a sum of its minimal ideals

THEOREM 1. *If M is an ideal of A_G, then the orthogonal complement $M^\perp$ is also an ideal.*

Proof. Let M be an ideal and suppose that $a \in M^\perp$, or $\operatorname{tr}(a^\dagger M) = 0$. Then

$$\operatorname{tr}((bac)^\dagger M) = \operatorname{tr}(c^\dagger a^\dagger b^\dagger M) = \operatorname{tr}(a^\dagger b^\dagger M c^\dagger) = \operatorname{tr}(a^\dagger M) = 0.$$

It follows that $bac \in M^\perp$ for arbitrary b and c, which means that $M^\perp$ is an ideal. ∎

If A_G is not simple, then it contains a proper ideal M. Theorem 1 shows that $M^\perp$ is also an ideal and A_G decomposes into an orthogonal sum of its ideals:

$$A_G = M + M^\perp.$$

If M is not minimal, then it contains a smaller ideal M'. $M'^\perp \cap M$ is another ideal and $M = M' + M' \cap M$. The process can be carried further until a decomposition of A_G into an orthogonal sum of minimal ideals is reached:

$$A_G = \overset{(1)}{M} + \overset{(2)}{M} + \cdots + \overset{(f)}{M} = \sum_{i=1}^{f} \overset{(i)}{M}. \tag{1}$$

Let

$$e = \sum_{i=1}^{f} \overset{(i)}{e}$$

be the unique decomposition of the identity e into elements $\overset{(i)}{e} \in \overset{(i)}{M}$. For an arbitrary element $a \in A_G$ we have

$$a = ae = ea = \sum_{i=1}^{f} a\overset{(i)}{e} = \sum_{i=1}^{f} \overset{(i)}{e}a.$$

The elements $a\overset{(i)}{e}$ and $\overset{(i)}{e}a$ are in $\overset{(i)}{M}$ since $\overset{(i)}{M}$ is an ideal. It follows that

$$a = \sum_{i=1}^{f} a\overset{(i)}{e} \quad \text{and} \quad a = \sum_{i=1}^{f} \overset{(i)}{e}a$$

are decompositions of a into elements in the minimal ideals $\overset{(i)}{M}$. This decomposition is unique hence $a\overset{(i)}{e} = \overset{(i)}{e}a$. If $a \in \overset{(i)}{M}$, then $a\overset{(i)}{e} = a$. In particular $\overset{(i)}{e}\overset{(i)}{e} = \overset{(i)}{e}$.

Every ideal of A_G can be decomposed into a sum of minimal ideals $\overset{(i)}{M}$. If M is an ideal, then

$$M = Me = \sum_{i=1}^{f} M\overset{(i)}{e}$$

but $M\overset{(i)}{e} = \{0\}$ or $M\overset{(i)}{e} = \overset{(i)}{M}$. In particular, if M is a minimal ideal it must coincide with one of $\overset{(i)}{M}$. The decomposition (1) of A_G into a sum of orthogonal minimal ideals is thus unique.

If $\overset{(i)}{M}$ is one of the minimal ideals, so is $\overset{(i)}{M}{}^{\dagger}$. If $a \neq 0$ belongs to $\overset{(i)}{M}$, then $\mathrm{tr}(a^{\dagger}a) \neq 0$, but this is possible only if $a^{\dagger} \in \overset{(i)}{M}$ which means that $\overset{(i)}{M}{}^{\dagger} = \overset{(i)}{M}$. This argument shows that $M^{\dagger} = M$ for any ideal M.

If M is an ideal, then both M^* and $M^{\perp}$ are ideals. Usually $M^* \neq M$ and $M^{\perp} \neq M$. However $M^{\perp} = M^*$ since $M^{\dagger} = M$.

Let

$$e = \sum_{i=1}^{f} \overset{(i)}{e}$$

be the unique decomposition of the identity e into elements $\overset{(i)}{e} \in \overset{(i)}{M}$. For an arbitrary element $a \in A_G$ we have

$$a = ae = ea = \sum_{i=1}^{f} a\overset{(i)}{e} = \sum_{i=1}^{f} \overset{(i)}{e}a.$$

The elements $a\overset{(i)}{e}$ and $\overset{(i)}{e}a$ are in $\overset{(i)}{M}$ since $\overset{(i)}{M}$ is an ideal. It follows that

$$a = \sum_{i=1}^{f} a\overset{(i)}{e} \quad \text{and} \quad a = \sum_{i=1}^{f} \overset{(i)}{e}a$$

are decompositions of a into elements in minimal ideals $\overset{(i)}{M}$. This decomposition is unique, hence $a\overset{(i)}{e} = \overset{(i)}{e}a$. If $a \in \overset{(i)}{M}$, then $a\overset{(i)}{e} = a$. In particular $\overset{(i)}{e}\overset{(i)}{e} = \overset{(i)}{e}$. Taking Hermitian conjugates on both sides of

$$e = \sum_{i=1}^{f} \overset{(i)}{e}$$

we have

$$e = e^{\dagger} = \sum_{i=1}^{f} \overset{(i)}{e}{}^{\dagger}.$$

Since $\overset{(i)}{M}{}^{\dagger} = \overset{(i)}{M}$ we have $\overset{(i)}{e}{}^{\dagger} \in \overset{(i)}{M}$. It follows from the uniqueness of the decomposition that $\overset{(i)}{e}{}^{\dagger} = \overset{(i)}{e}$.

The properties of the elements $\overset{(i)}{e}$ called the *primitive generating units* of the minimal ideals $\overset{(i)}{M}$ are summarized in

THEOREM 2.

a) *The elements* $\overset{(i)}{e}$ *give projections on the minimal ideals* $\overset{(i)}{M}$ *:* $a\overset{(i)}{e} \in \overset{(i)}{M}$ *for arbitrary* a *and* $a\overset{(i)}{e} = a$ *for* $a \in \overset{(i)}{M}$.

b) $\overset{(i)}{e}$ *are idempotent and independent:* $\overset{(i)}{e}\overset{(j)}{e} = \overset{(i)}{e}\delta^{ij}$.

c) $\overset{(i)}{e}$ *are central:* $\overset{(i)}{e}a = a\overset{(i)}{e}$ *for arbitrary* a.

d) $\overset{(i)}{e}$ *are Hermitian:* $\overset{(i)}{e}{}^{\dagger} = \overset{(i)}{e}$.

THEOREM 3. *Generating units are uniquely characterized by property a) of Theorem* 2.

Proof. Let $\overset{(i)}{e}{}'$ be a generating unit of $\overset{(i)}{M}$: $a\overset{(i)}{e}{}' \in \overset{(i)}{M}$ for arbitrary a and $a\overset{(i)}{e}{}' = a$ for $a \in \overset{(i)}{M}$. For $a = e$ we obtain $e\overset{(i)}{e}{}' = \overset{(i)}{e}{}' \in \overset{(i)}{M}$, hence $\overset{(i)}{e}\overset{(i)}{e}{}' = \overset{(i)}{e}{}'$. But for $a = \overset{(i)}{e} \in \overset{(i)}{M}$ we have $\overset{(i)}{e}\overset{(i)}{e}{}' = \overset{(i)}{e}$. Hence $\overset{(i)}{e}{}' = \overset{(i)}{e}$. ∎

An arbitrary ideal M is a sum of minimal ideals. Taking the sum of the corresponding generating units we obtain the generating unit of M.

3. The structure of minimal ideals

THEOREM 4. *Let* I *be a left ideal of* A_G. *Then* $I^{\perp}$ *is also a left ideal.*

Proof. The proof is similar to that of Theorem 1. ∎

It follows immediately that A_G decomposes into a sum of its minimal left ideals. Let M be an ideal and I a minimal left ideal. $I \cap M$ is again a left ideal and $I \cap M \subset I$, hence $I \cap M = I$ or $I \cap M = \{0\}$. This argument shows that the decomposition of A_G into a sum of its minimal left ideals can be achieved by first decomposing A_G into minimal two-sided ideals $\overset{(i)}{M}$ and then decomposing each $\overset{(i)}{M}$ into a sum

$$\overset{(i)}{M} = \overset{(i)}{I}{}_1 + \overset{(i)}{I}{}_2 + \cdots + \overset{(i)}{I}{}_{g_i} \tag{2}$$

of minimal left ideals contained in $\overset{(i)}{M}$. Let $\overset{(i)}{e}$ be the generating unit of $\overset{(i)}{M}$ and let

$$\overset{(i)}{e} = \overset{(i)}{e}{}^1{}_1 + \overset{(i)}{e}{}^2{}_2 + \cdots + \overset{(i)}{e}{}^{g_i}{}_{g_i}$$

be the unique decomposition of $\overset{(i)}{e}$ corresponding to (2). For arbitrary a, $a\overset{(i)}{e}{}^A{}_A$ belongs to $\overset{(i)}{I}{}_A$ and if $a \in \overset{(i)}{I}{}_A$, then $a\overset{(i)}{e}{}^A{}_A = a$. It follows that $\overset{(i)}{e}{}^A{}_A\overset{(i)}{e}{}^A{}_A = \overset{(i)}{e}{}^A{}_A$, $\overset{(i)}{e}{}^A{}_A\overset{(i)}{e}{}^B{}_B = 0$ if $B \neq A$, and

$$a = \sum_A a\overset{(i)}{e}{}^A{}_A = a\overset{(i)}{e}{}^1{}_1 + a\overset{(i)}{e}{}^2{}_2 + \cdots + a\overset{(i)}{e}{}^{g_i}{}_{g_i}$$

is the decomposition of a corresponding to (2).

