THE ORIGIN OF VARIATIONAL PRINCIPLES

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1. Introduction. This note presents an attempt to provide a conceptual framework for variational formulations of classical physics. It is based on a study of static systems in collaboration with P. Urbański of Warsaw.

Variational principles of physics have all a common source in the *principle of virtual work* well known in statics of mechanical systems. This principle is presented here as the first step in characterizing local stable equilibria of static systems. An extended analysis of local equilibria is given for systems with configuration manifolds of finite dimensions. Numerous examples of the principle of virtual work and the Legendre transformation applied to static mechanical systems are provided. Configuration spaces for the dynamics of autonomous mechanical systems and for statics of continua are constructed in the final sections. These configuration spaces are not differential manifolds.

The final section of the present note is an initial part of a programme of geometric formulations of the foundations of statics of continua undertaken jointly with F. Cardin of Padova in collaboration with P. Urbański.

2. Configurations, processes, work. Experimental study of physical systems consists in subjecting a system to external control and observing the results. A theoretical description of the system should have the power to predict observed results. The formulation of a theoretical description must start with assigning to the system parameters characterizing its configuration. These parameters do not ever exhaust all characteristics of the system. The choice is dictated by the interactions to which the system will be subjected. Statics of mechanical systems assigns to a mechanical system points in a differential manifold characterizing the geometrical placement of the system in the physical space. The points are translated in parameters by means of coordinate systems. The

differential manifold is called the *configuration space*. Control of a static mechanical system consists in coupling the system with external devices with the same configuration space and observing the resulting configurations. At least two configuration spaces are involved. The *control configuration space* represents degrees of freedom which are being controlled. The observed configurations are elements of the *observed configuration space*. The two spaces are not necessarily the same. The configuration spaces considered represent configurations of stable local equilibrium.

It is sometimes necessary to redefine the control configuration space. It is sometimes found that the same control arrangements do not result in the same response. When this happens we must discover and include in the description additional hidden degrees of freedom participating in the process of control. The configuration space with these degrees of freedom included is the *internal configuration space*. We will assume initially that the inclusion of additional degrees of freedom is not necessary and that the internal configuration space is the same as the control configuration space.

An important form of control consists in subjecting a system to a *quasi static process* by coupling it to an external device with slowly changing configuration. The changes must be slow enough to justify the treatment of a process as a succession of configurations of static equilibrium configurations rather than a dynamic event. Criteria of equilibrium are based on the response to quasi static processes.

Quasistatic processes are represented as oriented embedded arcs in the configuration space. Let Q be a differential manifold of dimension m representing the configuration space of a static system. An *embedded arc* is a subset $\mathbf{c} \subset Q$ which is the image of an embedding $\gamma \colon [0, a] \to Q$ of a closed interval $[0, a] \subset \mathbb{R}$ defined as the restriction to $[0, a]$ of an embedding $\widetilde{\gamma} \colon \mathbb{R} \to Q$. The embedding γ is a representative of an embedded arc called its *parameterization*. Different embeddings may be parameterizations of the same arc. A parameterization $\gamma \colon [0, a] \to Q$ of an arc $\mathbf{c}$ induces an *orientation* of the arc. The arc is oriented from $\gamma(0)$ to $\gamma(a)$ if the two points are distinct. If $\gamma(0) = \gamma(a)$, then the arc is a cycle. We will exclude cyclic processes from the discussion. They are of no interest for developing criteria of equilibrium. Two parameterizations γ and γ' induce the same orientation if $\gamma'(0) = \gamma(0)$. The *boundary* $\partial \mathbf{c}$ of an arc $\mathbf{c}$ is a set of two points q and q' such that $\gamma(0) = q$ and $\gamma(a) = q'$ for some parameterization $\gamma \colon [0, a] \to Q$ of $\mathbf{c}$. The designation of one of the boundary points as the *initial configuration* of the process represented by an arc specifies an orientation. An *oriented embedded arc* is defined as a pair $(\mathbf{c}, q_0)$, where $\mathbf{c}$ is an embedded arc and q_0 is a point in the boundary $\partial \mathbf{c}$ designated as the initial configuration. The remaining boundary point is the *terminal configuration* of $(\mathbf{c}, q_0)$. A parameterization $\gamma \colon [0, a] \to Q$ of an arc $\mathbf{c}$ such that $\gamma(0) = q_0$ is said to be *compatible* with the orientation of the oriented arc $(\mathbf{c}, q_0)$. Compatible parameterizations will be used for oriented arcs.

Let $(\mathbf{c}, q_0)$ be an oriented arc. We refer to points in $\mathbf{c}$ as elements of $(\mathbf{c}, q_0)$. Elements of an oriented arc $(\mathbf{c}, q_0)$ are ordered by the relation $\leqslant$ defined in terms of any parameterization γ compatible with the orientation. The relation $\gamma(s') \leqslant \gamma(s)$ is equivalent to $s' \leqslant s$. The relations $<$, $\geqslant$, and $>$ are defined in a similar way. They reflect the corresponding relations between the values of the parameter.