Sets of the form $\overset{(i)}{J}{}^{A} = \overset{(i)}{e}{}^{A}{}_{A} A_G$ are right ideals. They are minimal; otherwise one could decompose $\overset{(i)}{e}{}^{A}{}_{A}$ further, which would result in a decomposition of $\overset{(i)}{I}{}_{A}$. To every decomposition (2) there corresponds a decomposition of $\overset{(i)}{M}$ into a sum of minimal right ideals

$$\overset{(i)}{M} = \sum_{A} \overset{(i)}{e}{}^{A}{}_{A} \overset{(i)}{M} = \overset{(i)}{J}{}^{1} + \overset{(i)}{J}{}^{2} + \cdots + \overset{(i)}{J}{}^{g_i}$$

and also a double decomposition

$$\overset{(i)}{M} = \sum_{A,B} \overset{(i)}{e}{}^{A}{}_{A} \overset{(i)}{M} \overset{(i)}{e}{}^{B}{}_{B} = \sum_{A,B} \overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{B}.$$

Let us investigate the properties of sets of the form $\overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{B}$. These sets are linear subspaces of $\overset{(i)}{M}$. Next we prove that they contain non-zero elements. In fact suppose that $\overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{B}$ contains only 0; then $\overset{(i)}{I}{}_{B} \subset (\overset{(i)}{J}{}^{A})^{\perp}$ and $\overset{(i)}{I}{}_{B} \overset{(i)}{M} \subset (\overset{(i)}{J}{}^{A})^{\perp} \neq \overset{(i)}{M}$. But $\overset{(i)}{I}{}_{B} \overset{(i)}{M}$ must be equal to $\overset{(i)}{M}$ because $\overset{(i)}{I}{}_{B} \overset{(i)}{M}$ is an ideal contained in $\overset{(i)}{M}$ and containing a non-zero element $\overset{(i)}{e}{}^{B}{}_{B}$. If $a \in \overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{B}$ and $b \in \overset{(i)}{J}{}^{C} \cap \overset{(i)}{I}{}_{D}$, then ab obviously belongs to $\overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{D}$. This means in particular that $\overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{A}$ is an algebra. We show in addition that it is a division algebra and therefore a one-dimensional space. Let $a \in \overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{A}$, $a \neq 0$. $\overset{(i)}{M} a$ is a left ideal which contains $\overset{(i)}{e} a = a \neq 0$ and is contained in $\overset{(i)}{I}{}_{A}$. Therefore $\overset{(i)}{M} a = \overset{(i)}{I}{}_{A}$. There exists an element $a' \in \overset{(i)}{M}$ such that $a'a = \overset{(i)}{e}{}^{A}{}_{A}$; but then $\overset{(i)}{e}{}^{A}{}_{A} a' \overset{(i)}{e}{}^{A}{}_{A} a = \overset{(i)}{e}{}^{A}{}_{A}$ which indicates that $\overset{(i)}{e}{}^{A}{}_{A} a' \overset{(i)}{e}{}^{A}{}_{A}$ is the inverse of a. This shows that $\overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{A}$ is a division algebra and $a = \alpha \overset{(i)}{e}{}^{A}{}_{A}$. Let $a \in \overset{(i)}{J}{}^{A}$, $a \neq 0$. The set of elements $x \in \overset{(i)}{I}{}_{A}$ satisfying $xa = 0$ is a left ideal different from $\overset{(i)}{I}{}_{A}$ because $\overset{(i)}{e} a = a \neq 0$. Hence $x = 0$.

We now prove that every minimal ideal $\overset{(i)}{M}$ is isomorphic to the algebra of matrices of order g_i, where g_i is the number of minimal left (right) ideals contained in $\overset{(i)}{M}$. Select arbitrary non-zero elements $\overset{(i)}{e}{}^{1}{}_{A} \in \overset{(i)}{J}{}^{1} \cap \overset{(i)}{I}{}_{A}$, $A \neq 1$. Then select $\overset{(i)}{e}{}^{B}{}_{1} \in \overset{(i)}{J}{}^{B} \cap \overset{(i)}{I}{}_{1}$, $B \neq 1$ in such a way that $\overset{(i)}{e}{}^{1}{}_{A} \overset{(i)}{e}{}^{A}{}_{1} = \overset{(i)}{e}{}^{1}{}_{1}$. This is clearly possible because for any non-zero element $a \in \overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{1}$ we have $\overset{(i)}{e}{}^{1}{}_{A} a = \alpha \overset{(i)}{e}{}^{1}{}_{1}$ with $\alpha \neq 0$. Define elements $\overset{(i)}{e}{}^{A}{}_{B} \in \overset{(i)}{J}{}^{A} \cap \overset{(i)}{I}{}_{B}$ by

$$\overset{(i)}{e}{}^{A}{}_{B} = \overset{(i)}{e}{}^{A}{}_{1} \overset{(i)}{e}{}^{1}{}_{B}$$

for $B \neq A$. The relation

$$\overset{(i)}{e}{}^{A}{}_{A} = \overset{(i)}{e}{}^{A}{}_{1} \overset{(i)}{e}{}^{1}{}_{A}$$

holds identically. The elements $\overset{(i)}{e}{}^{A}{}_{B}$ satisfy the relations

$$\overset{(i)}{e}{}^{A}{}_{B}\,\overset{(i)}{e}{}^{C}{}_{D} = \varepsilon^{C}{}_{B}\,\overset{(i)}{e}{}^{A}{}_{D}.$$

For an arbitrary element $a \in \overset{(i)}{M}$ introduce numbers $\overset{(i)}{\alpha}{}^{A}{}_{B}$ characterized by

$$\overset{(i)}{e}{}^{1}{}_{B}\,a\,\overset{(i)}{e}{}^{A}{}_{1} = \overset{(i)}{\alpha}{}^{A}{}_{B}\,\overset{(i)}{e}{}^{1}{}_{1}.$$

Then

$$a = \sum_{A,B} \overset{(i)}{e}{}^{B}{}_{B} a\,\overset{(i)}{e}{}^{A}{}_{A} = \sum_{A,B} \overset{(i)}{e}{}^{B}{}_{1}\,\overset{(i)}{e}{}^{1}{}_{B} a\,\overset{(i)}{e}{}^{A}{}_{1}\,\overset{(i)}{e}{}^{1}{}_{A} = \overset{(i)}{\alpha}{}^{A}{}_{B}\,\overset{(i)}{e}{}^{B}{}_{A},$$

where summation over A and B is understood. For $a = \overset{(i)}{\alpha}{}^{A}{}_{B}\,\overset{(i)}{e}{}^{B}{}_{A}$ and $b = \overset{(i)}{\beta}{}^{A}{}_{B}\,\overset{(i)}{e}{}^{B}{}_{A}$ we have

$$ab = \overset{(i)}{\beta}{}^{A}{}_{B}\,\overset{(i)}{\alpha}{}^{C}{}_{D}\,\overset{(i)}{e}{}^{D}{}_{C}\,\overset{(i)}{e}{}^{B}{}_{A} = \overset{(i)}{\beta}{}^{A}{}_{B}\,\overset{(i)}{\alpha}{}^{C}{}_{D}\varepsilon^{B}{}_{C}\,\overset{(i)}{e}{}^{D}{}_{A} = \overset{(i)}{\beta}{}^{A}{}_{C}\,\overset{(i)}{\alpha}{}^{C}{}_{B}\,\overset{(i)}{e}{}^{B}{}_{A}$$

which proves our proposition. The elements $\overset{(i)}{e}{}^{A}{}_{B}$ form a linear basis for $\overset{(i)}{M}$ which we call the *canonical basis*.