Let $(\mathbf{c}, q_0)$ and $(\mathbf{c}', q_0')$ be oriented arcs such that $\mathbf{c}' \subset \mathbf{c}$. There are two ordering relations in $(\mathbf{c}', q_0')$ since elements of $(\mathbf{c}', q_0')$ are also elements of $(\mathbf{c}, q_0)$. The arc $(\mathbf{c}', q_0')$ is said to be *included* in the arc $(\mathbf{c}, q_0)$ if the two ordering relations coincide. The inclusion relation is denoted by $(\mathbf{c}', q_0') \subset (\mathbf{c}, q_0)$. The process represented by $(\mathbf{c}', q_0')$ is a *subprocess* of the process represented by $(\mathbf{c}, q_0)$ if $(\mathbf{c}', q_0') \subset (\mathbf{c}, q_0)$. If $(\mathbf{c}^1, q_0^1)$ and $(\mathbf{c}^2, q_0^2)$ are oriented arcs and if one of the pairs $(\mathbf{c}^1 \cup \mathbf{c}^2, q_0^1)$ or $(\mathbf{c}^1 \cup \mathbf{c}^2, q_0^2)$ is an oriented arc, then it is considered the *union* of $(\mathbf{c}^1, q_0^1)$ and $(\mathbf{c}^2, q_0^2)$ and is denoted by $(\mathbf{c}^1, q_0^1) \cup (\mathbf{c}^2, q_0^2)$. Similarly, if one of the pairs $(\mathbf{c}^1 \cap \mathbf{c}^2, q_0^1)$ or $(\mathbf{c}^1 \cap \mathbf{c}^2, q_0^2)$ is an oriented arc, then it is considered the *intersection* of $(\mathbf{c}^1, q_0^1)$ and $(\mathbf{c}^2, q_0^2)$ and is denoted by $(\mathbf{c}^1, q_0^1) \cap (\mathbf{c}^2, q_0^2)$.

Let $(\mathbf{c}, q_0)$ be an oriented arc. A function $a \colon \mathbf{c} \to \mathbb{R}$ is considered a function on $(\mathbf{c}, q_0)$. This function is said to be *differentiable* if it the restriction to $\mathbf{c}$ of a differentiable function on Q. Let $\mathsf{A}Q$ be the algebra of differentiable functions on Q and let $\mathsf{I}_0(Q, \mathbf{c}) \subset \mathsf{A}Q$ be the ideal of functions vanishing on $\mathbf{c}$. Two functions f and f' on Q have the same restrictions to $\mathbf{c}$ if and only if $f' - f \in \mathsf{I}_0(M, S)$. It follows that the algebra $\mathsf{A}(\mathbf{c}, q_0)$ of differentiable functions on $(\mathbf{c}, q_0)$ is canonically isomorphic to the quotient algebra $\mathsf{A}Q / \mathsf{I}_0(Q, \mathbf{c})$. The orientation of $(\mathbf{c}, q_0)$ makes it possible to single out increasing functions. A function $h \in \mathsf{A}\mathbf{c}$ is *increasing* if $q' > q$ implies $h(q') > h(q)$.

An oriented arc represents a quasi static process. We use these two terms interchangeably. We denote by $\mathsf{P}Q$ the set of all processes in Q. An essential part of the characterization of a static system is the specification of a set $\mathsf{C} \subset \mathsf{P}Q$ of *admissible processes* in the configuration space Q. Admissible processes are the processes that can be actually induced using the control devices at our disposal. The following conditions are satisfied.

(1) A subprocess of an admissible process is admissible.
(2) The union of admissible processes is admissible.
(3) If a process can be approximated with admissible processes, then it is admissible.

The last condition is made more precise by the following statement. Let $q^{\sharp}_1, q^{\sharp}_2, \ldots,$ $q^{\sharp}_i, \ldots$ be a sequence of interior configurations in a process $(\mathbf{c}, q_0)$ converging to the terminal configuration q_1. If processes $(\mathbf{c}_{q^{\sharp}_i |}, q_0)$ are admissible, then $(\mathbf{c}, q_0)$ is admissible. The symbol $(\mathbf{c}_{q|}, q_0)$ denotes the subprocess of $(\mathbf{c}, q_0)$ with $q \in \mathbf{c}$ as the terminal configuration. It is based on the arc

$$\mathbf{c}_{q|} = \{ q' \in \mathbf{c}; \ q' \leqslant q \}. \tag{1}$$

Another object characterizing a static system is a function

$$W \colon \mathsf{C} \to \mathbb{R} \tag{2}$$

associating with each admissible process $(\mathbf{c}, q_0)$ the *work* $W(\mathbf{c}, q_0)$ of the process. Since subprocesses of admissible processes are admissible we can associate with each admissible process $(\mathbf{c}, q_0)$ the function

$$w_{(\mathbf{c}, q_0)} \colon \mathbf{c} \to \mathbb{R}, \qquad q \mapsto \begin{cases} W(\mathbf{c}_{q|}, q_0) & \text{if } q \neq q_0, \\ 0 & \text{if } q = q_0. \end{cases} \tag{3}$$

The work function will be assumed to satisfy the following conditions.

(1) Work is additive in the sense that if $(\mathbf{c}, q_0)$ is the union of admissible processes

$$(\mathbf{c}^0, q_0^0), (\mathbf{c}^1, q_0^1), \ldots, (\mathbf{c}^n, q_0^n) \tag{4}$$

such that for $i = 0, 1, \ldots, n - 1$ the terminal configuration of $(\mathbf{c}^i, q_0^i)$ is the initial configuration of $(\mathbf{c}^{i+1}, q_0^{i+1})$, then

$$W(\mathbf{c}, q_0) = \sum_{i=0}^{n} W(\mathbf{c}^i, q_0^i). \tag{5}$$

(2) For each admissible process $(\mathbf{c}, q_0)$ the function $w_{(\mathbf{c}, q_0)}$ is differentiable.

3. An alternative definition of orientation. For the purpose of constructing an orientation of an embedded arc $\mathbf{c}$ we introduce the *tangent bundle* $\mathsf{T}\mathbf{c}$. Let $\gamma \colon [0, a] \to Q$ be a parameterization of $\mathbf{c}$ and let $\widetilde{\gamma} \colon \mathbb{R} \to Q$ be an embedding such that $\gamma = \widetilde{\gamma}|[0, a]$. The image $\widetilde{\mathbf{c}} = \widetilde{\gamma}(\mathbb{R})$ is a submanifold of Q and has a well defined tangent bundle $\mathsf{T}\widetilde{\mathbf{c}} \subset \mathsf{T}Q$. The tangent bundle $\mathsf{T}\mathbf{c}$ is the restriction

$$\mathsf{T}\mathbf{c} = \mathsf{T}\widetilde{\mathbf{c}} \cap \tau_Q^{-1}(\mathbf{c}) \tag{6}$$

of $\mathsf{T}C$ to $\mathbf{c}$. The result of this construction is independent of the choice of the parameterization. An *orientation* of the arc $\mathbf{c}$ is a subset $o \in \mathsf{T}\mathbf{c}$ with the following properties.

(1) $\tau_Q(o) = \mathbf{c}$.
(2) If $v \in o$ then $kv \in o$ for each $k > 0$.
(3) If $v \in o$ then $kv \notin o$ for $k \geqslant 0$.
(4) There is a parameterization $\gamma \colon [0, a] \to Q$ such that $\mathsf{t}\gamma(s) \in o$ for each $s \in]0, a[$.

The symbol $\mathsf{t}\gamma(s)$ denotes the vector tangent to the curve γ at $\gamma(s)$. There are obviously two different orientations of each embedded arc. An *oriented embedded arc* is a pair $(\mathbf{c}, o)$ of an embedded arc $\mathbf{c}$ and one of its orientations o. Parameterizations of an arc satisfying the condition (4) above are said to be *compatible* with the orientation o. A parameterization $\gamma \colon [0, a] \to Q$ can be used to generate an orientation of an arc. The set

$$o = \{ v \in \mathsf{T}\mathbf{c}; \ \exists_{s \in [0, a], \, k > 0} v = k\mathsf{t}\gamma(s) \} \tag{7}$$

is an orientation of the arc $\mathbf{c} = \gamma([0, a])$ and the parameterization γ is compatible with this orientation. Vectors $\mathsf{t}\gamma(0)$ an $\mathsf{t}\gamma(a)$ are obtained as limits. Specifying the initial configuration q_0 determines an orientation o of the arc unless the terminal configuration coincides with the initial configuration. If such are excluded, then the pair $(\mathbf{c}, q_0)$ is a representation of the oriented arc equivalent to $(\mathbf{c}, o)$.

4. Stable local equilibrium configurations. Let $C^0 \subset Q$ be the set of initial configurations of all admissible processes for a static system. A point $q_0 \in C^0$ is called a *stable local equilibrium configuration* of the system if for each admissible process $(\mathbf{c}, q_0)$ initiating at q_0 the function $w_{(\mathbf{c}, q_0)}$ has a local minimum at q_0.

The function $w_{(\mathbf{c}, q_0)}$ will have a local minimum at q_0 if there is a point $q \in \mathbf{c}$ such that the function $w_{(\mathbf{c}, q_0)}$ restricted to the arc

$$\mathbf{c}_{q|} = \{ q' \in \mathbf{c}; \ q' \leqslant q \} \tag{8}$$

is increasing. This observation implies that equilibrium at q_0 is a property of the germ of $w_{(\mathbf{c}, q_0)}$ at q_0. We will state the definition of equilibrium in terms of germs.

In the set $\mathsf{P}Q$ of oriented embedded arcs in Q we introduce an equivalence relation. Arcs $(\mathbf{c}, q_0)$ and $(\mathbf{c}', q_0')$ are equivalent if $q_0' = q_0$ and if there is an open neighbourhood $U \subset Q$ of q_0 such that $\mathbf{c}' \cap U = \mathbf{c} \cap U$. Equivalence classes are called *germs* of processes. The set of germs is denoted by $\mathsf{P}^{\mathfrak{c}}Q$. The germ of a process $(\mathbf{c}, q_0)$ is denoted by $\mathsf{j}^{\mathfrak{c}}(\mathbf{c}, q_0)$. Germs of processes can be defined in terms of ideals. In the algebra $\mathsf{A}Q$ we introduce an ideal $\mathsf{I}_{\mathfrak{c}}(Q, q)$ for each $q \in Q$. A function f is in $\mathsf{I}_{\mathfrak{c}}(Q, q)$ if there is a neighbourhood $U \subset Q$ of q such that $f|U = 0$. Arcs $(\mathbf{c}, q_0)$ and $(\mathbf{c}', q_0')$ are equivalent if

$$\mathsf{I}_{\mathfrak{c}}(Q, q_0') + \mathsf{I}_0(Q, \mathbf{c}') = \mathsf{I}_{\mathfrak{c}}(Q, q_0) + \mathsf{I}_0(Q, \mathbf{c}). \tag{9}$$

This is the same equivalence relation as the one defined above. The symbol $C_{q_0}^{\mathfrak{c}}$ will denote the set of germs of admissible processes initiating at q_0.

With a process $(\mathbf{c}, q_0)$ we associate an ideal $\mathsf{I}_{\mathfrak{c}}(\mathbf{c}, q_0)$ in the algebra $\mathsf{A}(\mathbf{c}, q_0)$ of differentiable functions on $(\mathbf{c}, q_0)$. A function h is in this ideal if there is a neighbourhood $V \in \mathbf{c}$ of the initial configuration q_0 such that $h|V = 0$. The algebra $\mathsf{A}(\mathbf{c}, q_0)$ is canonically isomorphic to the quotient algebra $\mathsf{A}Q/\mathsf{I}_0(Q, \mathbf{c})$ and the ideal $\mathsf{I}_{\mathfrak{c}}(\mathbf{c}, q_0)$ is isomorphic to the quotient

$$(\mathsf{I}_{\mathfrak{c}}(Q, q_0) + \mathsf{I}_0(Q, \mathbf{c}))/\mathsf{I}_0(Q, \mathbf{c}). \tag{10}$$

We will adopt the following identification:

$$\mathsf{A}^{\mathfrak{c}}(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_{\mathfrak{c}}(\mathbf{c}, q_0) \quad \text{is identified with} \quad \mathsf{A}Q/(\mathsf{I}_{\mathfrak{c}}(Q, q_0) + \mathsf{I}_0(Q, \mathbf{c})). \tag{11}$$

As a consequence of this identification the algebra $\mathsf{A}^{\mathfrak{c}}(\mathbf{c}, q_0)$ is associated with the germ $\mathsf{j}^{\mathfrak{c}}(\mathbf{c}, q_0)$ rather than with the process $(\mathbf{c}, q_0)$ since the algebra $\mathsf{A}^{\mathfrak{c}}(\mathbf{c}', q_0')$ is the same as the algebra $\mathsf{A}^{\mathfrak{c}}(\mathbf{c}, q_0)$ if $(\mathbf{c}', q_0')$ and $(\mathbf{c}, q_0)$ are equivalent.