4. Properties of the canonical basis. Take a minimal ideal M, its generating unit e, and a canonical basis $e^{A}{}_{B}$, $A, B = 1, 2, \ldots, g$. We prove that the canonical basis is defined up to a transformation

$$e^{A'}{}_{B'} = \varepsilon^{A'}{}_{C}\, e^{C}{}_{D}\, \varepsilon^{D}{}_{B'}.$$

Two bases are always connected by a linear transformation

$$e^{A'}{}_{B'} = e^{C}{}_{D}\, \varepsilon^{D}{}_{B'}\, e^{A'}{}_{C}.$$

The relations

$$e^{A}{}_{B}\, e^{C}{}_{D} = \varepsilon^{C}{}_{B}\, e^{A}{}_{D}$$

and

$$e^{A'}{}_{B'}\, e^{C'}{}_{D'} = \varepsilon^{C'}{}_{B'}\, e^{A'}{}_{D'}$$

give

$$\varepsilon^{A'}{}_{B'}\, \varepsilon^{D}{}_{E'}\, e^{F'}{}_{G} = \varepsilon^{D}{}_{E'}\, e^{A'}{}_{H}\, \varepsilon^{H}{}_{B'}\, e^{F'}{}_{G}.$$

Taking determinants of both sides with respect to indices A', B' and introducing

$$\varepsilon^{A}{}_{B'} = \det(\varepsilon|^{A}{}_{B'}|^{C'}{}_{D}),$$
$$\varepsilon^{A'}{}_{B} = \det(\varepsilon^{C}{}_{D'})^{A'}{}_{B},$$

we arrive at

$$\varepsilon^{D}{}_{B'}\, e^{A'}{}_{C} = \varepsilon^{A'}{}_{C}\, \varepsilon^{D}{}_{B'}.$$

The relation

$$e = \sum_{A'} e^{A'}{}_{A'} = \sum_{A'} \varepsilon^{A'}{}_{C}\, e^{C}{}_{D}\, \varepsilon^{D}{}_{A'} = \varepsilon^{D}{}_{C}\, e^{C}{}_{D}$$

gives

$$\varepsilon^{A}{}_{C'}\, \varepsilon^{C'}{}_{B} = \varepsilon^{A}{}_{B}.$$

The basis elements ${e^A}_B$ have a number of important properties.

a) $$\operatorname{tr}({e^A}_B) = \frac{\operatorname{tr}(e)}{g}{\varepsilon^A}_B.$$

Proof.

$$\operatorname{tr}({e^A}_B\,{e^C}_D) = \operatorname{tr}({e^C}_D\,{e^A}_B)$$

or

$$\varepsilon^C{}_B\operatorname{tr}({e^A}_D) = \varepsilon^A{}_D\operatorname{tr}({e^C}_B).$$

Hence $\operatorname{tr}({e^A}_B) = k\,{\varepsilon^A}_B$. Summing over $A = B$ on both sides we finally arrive at

$$k = \operatorname{tr}(e)/g.$$

b) $$\alpha^A{}_B = \frac{g}{\operatorname{tr}(e)}\operatorname{tr}(a\,{e^A}_B).$$

This follows immediately from a) and $a\,{e^A}_B = \alpha^A{}_C\,{e^C}_B$.

Group elements r and s are represented by matrices $\rho^A{}_B$ and $\sigma^A{}_B$ such that

$$re = \rho^A{}_B\,{e^B}_A \quad \text{and} \quad se = \sigma^A{}_B\,{e^B}_A.$$

Further properties:

c) $$\rho^A{}_B = \frac{g}{\operatorname{tr}(e)}\operatorname{tr}(re^A{}_B) = \frac{g}{\operatorname{tr}(e)}{\varepsilon^A}_B(r^{-1}).$$

d) $$\operatorname{tr}(a) = \frac{\operatorname{tr}(e)}{g}\alpha^A{}_A.$$

e) $$\operatorname{tr}(e) = \frac{g^2}{h}.$$

Proof.

$$\operatorname{tr}(e) = \frac{1}{g}\operatorname{tr}\Big(\sum_{A,B} e^A{}_B\,e^B{}_A\Big) = \frac{1}{g}\sum_{s\in G}\varepsilon^A{}_B(s^{-1})\,\varepsilon^B{}_A(s)$$
$$= \frac{(\operatorname{tr}(e))^2}{g^3}\sum_{s\in G}\sigma^A{}_B\,\sigma^{-1\,B}{}_A = \frac{h(\operatorname{tr}(e))^2}{g^2}.$$

The solution $\operatorname{tr}(e) = 0$ is excluded since it would lead to $\operatorname{tr}(a) = 0$ and $\operatorname{tr}(a^\dagger a) = 0$ for any $a \in M$.

Starting with the base ${e^A}_B$ we introduce new bases

$$e^{\dot A}{}_{\dot B} = ({e^A}_B)^*, \quad e_A{}^B = ({e^A}_B)^\mathsf{T}, \quad e_{\dot A}{}^{\dot B} = ({e^A}_B)^\dagger,$$

which satisfy

$$e^{\dot A}{}_{\dot B}\,e^{\dot C}{}_{\dot D} = \varepsilon^{\dot C}{}_{\dot B}\,e^{\dot A}{}_{\dot D}, \quad e_A{}^B\,e_C{}^D = \varepsilon_C{}^B\,e_A{}^D, \quad e_{\dot A}{}^{\dot B}\,e_{\dot C}{}^{\dot D} = \varepsilon_{\dot C}{}^{\dot B}\,e_{\dot A}{}^{\dot D},$$

and therefore

$$e^{\dot A}{}_{\dot B} = \gamma^{\dot A}{}_C\,e^C{}_D\,\gamma^D{}_{\dot B}, \quad \gamma^{\dot A}{}_C\,\gamma^C{}_{\dot B} = \varepsilon^{\dot A}{}_{\dot B},$$
$$e_A{}^B = \delta_{AC}\,e^C{}_D\,\delta^{DB}, \quad \delta_{AC}\,\delta^{CB} = \varepsilon_A{}^B,$$
$$e_{\dot A}{}^{\dot B} = \varepsilon_{\dot A C}\,e^C{}_D\,\varepsilon^{D\dot B}, \quad \varepsilon_{\dot A C}\,\varepsilon^{C\dot B} = \varepsilon_{\dot A}{}^{\dot B}.$$

III. Group algebra of a compact topological group

1. Definition. The notion of a group algebra can be easily extended to compact topological groups. Without going into details of the definition of a compact group we state properties of such groups essential for the existence of a group algebra. On a compact topological group G one can introduce an integral such that every continuous function $\alpha(s)$, $s \in G$ is integrable:

$$\int_G \alpha(s)\mathrm{d}s < \infty.$$

The integral is invariant under left as well as right translations:

$$\int_G \alpha(rs)\mathrm{d}s = \int_G \alpha(sr)\mathrm{d}s = \int_G \alpha(s)\mathrm{d}s.$$

The volume of the group is 1:

$$\int_G 1\mathrm{d}s = 1.$$

This last formula replaces

$$\sum_G 1 = h$$

of the finite case. Other properties of the integral have obvious counterparts in the finite case.

We can now define the group algebra as a set A_G of elements $a, b, \ldots$ represented by continuous functions $\alpha(s), \beta(s), \ldots$ in such a way that

a) $\lambda a + \mu b$ is represented by $\lambda\alpha(s) + \mu\beta(s)$,
b) ab is represented by $\int_G \alpha(r)\beta(r^{-1}s)\mathrm{d}r = \int_G \alpha(sr^{-1})\beta(r)\mathrm{d}r$,
c) a^* is represented by $\alpha^*(s) = (\alpha(s))^*$,
d) a^T is represented by $\alpha^\mathsf{T}(s) = \alpha(s^{-1})$,
e) $a^\dagger = a^{\mathsf{T}*}$ is represented by $\alpha^\dagger(s) = \alpha(s^{-1})^*$,
f) $\mathrm{tr}(a) = \alpha(1)$,
g) $\mathrm{tr}(a^\dagger b) = \int_G \alpha^*(s)\beta(s)\mathrm{d}s$,
h) $a \times b$ is represented by $\alpha(s)\beta(s)$.

An element $a \in A_G$ is central if $\alpha(r^{-1}sr) = \alpha(s)$.

2. Decomposition of the group algebra into a sum of its minimal ideals and the structure of minimal ideals. All minimal ideals of the group algebra of a compact group are finite dimensional. They form an infinite discrete set

$$\{\overset{(0)}{M}, \overset{(1)}{M}, \ldots, \overset{(i)}{M}, \ldots\}$$

and the algebra A_G is the sum

$$A_G = \overset{(0)}{M} + \overset{(1)}{M} + \cdots + \overset{(i)}{M} + \cdots = \sum_{i=0}^{\infty} \overset{(i)}{M}.$$

The last relation is to be understood in the sense that every element $a \in A_G$ defines an infinite series of elements $\overset{(i)}{a} \in \overset{(i)}{M}$ such that

$$\lim_{n\to\infty} \Big\| a - \sum_{i=0}^{n} \overset{(i)}{a} \Big\| = 0, \quad \text{where} \quad \|a\| = \sqrt{\mathrm{tr}(a^\dagger a)}.$$

Minimal ideals have exactly the same properties and structure as in the finite case. In all formulae h has to be put equal to 1.