Each function $h \in \mathsf{A}(\mathbf{c}, q_0)$ has a germ $\mathsf{j}^{\mathfrak{c}}h \in \mathsf{A}^{\mathfrak{c}}(\mathbf{c}, q_0)$. The germ $\mathsf{j}^{\mathfrak{c}}w_{(\mathbf{c}, q_0)}$ of the work function $w_{(\mathbf{c}, q_0)}$ along an admissible process $(\mathbf{c}, q_0)$ is the same as the germ $\mathsf{j}^{\mathfrak{c}}w_{(\mathbf{c}', q_0')}$ if $(\mathbf{c}', q_0')$ and $(\mathbf{c}, q_0)$ are equivalent. An element of $\mathsf{A}^{\mathfrak{c}}(\mathbf{c}, q_0)$ is said to be *increasing* if it has an increasing representative in $\mathsf{A}(\mathbf{c}, q_0)$.

The criterion of stable local equilibrium is based on response of the system to the initial phase of a process. It is therefore correct to adopt the following definition of equilibrium formulated in terms of germs.

A point $q_0 \in C^0$ is called a *stable local equilibrium configuration* if for each germ $\mathsf{j}^{\mathfrak{c}}(\mathbf{c}, q_0) \in C_{q_0}^{\mathfrak{c}}$ the germ $\mathsf{j}^{\mathfrak{c}}w_{(\mathbf{c}, q_0)}$ is increasing.

We have excluded constant processes represented by constant arcs. In a complete discussion of equilibrium configurations *isolated admissible configurations* and *admissible constant processes* should be considered. Only constant admissible configurations can initiate at an isolated admissible configuration. Such configurations ar obviously stable local equilibrium configurations. No further discussion is necessary.

5. Differential criteria of local equilibrium

5.1. *Jets of processes and functions.* We will denote by $\mathbb{K}$ the set $\mathbb{N} \cup \{\infty, \mathfrak{c}\}$, where $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$. The ordering relations $\leqslant, <, \geqslant$, and $>$ have in $\mathbb{K}$ the usual meaning of inequalities of cardinal numbers. Let q be a point in a differential manifold Q. In the algebra $\mathsf{A}Q$ of differentiable functions on Q we introduce a sequence of ideals

$$\mathsf{I}_0(Q, q), \; \mathsf{I}_1(Q, q), \; \ldots, \; \mathsf{I}_\infty(Q, q), \; \mathsf{I}_{\mathfrak{c}}(Q, q). \tag{12}$$

The ideal $\mathsf{I}_0(Q,q)$ associated with q is maximal in the sense that it is not a proper subset of any ideal except the trivial ideal $\mathsf{A}Q$. It is known that all maximal ideals in $\mathsf{A}Q$ are associated with points. For $k \in \mathbb{N}$, the ideal $\mathsf{I}_k(Q,q)$ is the power $(\mathsf{I}_0(Q,q))^{k+1}$ of the ideal $\mathsf{I}_0(Q,q)$. The ideal $\mathsf{I}_\infty(Q,q)$ is the intersection $\bigcap_{k \in \mathbb{N}} \mathsf{I}_k(Q,q)$. The ideal $\mathsf{I}_{\mathfrak{c}}(Q,q)$ is the set of functions each vanishing in a neighbourhood of q. This ideal was introduced earlier. The inclusion relations

$$\mathsf{I}_k(Q,q) \subset \mathsf{I}_{k'}(Q,q) \tag{13}$$

hold for all k' and k in $\mathbb{K}$ such that $k' \leqslant k$.

For each $k \in \mathbb{K}$ we introduce an equivalence relation in the set $\mathsf{P}Q$ of oriented embedded arcs in Q. Arcs $(\mathbf{c}, q_0)$ and $(\mathbf{c}', q_0')$ are equivalent if

$$\mathsf{I}_k(Q, q_0') + \mathsf{I}_0(Q, \mathbf{c}') = \mathsf{I}_k(Q, q_0) + \mathsf{I}_0(Q, \mathbf{c}). \tag{14}$$

Equivalence classes are called k-*jets* of processes. The $\mathfrak{c}$-jets are the germs introduced earlier. The set of k-jets will be denoted by $\mathsf{P}^k Q$. The k-jet of a process $(\mathbf{c}, q_0)$ will be denoted by $\mathsf{j}^k(\mathbf{c}, q_0)$. Inclusion relations imply the existence of projections

$$\pi^{k'}{}_{k\,Q} \colon \mathsf{P}^k Q \to \mathsf{P}^{k'} Q, \quad \mathsf{j}^k(\mathbf{c}, q_0) \mapsto \mathsf{j}^{k'}(\mathbf{c}, q_0), \tag{15}$$

for $k' \leqslant k$ in addition to

$$\pi_{k\,Q} \colon \mathsf{P}^k Q \to Q, \quad \mathsf{j}^k(\mathbf{c}, q_0) \mapsto q_0. \tag{16}$$

Corresponding to the sequence (12) of ideals we have the sequence

$$\mathsf{A}^0(Q,q) = \mathsf{A}Q/\mathsf{I}_0(Q,q), \dots, \ \mathsf{A}^\infty(Q,q) = \mathsf{A}Q/\mathsf{I}_\infty(Q,q), \ \mathsf{A}^{\mathfrak{c}}(Q,q) = \mathsf{A}Q/\mathsf{I}_{\mathfrak{c}}(Q,q) \tag{17}$$

of quotient algebras.