3. An example. As an example we consider the group of unitary unimodular 2×2 matrices ${u^A}_B$. In a parameterization considered canonical we have

$${u^A}_B = {\varepsilon^A}_B \cos\frac{u}{2} + i{\sigma^A}_{Br}\frac{u^r}{u}\sin\frac{u}{2}.$$

The parameters u^r are components of a vector in a Euclidean space of three dimensions, u is the norm of this vector, and

$$({\sigma^A}_{Br}) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

are the Pauli matrices. The parameters are restricted by $u < 4\pi$. For $u \to 4\pi$, ${u^A}_B \to {\varepsilon^A}_B$ and for $u = 2\pi$, ${u^A}_B = -{\varepsilon^A}_B$.

Continuous functions on the group are continuous functions $\alpha(u^r)$ of the parameters u^r such that for $u \to 4\pi$, $\alpha(u^r) \to \alpha(0)$. An invariant integral is defined by

$$\frac{1}{32\pi^2}\int_{u\leqslant 4\pi} \alpha(u^r)\frac{2(1-\cos u)}{u^2}\mathrm{d}^3u.$$

The Pauli σ matrices have the following properties:

$$\begin{aligned}
&{\sigma^A}_{Cr}\,{\sigma^C}_{Bs} = {\varepsilon^A}_B\,\varepsilon_{rs} + i{\sigma^A}_{Bt}\,{\varepsilon^t}_{rs}, && {\sigma^A}_{Ar} = 0,\\
&{\sigma^A}_{C(r}\,{\sigma^C}_{Bs)} = {\varepsilon^A}_B\,\varepsilon_{rs}, && \frac{1}{2}{\sigma^A}_{Br}\,{\sigma^B}_{As} = \varepsilon_{rs},\\
&{\sigma^A}_{C[r}\,{\sigma^C}_{Bs]} = i{\sigma^A}_{Bt}\,{\varepsilon^t}_{rs},
\end{aligned}$$

where ${\varepsilon^t}_{rs}$ is the Levi Civita tensor density. For the product of two unitary unimodular matrices

$${u^A}_B = {v^A}_C {w^C}_B$$

we get

$$\begin{aligned}
{\varepsilon^A}_B\cos\frac{u}{2} + i{\sigma^A}_{Br}\frac{u^r}{u}\sin\frac{u}{2} &= {\varepsilon^A}_B\left(\cos\frac{v}{2}\cos\frac{w}{2} - \varepsilon_{rs}\frac{v^r}{v}\frac{w^s}{w}\sin\frac{v}{2}\sin\frac{w}{2}\right)\\
&\quad + i{\sigma^A}_{Br}\left(\frac{v^r}{v}\sin\frac{v}{2}\cos\frac{w}{2} + \frac{w^r}{w}\sin\frac{w}{2}\cos\frac{v}{2} - {\varepsilon^r}_{st}\frac{v^s}{v}\frac{w^t}{w}\sin\frac{v}{2}\sin\frac{w}{2}\right),
\end{aligned}$$

from which the composition rule for parameters can be obtained.

An inner automorphism

$${u'^A}_B = {v^A}_C\,{u^C}_D\,{v^{-1D}}_B$$

results in an orthogonal transformation applied to parameters:

$${u'^r} = \left({\varepsilon^r}_s\cos v + {\varepsilon^r}_{st}\frac{v^t}{v}\sin v + \frac{v^r}{v}\frac{v^t}{v}\varepsilon_{ts}(1-\cos v)\right)u^s.$$

This shows that classes are composed of elements of equal values of u. It can be also shown that such elements always belong to one class.

One minimal ideal $\overset{(0)}{M}$ consists of functions constant on the group. The generating unit $\overset{(0)}{e}$ of this one-dimensional ideal is represented by

$$\overset{(0)}{\varepsilon}(u^r) = 1.$$

Another, four-dimensional, ideal $\overset{(1)}{M}$ is generated by $\overset{(1)}{e}$ represented by the function

$$\overset{(1)}{\varepsilon}(u^r) = 4\cos\frac{u}{2}$$

and its canonical basis is

$$\overset{(1)}{\varepsilon}{}^A{}_B = 2\left(\varepsilon^A{}_B\cos\frac{u}{2} + i\sigma^A{}_{Br}\frac{u^r}{u}\sin\frac{u}{2}\right).$$

More ideals can be obtained by forming symmetrical Kronecker products of $\overset{(1)}{e}{}^A{}_B$. The elements

$$\overset{(i)}{e}{}^{A_1\dots A_i}{}_{B_1\dots B_i} = \frac{i+1}{2^i}\,\overset{(1)}{e}{}^{(A_1}{}_{(B_1}\times\cdots\times\overset{(1)}{e}{}^{A_i)}{}_{B_i)}$$

form the canonical basis of the minimal ideal $\overset{(i)}{M}$ of dimension $(i+1)^2$. It can be shown that the $\overset{(i)}{M}$ give all minimal ideals of the considered group algebra.

The matrix $\overset{(1)}{\varepsilon}{}_{\dot A B}$ is the unit matrix:

$$\overset{(1)}{\varepsilon}{}_{\dot A B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrices $\overset{(i)}{\varepsilon}{}_{\dot A_1\dots\dot A_i B_1\dots B_i}$ can be easily expressed as Kronecker products of $\overset{(1)}{\varepsilon}{}_{\dot A B}$.

IV. The group algebra of a locally compact Abelian group

The notion of group algebra can be also extended to locally compact Abelian topological groups. We restrict the discussion to n-dimensional real linear spaces with addition of vectors as group operation. Let a basis in $S = G$ be chosen so that an element x is represented by n real components x^i, $i = 1,\dots,n$. If $\alpha(x^i)$ is a function on the group, an invariant integral is given by

$$\int \alpha(x^i)\mathrm{d}^n x;$$

however not all functions are integrable. The group algebra can be now defined as a set A_G of elements $a, b, \dots$ represented by square integrable functions $\alpha(x^i), \beta(x^i), \dots$,

$$\int \alpha(x^i)^*\alpha(x^i)\mathrm{d}^n x < \infty,\ \int \beta(x^i)^*\alpha(x^i)\mathrm{d}^n x < \infty, \dots$$

with

a) $\lambda a + \mu b$ represented by $\lambda\alpha(x^i) + \mu\beta(x^i)$,
b) ab represented by $\int \alpha(x^i - x'^i)\beta(x'^i)\mathrm{d}^n x'$,
c) a^* represented by $\alpha^*(x^i) = \alpha(x^i)^*$,

d) a^{T} represented by $\alpha^{\mathsf{T}}(x^i) = \alpha(-x^i)$,
e) $a^{\dagger}$ represented by $\alpha^{\dagger}(x^i) = \alpha(-x^i)^*$,
f) $\mathrm{tr}(a) = \alpha(0)$,
g) $\mathrm{tr}(a^{\dagger}b) = \int \alpha(x^i)^*\beta(x^i)\mathrm{d}^n x$,
h) $a \times b$ represented by $\alpha(x^i)\beta(x^i)$.

We introduce an n-parameter family of invariant function spaces $\overset{(p_j)}{M}$ consisting of periodic functions proportional to

$$\overset{(p_j)}{\varepsilon}(x^i) = \frac{1}{(2\pi)^n}\exp\bigl(i\,p_j x^j\bigr)\,.$$

The functions $\overset{(p_j)}{\varepsilon}(x^i)$ are not square integrable and therefore the spaces $\overset{(p_j)}{M}$ are not ideals of the group algebra. The situation is nevertheless similar to that for finite and compact groups because

$$\int \overset{(p_j)}{\varepsilon}(x^i - x'^i)\,\overset{(p'_j)}{\varepsilon}(x'^i)\mathrm{d}x' = \delta^n(p_j - p'_j)\,\overset{(p_j)}{\varepsilon}(x^i),$$

$$\alpha(x^i) = \int \alpha(p_j)\,\overset{(p_j)}{\varepsilon}(x^i)\mathrm{d}^n p,$$

or, if $\overset{(p_j)}{e}$ are abstract elements represented by $\overset{(p_j)}{\varepsilon}(x^i)$:

$$\overset{(p_j)}{e}\,\overset{(p'_j)}{e} = \delta^n(p_j - p'_j)\,\overset{(p_j)}{e},$$

$$a = \int \alpha(p_j)\,\overset{(p_j)}{e}\,\mathrm{d}^n p.$$

The function $\alpha(p_j)$ is a square integrable function with the following properties:

a) $\lambda a + \mu b$ represented by $\lambda\alpha(p_j) + \mu\beta(p_j)$,
b) ab represented by $\alpha(p_j)\beta(p_j)$,
c) a^* represented by $\alpha^*(p_j) = \alpha(p_j)^*$,
d) a^{T} represented by $\alpha^{\mathsf{T}}(p_j) = \alpha(-p_j)$,
e) $a^{\dagger}$ represented by $\alpha^{\dagger}(p_j) = \alpha(-p_j)^*$,
f) $\mathrm{tr}(a) = \alpha(0)$,
g) $\mathrm{tr}(a^{\dagger}b) = \int \alpha(p_j)^*\beta(p_j)\mathrm{d}^n x$,
h) $a \times b$ represented by $\int \alpha(p_j - p'_j)\beta(p'_j)\mathrm{d}^n p'$.

C. GROUP ALGEBRAS AND QUANTUM THEORY

I. Hilbert space and operator formulations of quantum mechanics

Quantum theories are usually expressed in Hilbert space formalism. Quantum states of a system are represented by vectors $\varphi, \psi, \ldots$ of the Hilbert space and dynamical variables by Hermitian operators $A, B, \ldots$ operating in that space. Vectors are usually normalized

$$(\varphi, \psi) = 1$$

and then the expectation values of an operator A in the state φ is given by

$$\langle A\rangle = (\varphi, A\varphi).$$

The probability that a system in state φ is observed in state ψ when a measurement is made is given by

$$(\varphi,\psi)(\psi,\varphi) = |(\varphi,\psi)|^2.$$

This formulation limits all considerations to pure states.

A slightly more general formulation is possible if a suitable algebra with Hermitian conjugation and trace is given. Then states, both pure and mixed, are represented by Hermitian, positive definite elements of the algebra and dynamical variables are represented by Hermitian elements of the same algebra. The normalization condition for a *density operator* ρ representing a state takes the form

$$\operatorname{tr}(\rho) = 1.$$

Expectation values are given by

$$\langle A\rangle = \operatorname{tr}(A\rho)$$

and the probability that a system in state ρ_1 is found in state ρ_2 is given by

$$\operatorname{tr}(\rho_1\rho_2).$$

If in the algebra a commutative $\times$ product of two operators is defined, it can be used to form density operators of systems composed of two other systems. For example $\rho_1 \times \rho_2$ represents the state of a composition of two systems in states ρ_1 and ρ_2.

Since a group algebra satisfies all the stated requirements, it can be used to formulate a quantum theory. The group has to be specially selected to give physically meaningful results. In subsequent sections we give examples of quantum theories in both Hilbert space and operator formulation.

II. Quantum theory of angular momentum

To formulate the theory of angular momentum we use the group algebra of the group of unitary unimodular matrices. In view of physical interpretation $\hbar$, the Planck constant divided by 2π is introduced. Unitary unimodular matrices are written in the form

$$u^A{}_B = \varepsilon^A{}_B \cos\frac{u}{2} + \frac{2i}{\hbar} S^A{}_{Br}\frac{u^r}{u}\sin\frac{u}{2}$$

with

$$S^A{}_{Br} = \frac{\hbar}{2}\sigma^A{}_{Br}, \qquad S^A{}_{Cr}\,S^C{}_{Bs} = \frac{\hbar^2}{4}\varepsilon^A{}_B\,\varepsilon_{rs} + \frac{i\hbar}{2}\sigma^A{}_{Bt}\,\varepsilon^t{}_{rs}.$$

The minimal ideal $\overset{(0)}{M}$ contains one normalized, Hermitian, positive definite element—the generating unit $\overset{(0)}{e}$ itself.

Hermitian elements of $\overset{(1)}{M}$ are of the form $\rho^A{}_B\,\overset{(1)}{e}{}^B{}_A$ with $\rho^{\dagger A}{}_B = \rho^A{}_B$. The Hermitian matrices form a four-dimensional space with the matrices $\varepsilon^A{}_B$ and $S^A{}_{Br}$ as elements of an orthogonal basis. The matrix $\rho^A{}_B$ is a combination

$$\rho^A{}_B = \rho\left(\varepsilon^A{}_B + \frac{2}{\hbar}P^r S^A{}_{Br}\right)$$

of these matrices with real coefficients. Normalization and positive definiteness conditions restrict ρ and P^r by

$$\rho = \frac{1}{4}, \quad P_r P^r \leqslant 1.$$

If $P_r P^r = 1$, then the state is pure and

$$\rho^A{}_B = \frac{1}{2}\psi^A \psi^*{}_B,$$

where

$$\psi^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+P^3}e^{i\alpha} \\ \\ \sqrt{1-P^3}e^{i\beta} \end{pmatrix}, \quad \tan(\alpha-\beta) = \frac{P^1}{P^2}.$$

The canonical basis $\overset{(2)}{e}{}^{A_1A_2}{}_{B_1B_2}$ for $\overset{(2)}{M}$ is defined by

$$\begin{aligned} \overset{(2)}{\varepsilon}{}^{A_1A_2}{}_{B_1B_2}(u^r) &= \frac{3}{4}\overset{(1)}{\varepsilon}{}^{(A_1}{}_{(B_1}(u^r)\overset{(1)}{\varepsilon}{}^{A_2)}{}_{B_2)}(u^r) \\ &= 3\Bigg(\varepsilon^{A_1A_2}{}_{B_1B_2}\frac{1+\cos u}{3} - \frac{2i}{\hbar}S^{A_1A_2}{}_{B_1B_2r}\frac{u^r}{u}\sin u \\ &\qquad - \frac{4}{\hbar^2}Q^{A_1A_2}{}_{B_1B_2rs}\frac{u^r}{u}\frac{u^s}{u}\frac{1-\cos u}{2}\Bigg) \end{aligned}$$

with

$$\varepsilon^{A_1A_2}{}_{B_1B_2} = \varepsilon^{(A_1}{}_{(B_1}\,\varepsilon^{A_2)}{}_{B_2)},$$

$$S^{A_1A_2}{}_{B_1B_2r} = S^{(A_1}{}_{(B_1r}\,\varepsilon^{A_2)}{}_{B_2)},$$

$$Q^{A_1A_2}{}_{B_1B_2rs} = S^{(A_1}{}_{(B_1r}\,\varepsilon^{A_2)}{}_{B_2)s}.$$

These matrices can be regarded as basis elements of the ideal and a density matrix $\rho^{A_1A_2}{}_{B_1B_2}$ can be written as a combination of the basis elements with real coefficients:

$$\rho^{A_1A_2}{}_{B_1B_2} = \rho\Bigg(\varepsilon^{A_1A_2}{}_{B_1B_2} + \frac{2}{\hbar}P^r S^{A_1A_2}{}_{B_1B_2r} + \frac{4}{\hbar^2}R^{rs}Q^{A_1A_2}{}_{B_1B_2rs}\Bigg).$$

The Kronecker product $\overset{(1)}{e}{}^{A_1}{}_{B_1} \times \overset{(1)}{e}{}^{A_2}{}_{B_2}$ is represented by

$$\begin{aligned} \overset{(1)}{\varepsilon}{}^{A_1}{}_{B_1}(u^r)\overset{(1)}{\varepsilon}{}^{A_2}{}_{B_2}(u^r) &= \overset{(1)}{\varepsilon}{}^{(A_1}{}_{(B_1}(u^r)\overset{(1)}{\varepsilon}{}^{A_2)}{}_{B_2)}(u^r) + \overset{(1)}{\varepsilon}{}^{[A_1}{}_{[B_1}(u^r)\overset{(1)}{\varepsilon}{}^{A_2]}{}_{B_2]}(u^r) \\ &= \frac{4}{3}\overset{(2)}{\varepsilon}{}^{A_1A_2}{}_{B_1B_2}(u^r) + 2\delta^{A_1A_2}\delta_{B_1B_2}, \end{aligned}$$

where

$$\delta^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \delta_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If

$$\rho_1{}^A{}_B = \frac{1}{4}\left(\varepsilon^A{}_B + \frac{2}{\hbar}P_1{}^r S^A{}_{Br}\right), \quad \rho_2{}^A{}_B = \frac{1}{4}\left(\varepsilon^A{}_B + \frac{2}{\hbar}P_2{}^r S^A{}_{Br}\right),$$

then the state composed of the two states is represented by

$$\begin{aligned}\rho(u^r) &= \rho^{A_1}{}_{B_1}\,\rho^{A_2}{}_{B_2}\left(\frac{4}{3}\overset{(2)}{\varepsilon}{}^{A_1A_2}{}_{B_1B_2}(u^r) + 2\delta^{A_1A_2}\delta_{B_1B_2}\right)\\ &= \frac{1}{12}\left[\left(1+\frac{1}{3}P_{1r}P_2{}^r\right)\varepsilon^{A_1A_2}{}_{B_1B_2} + (P_1{}^r + P_2{}^r)\frac{2}{\hbar}S^{A_1A_2}{}_{B_1B_2r}\right.\\ &\qquad\left. + \left(P_1{}^rP_2{}^s - \frac{1}{3}P_{1t}P_2{}^t\varepsilon^{rs}\right)\frac{4}{\hbar^2}Q^{A_1A_2}{}_{B_1B_2rs}\right]\overset{(2)}{\varepsilon}{}^{B_1B_2}{}_{A_1A_2}(u^r)\\ &\qquad + \frac{1}{4}(1 - P_{1r}P_2{}^r)\overset{(0)}{\varepsilon}(u^r).\end{aligned}$$