In the algebra $\mathsf{A}(\mathbf{c}, q_0)$ of differentiable functions on the oriented arc $(\mathbf{c}, q_0)$ we have ideals

$$\mathsf{I}_0(\mathbf{c}, q_0), \ \mathsf{I}_1(\mathbf{c}, q_0), \ \dots, \ \mathsf{I}_\infty(\mathbf{c}, q_0), \ \mathsf{I}_{\mathfrak{c}}(\mathbf{c}, q_0) \tag{18}$$

defined just as the ideals in the sequence (12) and the corresponding quotient algebras

$$\mathsf{A}^0(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_0(\mathbf{c}, q_0), \ \mathsf{A}^1(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_1(\mathbf{c}, q_0), \dots$$
$$\dots, \ \mathsf{A}^\infty(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_\infty(\mathbf{c}, q_0), \ \mathsf{A}^{\mathfrak{c}}(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_{\mathfrak{c}}(\mathbf{c}, q_0). \tag{19}$$

For each $k \in \mathbb{K}$ the ideal $\mathsf{I}_k(\mathbf{c}, q_0)$ is isomorphic to the quotient

$$(\mathsf{I}_k(Q, q_0) + \mathsf{I}_0(Q, \mathbf{c}))/\mathsf{I}_0(Q, \mathbf{c}). \tag{20}$$

This justifies the following identification:

$$\mathsf{A}^k(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_k(\mathbf{c}, q_0) \quad \text{is identified with} \quad \mathsf{A}Q/(\mathsf{I}_k(Q, q_0) + \mathsf{I}_0(Q, \mathbf{c})). \tag{21}$$

The identification implies that the algebra $\mathsf{A}^k(\mathbf{c}, q_0)$ is associated with the k-jet $\mathsf{j}^k(\mathbf{c}, q_0)$ rather than with the process $(\mathbf{c}, q_0)$ since the algebra $\mathsf{A}^k(\mathbf{c}', q_0')$ is the same as the algebra $\mathsf{A}^k(\mathbf{c}, q_0)$ if $(\mathbf{c}', q_0')$ and $(\mathbf{c}, q_0)$ are in the same jet.

An element of the quotient algebra $\mathsf{A}^k(\mathbf{c}, q_0) = \mathsf{A}(\mathbf{c}, q_0)/\mathsf{I}_k(\mathbf{c}, q_0)$ is said to be *increasing* if it has an increasing representative in $\mathsf{A}(\mathbf{c}, q_0)$.

5.2. *Jets of functions on* $[0, a] \subset \mathbb{R}$. The algebra $\mathsf{A}[0, a]$ of differentiable functions on $[0, a] \subset \mathbb{R}$, its ideals

$$\mathsf{I}_0([0, a], 0), \ \ \mathsf{I}_1([0, a], 0), \ \ \ldots, \ \ \mathsf{I}_\infty([0, a], 0), \ \ \mathsf{I}_\mathfrak{c}([0, a], 0), \tag{22}$$

and the quotient algebras

$$\mathsf{A}^0([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_0([0, a], 0), \ \ \mathsf{A}^1([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_1([0, a], 0), \ldots$$
$$\ldots, \ \mathsf{A}^\infty([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_\infty([0, a], 0), \ \ \mathsf{A}^\mathfrak{c}([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_\mathfrak{c}([0, a], 0) \tag{23}$$

deserve special attention. Elements of an algebra $\mathsf{A}^k([0, a], 0)$ are k-jets of functions on $[0, a]$.

A differentiable function on $[0, a]$ is the restriction to $[0, a]$ of a differentiable function on $\mathbb{R}$. A function is in an ideal $\mathsf{I}_k([0, a], 0)$ with $k < \mathfrak{c}$ if and only if it is the restriction to $[0, a]$ of a function in $\mathsf{I}_k(\mathbb{R}, 0)$. This is not true for $k = \mathfrak{c}$. As a consequence the quotient algebra $\mathsf{A}^k([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_k([0, a], 0)$ is isomorphic to $\mathsf{A}^k(\mathbb{R}, 0) = \mathsf{A}\mathbb{R}/\mathsf{I}_k(\mathbb{R}, 0)$ for $k < \mathfrak{c}$ but not for $k = \mathfrak{c}$. In each algebra $\mathsf{A}^k([0, a], 0)$ there is a sequence of ideals

$$\mathsf{I}^k{}_0([0, a], 0) = \mathsf{I}_0([0, a], 0)/\mathsf{I}_k([0, a], 0), \ \mathsf{I}^k{}_1([0, a], 0) = \mathsf{I}_1([0, a], 0)/\mathsf{I}_k([0, a], 0), \ldots$$
$$\ldots, \ \mathsf{I}^k{}_k([0, a], 0) = \mathsf{I}_k([0, a], 0)/\mathsf{I}_k([0, a], 0). \tag{24}$$

The quotient algebra $\mathsf{A}^k([0, a], 0)/\mathsf{I}^k{}_{k'}([0, a], 0)$ with $k' \leqslant k$ is isomorphic to $\mathsf{A}^{k'}([0, a], 0)$.

A function $g \in \mathsf{A}[0, a]$ is said to be *increasing* if the inequality $s' > s$ implies $q(s') > g(s)$. An element of a quotient algebra $\mathsf{A}^\mathfrak{c}([0, a], 0) = \mathsf{A}[0, a]/\mathsf{I}_\mathfrak{c}([0, a], 0)$ is said to be *increasing* if it has an increasing representative in $\mathsf{A}[0, a]$. A function $g \in \mathsf{A}[0, a]$ is in $\mathsf{I}^\mathfrak{c}([0, a], 0)$ if and only if there is a real number $\delta > 0$ such that $g(s) = 0$ for $s \leqslant \delta$. It follows that the germ $\mathsf{j}^\mathfrak{c}g(0)$ of a function $g \in \mathsf{A}[0, a]$ is increasing if there is a number $\delta > 0$ such that g is increasing in $[0, \delta]$.