The ideals $\overset{(0)}{M}$, $\overset{(1)}{M}$ and $\overset{(2)}{M}$ correspond to spin 0, $\frac{1}{2}$ and 1 (angular momentum 0, $\frac{\hbar}{2}$ and $\hbar$) respectively. The operators $S^A{}_{Br}$, $S^{A_1A_2}{}_{B_1B_2r}$ and $Q^{A_1A_2}{}_{B_1B_2rs}$ correspond to physical quantities known as multipole polarizations (dipole and quadrupole). The quantities P^r and R^{rs} are polarization coefficients and are directly related to expectation values of polarization operators. The result obtained for the Kronecker product of two spin $\frac{1}{2}$ states with polarizations $P_1{}^r$ and $P_2{}^r$ shows that the composite system can be found in states of spin 0 and 1 with probabilities

$$\frac{1}{4}\left(1 - P_{1r}P_2{}^r\right) \quad \text{and} \quad \frac{3}{4}\left(1+\frac{1}{3}P_{1r}P_2{}^r\right)$$

respectively.

Multipole polarization operators can be introduced for ideals $\overset{(j)}{M}$ corresponding to spin $\frac{j}{2}$ in a similar way.

Quantum theory of angular momentum can be also formulated in Hilbert space with pure states of angular momentum $l\hbar$ represented by vectors in a $(2l+1)$-dimensional space. Instead of multipole polarizations one introduces the angular momentum operators related to infinitesimal automorphisms and coinciding with dipole polarization operators.

III. Quantum theory of translational degrees of freedom

1. General formulation. To formulate the theory of a system with n degrees of freedom we use the algebra of square integrable functions on an n-dimensional linear space discussed in B.IV. The parameter $\hbar$ is again introduced and the functions $\overset{(p_j)}{\varepsilon}(x^i)$ redefined as

$$\overset{(p_j)}{\varepsilon}(x^i) = \frac{1}{(2\pi\hbar)^n}\exp\left(\frac{i}{\hbar}p_jx^j\right).$$

States are represented by Hermitian, positive definite, normalized elements, that is, by functions

$$\rho(x^i) = \frac{1}{(2\pi\hbar)^n}\int \exp\left(\frac{i}{\hbar}p_ix^i\right)\rho(p_i)\mathrm{d}^np$$

with $\rho(p_j)$ real and non-negative and

$$\rho(0) = \frac{1}{(2\pi\hbar)^n}\int \rho(p_j)\mathrm{d}^np = 1.$$

The physical quantities are the linear momenta p_j with expectation values

$$\langle p_j \rangle = \frac{1}{(2\pi\hbar)^n} \int p_j \rho(p_j) \mathrm{d}^n p = \frac{\hbar}{i} \frac{\partial}{\partial x^j} \rho(x^i)_{x^i=0}.$$

Every function $\rho(p_j)$ can be written in the form

$$\rho(p_j) = (2\pi\hbar)^n \psi(p_j) \psi^\dagger(p_j)$$

with

$$\int \psi^\dagger(p_j) \psi(p_j) \mathrm{d}^n p = 1.$$

This shows that in this case the Hilbert space formulation is just as general as the operator formulation. The Hilbert space consists of square integrable functions $\psi(p_j)$ and the scalar product is

$$(\varphi, \psi) = \int \varphi^\dagger(p_j) \psi(p_j) \mathrm{d}^n p.$$

The function $\rho(x^i)$ is now equal to

$$\int \psi^\dagger(x^i - x'^i) \psi(x'^i) \mathrm{d}^n x'$$

where

$$\psi(x^i) = \frac{1}{(2\pi\hbar)^{n/2}} \int \exp\left(\frac{i}{\hbar} p_i x^i\right) \psi(p_i) \mathrm{d}^n p.$$

The linear momenta have expectation values

$$\langle p_j \rangle = \int \psi^\dagger(p_j) p_j \psi(p_j) \mathrm{d}^n p = \int \psi(x^i)^* \frac{\hbar}{i} \frac{\partial}{\partial x^j} \psi(x^i) \mathrm{d}^n x.$$

In addition to linear momenta, position operators x^j with expectation values

$$\langle x^j \rangle = \int \psi(x^i)^* x^i \psi(x^i) \mathrm{d}^n x = \int \psi(p_j)^* i\hbar \frac{\partial}{\partial p_j} \psi(p_j) \mathrm{d}^n p$$

are introduced.

2. Relativistic dynamics. We now proceed to formulate the theory of a free particle of mass m and no internal degrees of freedom. This is done by taking the four-dimensional Minkowskian space with coordinates x^μ, $\mu = 0, 1, 2, 3$, and metric tensor $\eta_{\mu\nu}$ and constructing the algebra of functions of the form

$$\alpha(x) = \frac{1}{(2\pi\hbar)^4} \int \exp\left(\frac{i}{\hbar} px\right) \alpha(p)\, \varepsilon(p)\, \delta(\sqrt{p^2} - m) \mathrm{d}^4 p, \qquad \varepsilon(p) = \frac{p_0}{|p_0|}.$$

These functions are not square integrable. Instead we demand that[1]

$$\frac{1}{(2\pi\hbar)^4} \int \alpha(p)^* \alpha(p)\, \varepsilon(p)\, \delta(\sqrt{p^2} - m) \mathrm{d}^4 p = \frac{\pi\hbar^2}{im} \int \alpha(x)^* \overleftrightarrow{\partial}_\mu \alpha(x) \mathrm{d}\sigma^\mu < \infty.$$

[1]The integral in the following formula extends over a spacelike section of space-time. The three-dimensional volume element $\mathrm{d}\sigma^\mu$ could be defined as $g^{\mu\nu}\partial_n \rfloor \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

The algebraic rules are slightly modified. The product ab is now represented by $\alpha(p)\beta(p)$ or by

$$\frac{\pi\hbar^2}{im}\int \alpha(x-x')\overset{\rightleftarrows}{\partial}_{\mu'}\beta(x')\mathrm{d}\sigma^{\mu'},$$

the trace is given by

$$\mathrm{tr}(a)=\frac{1}{(2\pi\hbar)^4}\int \alpha(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p=\alpha(x)|_{x=0}$$

and the scalar product by

$$\mathrm{tr}(a^\dagger b)=\frac{1}{(2\pi\hbar)^4}\int \alpha(p)^*\beta(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p=\frac{\pi\hbar^2}{im}\int \alpha(x)^*\overset{\rightleftarrows}{\partial}_{\mu}\beta(x)\mathrm{d}\sigma^{\mu}.$$

Although the scalar product is no longer positive definite it has been successfully used in relativistic quantum mechanics. The identity of this algebra is the singular function

$$\Delta(x)=\frac{1}{(2\pi\hbar)^4}\int \exp\left(\frac{i}{\hbar}px\right)\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p$$

known in connection with the Klein-Gordon equation.

States are represented by real non-negative normalized functions

$$\rho(p)=(2\pi\hbar)^4\psi(p)\psi^\dagger(p),$$

$$\frac{1}{(2\pi\hbar)^4}\int \rho(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p=\int \psi(p)^*\psi(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p=1$$

or by

$$\rho(x)=\frac{\pi\hbar^2}{im}\int \psi^\dagger(x-x')\overset{\rightleftarrows}{\partial}_{\mu'}\psi(x')\mathrm{d}\sigma^{\mu'},$$

$$\psi(x)=\frac{1}{(2\pi\hbar)^2}\int \exp\left(\frac{i}{\hbar}px\right)\psi(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p.$$

The physical quantities are the momenta p_μ with expectation values

$$\begin{aligned}\langle p_\mu\rangle&=\frac{1}{(2\pi\hbar)^4}\int p_\mu\rho(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p\\&=\frac{1}{(2\pi\hbar)^4}\int \psi^\dagger(p)p_\mu\psi(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p\\&=\frac{\hbar}{i}\partial_\mu\rho(x)|_{x=0}=\frac{\pi\hbar^3}{m}\int \partial_\mu\psi(x)^*\overset{\rightleftarrows}{\partial}_{\nu}\psi(x)\mathrm{d}\sigma^{\nu}.\end{aligned}$$

Using the Klein-Gordon equation

$$\partial_\mu\partial^\mu\psi(x)+\frac{m^2}{\hbar^2}\psi(x)=0$$

and integrating by parts one derives a completely symmetric expression

$$\begin{aligned}\langle p_\mu\rangle=\frac{\pi\hbar^3}{m}\int\Big[&\partial_\mu\psi(x)^*\partial_\nu\psi(x)+\partial_\nu\psi(x)^*\partial_\mu\psi(x)\\&-\eta_{\mu\nu}\left(\partial_\kappa\psi(x)^*\partial^\kappa\psi(x)-\frac{m^2}{\hbar^2}\psi(x)^*\psi(x)\right)\Big]\mathrm{d}\sigma^{\nu}.\end{aligned}$$

The positions given by x^μ or in momentum representation by $i\hbar\frac{\partial}{\partial p_\mu}$ are not well defined, because $\rho(p)$ is not defined off the mass shell $\sqrt{p^2} = m$. Only derivatives in directions tangent to the mass shell can be considered, i.e. operators

$$i\hbar\left(\frac{\partial}{\partial p_\mu} - \frac{1}{2}\frac{p^\mu p_\nu}{m^2}\frac{\partial}{\partial p_\nu} - \frac{1}{2}\frac{\partial}{\partial p_\nu}\frac{p^\mu p_\nu}{m^2}\right).$$

Infinitesimal Lorentz transformations define orbital angular momentum operators $M_{\mu\nu}$ represented by

$$i\hbar\left(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}\right)$$

or

$$-i\hbar\left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right).$$

IV. Relativistic theory of particles with spin $\frac{1}{2}$ and mass m. In the present section we combine results derived separately for angular momentum $\frac{1}{2}\hbar$ and mass m to formulate the theory of particles with spin $\frac{1}{2}$ and mass m. For this purpose we introduce the generalized σ matrices

$$\sigma^{A\dot{B}}{}_\mu = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),$$

$$\sigma_{\dot{A}B\,\mu} = \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, -\begin{pmatrix}0&1\\1&0\end{pmatrix}, -\begin{pmatrix}0&-i\\i&0\end{pmatrix}, -\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),$$

$$\sigma^{A\dot{C}}{}_\mu\,\sigma_{\dot{C}B\,\nu} = \begin{pmatrix} \begin{pmatrix}1&0\\0&1\end{pmatrix} & -\begin{pmatrix}0&1\\1&0\end{pmatrix} & -\begin{pmatrix}0&-i\\i&0\end{pmatrix} & -\begin{pmatrix}1&0\\0&-1\end{pmatrix} \\ \begin{pmatrix}0&1\\1&0\end{pmatrix} & -\begin{pmatrix}1&0\\0&1\end{pmatrix} & -i\begin{pmatrix}1&0\\0&-1\end{pmatrix} & i\begin{pmatrix}0&-i\\i&0\end{pmatrix} \\ \begin{pmatrix}0&-i\\i&0\end{pmatrix} & i\begin{pmatrix}1&0\\0&-1\end{pmatrix} & -\begin{pmatrix}1&0\\0&1\end{pmatrix} & -i\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ \begin{pmatrix}1&0\\0&-1\end{pmatrix} & -i\begin{pmatrix}0&-i\\i&0\end{pmatrix} & i\begin{pmatrix}0&1\\1&0\end{pmatrix} & -\begin{pmatrix}1&0\\0&1\end{pmatrix} \end{pmatrix}.$$

We note that

$$\sigma^{A\dot{C}}{}_{(\mu}\,\sigma_{\dot{C}B\,\nu)} = \varepsilon^A{}_B\eta_{\mu\nu}.$$

A special symbol is introduced for the antisymmetric part

$$\sigma^A{}_{B\mu\nu} = \sigma^{A\dot{C}}{}_{[\mu}\,\sigma_{\dot{C}B\,\nu]}.$$

We calculate the product of two $\sigma^A{}_{B\mu\nu}$ matrices:

$$\begin{aligned}\sigma^A{}_{C\mu\nu}\sigma^C{}_{B\kappa\lambda} = {}&\varepsilon^A{}_B\left(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\kappa} - i\,\varepsilon_{\mu\nu\kappa\lambda}\right)\\ &+ i\left(\sigma^A{}_{B\mu\lambda}\eta_{\kappa\nu} + \sigma^A{}_{B\nu\kappa}\eta_{\mu\lambda} + \sigma^A{}_{B\kappa\mu}\eta_{\nu\lambda} + \sigma^A{}_{B\lambda\nu}\eta_{\mu\kappa}\right).\end{aligned}$$

We now consider an algebra of elements $a, b, \ldots$ represented by functions $\alpha^A{}_B(p)$, $\beta^A{}_B(p), \ldots$ with the product rule: ab is represented by $\alpha^A{}_C(p)\,\beta^C{}_B(p)$. The trace is

defined by

$$\mathrm{tr}(a) = \frac{1}{(2\pi\hbar)^4}\int \alpha^A{}_A(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p$$

and Hermitian conjugate $a^\dagger$ by

$$\alpha^{\dagger A}{}_B(p) = \sigma^{A\dot{C}}{}_\mu \frac{p^\mu}{m}\left(\alpha^D{}_C(p)\right)^* \sigma_{\dot{D}B\nu}\frac{p^\nu}{m}.$$

The scalar product is

$$\mathrm{tr}(a^\dagger b) = \frac{1}{(2\pi\hbar)^4}\int \alpha^A{}_B(p)^*\beta^B{}_A(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p.$$

The Hermitian basis for multipole representation of density operators $\rho^A{}_B(p)$ is provided by $\varepsilon^A{}_B$ and

$$-i\sigma^A{}_{B\mu\nu}\frac{p^\nu}{m} = \sigma^A{}_{B\nu}\left(A^\nu{}_\mu - \frac{p^\nu p_\mu}{m^2}\right).$$

Density operators are expressed in the form

$$\rho^A{}_B(p) = \rho(p)\left(\varepsilon^A{}_B + \frac{2}{\hbar}P^\mu(p)S^A{}_{B\mu}\right)$$

with

$$S^A{}_{B\mu} = \frac{\hbar}{2}\sigma^A{}_{B\mu} = \frac{\hbar}{2}\sigma^{A\dot{C}}{}_\mu\sigma_{\dot{C}B\mu}\frac{p^\nu}{m} \quad\text{and}\quad P^\mu(p)p_\mu = 0.$$

Normalization and positive definiteness require that

$$\rho(p)\varepsilon(p) \geqslant 0, \qquad P_\mu(p)P^\mu(p) \geqslant -1,$$
$$\frac{1}{(2\pi\hbar)^4}\int \rho(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p = \frac{1}{4}.$$

A Lorentz transformation applied to $\rho^A{}_B(p)$ gives

$$\rho'^A{}_B(p) = \Lambda^A{}_C\ \rho^C{}_D(p')\ \Lambda^D{}_B$$

where

$$p'_\mu = p_\nu\, L^\nu{}_\mu,$$

$\Lambda^A{}_B$ is a unimodular matrix and

$$L^\mu{}_\nu = \frac{1}{2}\sigma_{\dot{A}B}{}^\mu\,\Lambda^B{}_C\,\sigma^{C\dot{D}}{}_\nu\,\Lambda^\dagger{}_{\dot{D}}{}^{\dot{A}}$$

is a Lorentz transformation matrix. An infinitesimal Lorentz transformation

$$\Lambda^A{}_B = \varepsilon^A{}_B - \frac{i}{\hbar}S^A{}_{B\mu\nu}\lambda^{\mu\nu}, \qquad L^\mu{}_\nu = A^\mu{}_\nu + 2\lambda^\mu{}_\nu$$

results in

$$\rho'^A{}_B(p) = \rho^A{}_B(p) - \frac{i}{\hbar}\left[i\hbar\left(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}\right)\rho^A{}_B(p) + S^A{}_{C\mu\nu}\,\rho^C{}_B(p) - \rho^A{}_C(p)S^C{}_{B\mu\nu}\right]\lambda^{\mu\nu}.$$

Pure states are those for which $P_\mu P^\mu = -1$. For such states

$$\rho^A{}_B(p) = \frac{1}{2}(2\pi\hbar)^4\psi^A(p)\psi^\dagger{}_B(p), \qquad \text{where}\quad \psi^\dagger{}_B(p) = (\psi^C(p))^*\sigma_{\dot{C}B\mu}\frac{p^\mu}{m}.$$

We introduce

$$\psi^A(x) = \frac{1}{(2\pi\hbar)^2}\int \exp\left(\frac{i}{\hbar}px\right)\psi^A(p)\,\varepsilon(p)\,\delta(\sqrt{p^2}-m)\mathrm{d}^4p$$

$$\psi^\dagger{}_A(x) = \frac{\hbar}{im}\partial_\mu(\psi^B(-x))^*\sigma_{\dot B A}{}^\mu.$$