PROPOSITION 1. *For* $k \in \mathbb{N}$ *a function* $g \in \mathsf{A}[0, a]$ *is in* $\mathsf{I}_k([0, a], 0)$ *if and only if* $\mathrm{D}^i g(0) = 0$ *for each* $i \leqslant k$.

Proof. The derivatives of a function $g \in \mathsf{A}[0, a]$ at 0 are well defined and the function can be represented by the Taylor formula

$$g = e_0(g) + e_1(g)s + \ldots + e_k(g)s^k + rs^{k+1}, \tag{25}$$

where

$$e_i(g) = \frac{1}{k!}\mathrm{D}^i g(0), \tag{26}$$

$s\colon [0, a] \to \mathbb{R}$ is the canonical injection, and r is a differentiable function on $[0, a]$. The function s is in $\mathsf{I}_0([0, a], 0)$ and the power s^{k+1} is in $\mathsf{I}_k([0, a], 0)$. If $\mathrm{D}^i g(0) = 0$ for each $i \leqslant k$, then $g = rs^{k+1}$ is in $\mathsf{I}_k([0, a], 0)$.

A function $g \in \mathsf{I}_l([0, a], 0)$ is a combination of products $g_0 g_1 \cdots g_l$ of elements of $\mathsf{I}_0([0, a], 0)$. The derivative $\mathrm{D}g$ is a combination of products of functions with each product containing at least l factors in $\mathsf{I}_0([0, a], 0)$. It follows that the derivative $\mathrm{D}g$ of a function $g \in \mathsf{I}_l([0, a], 0)$ is in $\mathsf{I}_{l-1}([0, a], 0)$. If $g \in \mathsf{I}_k([0, a], 0)$, then $\mathrm{D}^0 g = g \in \mathsf{I}_k([0, a], 0)$, $\mathrm{D}^1 g = \mathrm{D}g \in \mathsf{I}_{k-1}([0, a], 0)$, $\mathrm{D}^2 g \in \mathsf{I}_{k-2}([0, a], 0)$, $\ldots$, $\mathrm{D}^k g \in \mathsf{I}_0([0, a], 0)$. Hence, $\mathrm{D}^i g(0) = 0$ for each $i \leqslant k$. ∎

A function $g \in \mathsf{A}[0, a]$ is in $\mathsf{I}^\infty([0, a], 0)$ if and only if $\mathrm{D}^i g(0) = 0$ for each $i \in \mathbb{N}$.

It follows from Proposition 1 that for $k < \mathfrak{c}$ the jet $\mathsf{j}^k g(0) \in \mathsf{A}^k([0,a],0)$ is fully represented by the sequence

$$e_0(g),\; e_1(g),\ldots,\; e_k(g) \tag{27}$$

of derivatives

$$e_i(g) = \frac{1}{k!}\mathrm{D}^i g(0) \tag{28}$$

of its representative $g \in \mathsf{A}[0,a]$. If $k \in \mathbb{N}$, then the polynomial

$$e_0(g) + e_1(g)s + \ldots + e_k(g)s^k \tag{29}$$

represents the jet $\mathsf{j}^k g(0)$ and the product of two jets is represented by the product of the corresponding polynomials truncated after the first $k + 1$ terms.

For $k < \mathfrak{c}$ an element of the ideal $\mathsf{I}^k{}_0([0,a],0) \subset \mathsf{A}^k([0,a],0)$ is represented by the sequence $e_0, e_1, \ldots, e_k$ with $e_0 = 0$. This element is said to be *positive* if the first non-zero element in the sequence is positive. The element is said to be *negative* if the first non-zero element in the sequence is negative. Each element of the ideal $\mathsf{I}^k{}_0([0,a],0)$ is either positive or negative if it is not zero. There are obvious relations $>$, $<$, $\geqslant$, and $\leqslant$ between elements of $\mathsf{I}^k{}_0([0,a],0)$.

PROPOSITION 2. *Let* $k < \mathfrak{c}$. *If a jet* $\mathsf{j}^k g(0)$ *in the ideal* $\mathsf{I}^k{}_0([0,a],0)$ *is positive, then the germ* $\mathsf{j}^\mathfrak{c} g(0)$ *is increasing. If the jet* $\mathsf{j}^k g(0)$ *is negative, then the germ* $\mathsf{j}^\mathfrak{c} g(0)$ *is decreasing.*

Proof. If $e_l(g)$ is the first non-zero element in the sequence (27) of derivatives, then the function g is represented by the Taylor formula

$$g(s) = e_l(g)s^l + r(s)s^{l+1}. \tag{30}$$

From

$$\lim_{s \to 0}((l+1)r(s)s + \mathrm{D}r(s)s^2) = 0 \tag{31}$$

it follows that there is a number $\delta > 0$ such that

$$|(l+1)r(s)s + \mathrm{D}r(s)s^2| < |le_l(g)| \tag{32}$$

for $|s| < \delta$. If $e_l(g) > 0$, then the function g is increasing in the interval $[0,\delta]$ since the derivative

$$\begin{aligned}\mathrm{D}g &= le_l s^{l-1} + (l+1)r(s)s^l + \mathrm{D}r(s)s^{l+1} \\ &= (le_l + (l+1)r(s)s + \mathrm{D}r(s)s^2)s^{l-1}\end{aligned} \tag{33}$$

is positive for $0 < s < \delta$. This is a consequence of the Lagrange mean value theorem. It follows that the germ $\mathsf{j}^\mathfrak{c} g(0)$ is increasing. It is shown in a similar way that if e_l is negative, then the germ is decreasing. ∎

It follows from the proposition that if the germ $\mathsf{j}^\mathfrak{c} g(0)$ of a function g is increasing, then the jet $\mathsf{j}^k g(0)$ is non-negative for each $k < \mathfrak{c}$.

5.3. *Conditions of equilibrium.* Let h be a function in the algebra $\mathsf{A}(\mathbf{c}, q_0)$ and let $\gamma\colon [0,a] \to Q$ be a parameterization of the process $(\mathbf{c}, q_0)$. The parameterization induces the mapping $\mathbf{c}|\gamma\colon [0,a] \to \mathbf{c}\colon s \mapsto \gamma(s)$. The composition $h \circ \mathbf{c}|\gamma$ is a function on $[0,a]$. The function h is increasing if and only if $h \circ \mathbf{c}|\gamma$ is increasing. Let h be in the ideal $\mathsf{I}_0(\mathbf{c}, q_0) \subset \mathsf{A}(\mathbf{c}, q_0)$. For $k < \mathfrak{c}$, the jet $\mathsf{j}^k h \in \mathsf{I}^k{}_0(\mathbf{c}, q_0)$ is said to be *positive* if the jet

$j^k(h \circ \mathbf{c}|\gamma)(0)$ is positive. The jet $\mathsf{j}^k h$ is said to be *negative* if the jet $j^k(h \circ \mathbf{c}|\gamma)(0)$ is negative. Being positive or negative is a property of the jet $\mathsf{j}^k h$ independent of the choice of the parameterization. The following proposition is an adaptation of Proposition 2.

PROPOSITION 3. *Let $k < \mathfrak{c}$. If a jet $\mathsf{j}^k h$ in the ideal $\mathsf{I}^k{}_0(\mathbf{c}, q_0)$ is positive, then the germ $\mathsf{j}^\mathfrak{c} h$ is increasing. If the jet $\mathsf{j}^k h$ is negative, then the germ $\mathsf{j}^\mathfrak{c} h$ is decreasing.*

The proposition implies that if the germ $\mathsf{j}^\mathfrak{c} h$ of a function h is increasing, then the jet $\mathsf{j}^k h$ is non-negative for each $k < \mathfrak{c}$.

Let q_0 be a configuration in the set C^0 an let $k \in \mathbb{N} \cup \{\infty\}$. If for each jet $j^k(\mathbf{c}, q_0)$ of an admissible process $(\mathbf{c}, q_0)$ the jet $\mathsf{j}^k w_{(\mathbf{c}, q_0)}$ is positive, then the germ $\mathsf{j}^\mathfrak{c} w_{(\mathbf{c}, q_0)}$. Hence, q_0 is a stable equilibrium configuration. We have obtained a sufficient condition for a point $q_0 \in C^0$ to be a stable local equilibrium configuration for each $k \in \mathbb{N} \cup \{\infty\}$.

If q_0 is a stable equilibrium configuration, then the germ $\mathsf{j}^\mathfrak{c} w_{(\mathbf{c}, q_0)}$ is increasing for each germ $j^\mathfrak{c}(\mathbf{c}, q_0)$ of an admissible process $(\mathbf{c}, q_0)$. It follows that for each $k \in \mathbb{N} \cup \{\infty\}$ and each jet $j^k(\mathbf{c}, q_0)$ of an admissible process $(\mathbf{c}, q_0)$ the jet $\mathsf{j}^k w_{(\mathbf{c}, q_0)}$ is non-negative. This results in a series of necessary conditions for a point $q_0 \in C^0$ to be a stable local equilibrium configuration. Neither of these conditions is sufficient. Variational principles of classical physics are based on the necessary equilibrium condition of order $k = 1$.

6. Composition of static systems and control.

6. Composition of static systems and control. Static systems can be composed if they share the same control configuration space. The equality of configuration spaces is usually the result of a suitable choice. Let two systems be characterized by work functions $W_1 \colon \mathsf{C}_1 \to \mathbb{R}$ and $W_2 \colon \mathsf{C}_2 \to \mathbb{R}$ defined on sets C_1 and C_2 of admissible processes in the same configuration manifold Q. The set $\mathsf{C} = \mathsf{C}_1 \cap \mathsf{C}_2$ is the set of admissible processes of the composed system and the work is the function $W = W_1|\mathsf{C} + W_2|\mathsf{C}$. Coupling the system to other systems with the same configuration space is a form of control. The function W is used to find the equilibrium configurations of the controlled system.

Composition of constrained systems may encounter difficulties. Examples will be given.

7. Partial control. We have considered control of static systems through interaction with systems with the same configuration space. This is not always the case. One can in general associate three distinct configuration spaces with a static system: the *internal configuration space* $\overline{Q}$, the *control configuration space* Q, and the *observed configuration space* $\widetilde{Q}$. There are differential relations connecting the three spaces. We will consider the cases when a static system with a configuration space $\widetilde{Q}$ is controlled by external devices in a configuration space $\overline{Q}$. The relation between the two spaces is a differential fibration $\eta \colon \overline{Q} \to Q$. The configuration space $\overline{Q}$ of the controlled system is the internal configuration space and the configuration space Q of the controlling devices is the control configuration space. We will refer to such situations as cases of *partial control*. The observed configuration space $\widetilde{Q}$ will coincide either with Q or with $\overline{Q}$.