The physical quantities are the linear momenta p_μ, the modified positions

$$i\hbar\left(\frac{\partial}{\partial p_\mu} - \frac{1}{2}\frac{\partial}{\partial p_\nu}\frac{p^\mu p_\nu}{m^2} - \frac{1}{2}\frac{p^\mu p_\nu}{m^2}\frac{\partial}{\partial p_\nu}\right) + \frac{p_\nu}{m}S^A{}_B{}^{\nu\mu},$$

and the angular momenta

$$i\hbar\left(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}\right)\varepsilon^A{}_B + S^A{}_{B\mu\nu}\frac{p^\kappa p_\mu}{m^2}.$$

One can also introduce internal angular momentum operators

$$S^A{}_{B\mu\nu} + S^A{}_{B\kappa\mu}\frac{p^\kappa p_\nu}{m^2} + S^A{}_{B\nu\kappa}\frac{p^\kappa p_\mu}{m^2}$$

closely related to dipole polarization operators. The scalar product has the form

$$\frac{\pi\hbar^2}{im}\int \varphi^\dagger{}_A(-x)\overrightarrow{\partial}_\mu\psi^A(x)\mathrm{d}\sigma^\mu.$$

In the usual formulation of spin $\frac{1}{2}$ mass m theory states are represented by bispinor wave functions $\psi(x)$ satisfying the Dirac equation

$$\frac{\hbar}{i}\gamma^\mu\partial_\mu\psi(x) = m\psi(x),$$

and the scalar product has the form

$$\pi\hbar\int\bar\psi(x)\gamma_\mu\psi(x)\mathrm{d}\sigma^\mu,$$

where γ^μ is the set of Dirac matrices and

$$\bar\psi(x) = \psi(x)^*\gamma_0.$$

To show the equivalence of this formulation to the one presented in this section we write

$$\psi(x) = \begin{pmatrix}\psi^A(x)\\ \psi_{\dot B}(x)\end{pmatrix},\quad \gamma_\mu = \begin{pmatrix}0 & \sigma^{A\dot B}{}_\mu\\ \sigma_{\dot C D\mu} & 0\end{pmatrix},\quad \bar\psi(x) = (\psi^\dagger{}_A(-x), \psi^{\dagger\dot B}(-x)).$$

The Dirac equation takes the form

$$\frac{\hbar}{i}\begin{pmatrix}0 & \sigma^{A\dot B\mu}\\ \sigma_{\dot C D}{}^\mu & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\psi^D(x)\\ \psi_{\dot B}(x)\end{pmatrix} = m\begin{pmatrix}\psi^A(x)\\ \psi_{\dot C}(x)\end{pmatrix}$$

equivalent to

$$\frac{\hbar}{i}\sigma^{A\dot B\mu}\partial_\mu\psi_{\dot B}(x) = m\psi^A(x),\qquad \frac{\hbar}{i}\sigma_{\dot C D}{}^\mu\partial_\mu\psi^D(x) = m\psi_{\dot C}(x).$$

The scalar product is

$$\pi\hbar\int(\varphi^\dagger{}_A(-x)\sigma^{A\dot B}{}_\mu\psi_{\dot B}(x) + \varphi^{\dagger\dot A}(-x)\sigma_{\dot A B\mu}\psi^B(x))\mathrm{d}\sigma^\mu = \int\varphi^\dagger{}_A(-x)\overrightarrow{\partial}_\mu\psi^A(x)\mathrm{d}\sigma^\mu,$$

and

$$\bar{\psi}(x) = (\psi^{\dagger}{}_{A}(-x), \psi^{\dagger\dot{B}}(-x)) = (\psi_{\dot{A}}(x)^*, \psi^{B}(x)^*) = \psi^* A$$

with

$$A = \begin{pmatrix} 0 & \varepsilon^{A}{}_{B} \\ \varepsilon_{\dot{C}}{}^{\dot{D}} & 0 \end{pmatrix}$$

numerically equal to γ_0.

D. CLASSICAL APPLICATIONS

In the present chapter we draw classical conclusions from quantum mechanical results of the preceding chapter. Operators will be denoted by $\hat{A}, \hat{B}, \ldots$ to distinguish them from their classical counterparts $A, B, \ldots$.

The method of classical correspondence consists in letting $\hbar$ go to zero, and calculating zeroth order, or sometimes first order terms in the asymptotic expansion of all quantum mechanical expressions. The details of the classical correspondence are discussed separately. Here we note that to the zeroth order in $\hbar$ all operators commute and go over into corresponding classical quantities. For systems with translational degrees of freedom the basic operators are $\hat{x}^i$ and $\hat{p}_j$ with commutators

$$\left[\hat{x}^i, \hat{p}_j\right] = i\hbar\, \varepsilon^{i}{}_{j}.$$

All other operators are functions of the basic set and first order expressions of their commutators coincide with Poisson brackets multiplied by $i\hbar$. Here by functions of noncommuting operators we mean algebraic functions with symmetrized products, e.g. $\hat{A}\hat{B}$ means $\frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$. For internal degrees of freedom there is usually no need to introduce functions of the basic operators (angular momenta and polarizations) sufficiently complicated to necessitate deriving a special formalism for their commutators.

First we discuss the theory angular momentum. No operators had to be introduced for angular momentum 0. For angular momentum $\frac{1}{2}\hbar$ we had to introduce dipole polarization, that is, angular momentum operators, and for angular momentum $\hbar$, additionally quadrupole polarization operators. For higher angular momenta higher polarization operators would have to be introduced. We thus conclude that the degrees of freedom of systems with angular momentum coincide with those of multipole particles at rest.

In the case of a system with mass m the operators introduced were in the first place momenta $\hat{p}_\mu$ represented by p_μ or $-i\hbar\partial_\mu$ and related to infinitesimal translations in spacetime. The complementary set of operators is given by positions $\hat{x}^\mu$ represented by $i\hbar\frac{\partial}{\partial p_\mu}$ or x^μ and related to infinitesimal translations in momentum space. These operators are well defined only when no restrictions on states are imposed. If the states are restricted to those of mass m, only translations tangent to the mass shell can be considered. This leads to modified operators

$$\hat{x}^{\mu}{}_{0} = \hat{x}^{\mu}\left(A^{\mu}{}_{\nu} - \frac{p^\mu p_\nu}{m^2}\right)$$

which commute with $\hat{m} = \sqrt{\hat{p}^2}$. If states of mass m are regarded as special cases of states

with arbitrary mass, we can use operators $\hat{x}^\mu$ or supplement ${\hat{x}^\mu}_0$ with

$$\hat{s} = \frac{p_\mu}{m}\ \hat{x}^\mu, \quad [\hat{s}, \hat{m}] = i\hbar.$$

The relation between $\hat{x}^\mu$ and ${\hat{x}^\mu}_0$ is expressed by

$$\hat{x}^\mu = {\hat{x}^\mu}_0 + \hat{s}\ \frac{\hat{p}^\mu}{\hat{m}}.$$

The corresponding classical system is a simple pole particle of mass m described by a straight time-like world line in space-time. The world line is characterized by its direction $\frac{p^\mu}{m}$ and orthogonal distance ${x^\mu}_0$ from the origin of the coordinate system. If the world line is thought of as composed of points of space-time, position on the line can be specified either directly by x^μ or by ${x^\mu}_0$ and the proper time s. The relation between these quantities is

$$x^\mu = {x^\mu}_0 + s\ \frac{p^\mu}{m}.$$

The degrees of freedom of a spin $\frac{1}{2}$, mass m system are clearly those of a pole-dipole particle. The linear momentum has the same properties as in the spinless case, the operator $i\hbar\frac{\partial}{\partial p_\mu}$ is not well defined even if mass is not restricted. The reason for this is that a translation in momentum space has to be accompanied by a suitable Lorentz rotation applied to internal degrees of freedom in order to maintain orthogonality between momentum and dipole polarization. This leads to an operator

$$i\hbar\frac{\partial}{\partial p_\mu}\ {\varepsilon^A}_B + \frac{p_\nu}{m}\ {S^A}_B{}^{\nu\mu}$$

which we interpret as the position $\hat{x}^\mu$. Mass restriction requires that $\hat{x}^\mu$ be projected on the mass shell:

$${\hat{x}^\mu}_0 = \hat{x}^\mu \left({A^\mu}_\nu - \frac{p^\mu p_\nu}{m^2} \right).$$

The relation between $\hat{x}^\mu$ and ${\hat{x}^\mu}_0$ is again

$$\hat{x}^\mu = {\hat{x}^\mu}_0 + \hat{s}\ \frac{\hat{p}^\mu}{\hat{m}}.$$

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