Let a controlled system be represented by a work function $W \colon \mathsf{C} \to \mathbb{R}$ defined on a set of admissible processes $\mathsf{C} \subset \mathsf{P}\overline{Q}$ and let a controlling device be represented by a work function $W' \colon \mathsf{C}' \to \mathbb{R}$ defined on a set of admissible processes $\mathsf{C}' \subset \mathsf{P}Q$. An admissible

process of the combined system is a process $\mathbf{c} \in \mathsf{C}$ such that $\mathbf{c}' = \eta(\mathbf{c}) \in \mathsf{C}'$. Let $\overline{\mathsf{C}} \subset \mathsf{C}$ be the set of such processes. The work of the combined system is the function

$$\overline{W} \colon \overline{\mathsf{C}} \to \mathbb{R}, \quad \mathbf{c} \mapsto W(\mathbf{c}) + W'(\eta(\mathbf{c})). \tag{34}$$

We denote by $\mathbf{P}_V Q$ the set of processes $(\mathbf{c}, q_0)$ such that $\eta(\mathbf{c})$ is a single point set. Let C_V be the intersection $C \cap \mathbf{P}_V Q$. We introduce the concept of the *critical set*. A configuration $q_0 \in C^0$ is in the critical set $Cr(W, \eta)$ if for each germ $\mathrm{j}^{\mathfrak{c}}(\mathbf{c}, q_0) \in C_{V q_0}{}^{\mathfrak{c}}$ the germ $\mathrm{j}^{\mathfrak{c}} w_{(\mathbf{c}, q_0)}$ is increasing.

8. Realistic static systems

8.1. *Constraints determined by differential equations.* From the set $\mathsf{C} \subset \mathsf{P}Q$ of admissible processes of a static system we extract the sequence of sets

$$C^0 \subset Q, \ C^1 \subset \mathsf{P}^1 Q, \ldots, \ C^k \subset \mathsf{P}^k Q, \ldots, \ C^\infty \subset \mathsf{P}^\infty Q, \ C^{\mathfrak{c}} \subset \mathsf{P}^{\mathfrak{c}} Q. \tag{35}$$

The k-jet of a process is in C^k if it has a representative in C. The set C^0 was already introduced earlier. The relations

$$\pi^{k'}{}_{k\,Q}(C^k) = C^{k'} \tag{36}$$

hold for $k' \leqslant k$ and

$$\pi_{k\,Q}(C^k) = C^0 \tag{37}$$

for $k > 0$.

Let $(\mathbf{c}, q_0)$ be a process with the property that for each configuration $q \in (\mathbf{c}, q_0)$ with the exclusion of the terminal configuration q_1 the jet $\mathrm{j}^k(\mathbf{c}_{|q}, q)$ of the process $(\mathbf{c}_{|q}, q)$ based on the arc

$$\mathbf{c}_{|q} = \{q' \in \mathbf{c}; \ q' \geqslant q\}. \tag{38}$$

is in C^k. We denote by C^k the class of all such processes. If a process $(\mathbf{c}, q_0)$ with terminal point q_1 is in C^k, then $\mathrm{j}^k(\mathbf{c}_{|q}, q) \in C^k$ for each $q \in \mathbf{c} \setminus \{q_1\}$. It follows that $\mathrm{j}^{k'}(\mathbf{c}_{|q}, q) = \pi^{k'}{}_{k\,Q}(\mathrm{j}^k(\mathbf{c}_{|q}, q)) \in C^{k'}$ for each $q \in \mathbf{c} \setminus \{q_1\}$. Hence, $(\mathbf{c}, q_0) \in \mathsf{C}^{k'}$. We have established the inclusion

$$\mathsf{C}^{k'} \supset \mathsf{C}^k \tag{39}$$

for $k' \leqslant k$.

If $(\mathbf{c}, q_0)$ is an admissible process, then for each configuration $q \in (\mathbf{c}, q_0)$ with the exclusion of the terminal configuration q_1 the process $(\mathbf{c}_{|q}, q)$ is admissible and the germ $\mathrm{j}^{\mathfrak{c}}(\mathbf{c}_{|q}, q)$ is in $C^{\mathfrak{c}}$. It follows that $\mathsf{C} \subset \mathsf{C}^{\mathfrak{c}}$. Let the germ $\mathrm{j}^{\mathfrak{c}}(\mathbf{c}_{|q}, q)$ be in $C^{\mathfrak{c}}$ for each configuration $q \in \mathbf{c} \setminus \{q_1\}$. We show that the process $(\mathbf{c}_{q^\sharp|}, q_0)$ is admissible for each configuration $q^\sharp$ in the interior of $\mathbf{c}$. For each $q \in \mathbf{c} \setminus \{q_1\}$ there is a configuration $q' \in (\mathbf{c}, q_0)$ such that $q' > q$ and the process $(\mathbf{c}_{|qq'|}, q)$ based on the arc

$$\mathbf{c}_{|qq'|} = \{q'' \in \mathbf{c}; \ q' \geqslant q'' \geqslant q\} \tag{40}$$

is admissible. This is a consequence of $\mathrm{j}^{\mathfrak{c}}(\mathbf{c}_{|q}, q) \in C^{\mathfrak{c}}$. The union

$$\bigcup_{q \in \mathbf{c} \setminus \{q_1\}} (\mathbf{c}_{|qq'|}, q) \tag{41}$$

is an admissible process and for each $q^\sharp$ in the interior of $\mathbf{c}$ the process $(\mathbf{c}_{q^\sharp|}, q_0)$ is a subprocess of the union. Hence, $(\mathbf{c}_{q^\sharp|}, q_0)$ is admissible. The process $(\mathbf{c}, q_0)$ itself is admissible since it can be approximated by admissible processes. We have proved the inclusion $\mathsf{C}^{\mathfrak{c}} \subset \mathsf{C}$.

It may happen that the equality $\mathsf{C} = \mathsf{C}^k$ holds for $k < \mathfrak{c}$. If it holds for k and not for $k-1$, then the set C is said to represent *constraints of order k*. The set C^k is interpreted as a differential equation of order k. Admissible processes are obtained by solving this equation. Constraints of order 0 are said to be *holonomic*. Constraints of order 1 are said to be *non-holonomic*. Constraints of higher order are not usually discussed since their presence is not apparent if only first order criteria of equilibrium are considered.

8.2. *Jets of processes with volume, the work integral.* A *k-jet of a process with volume* is a pair $(\mathrm{j}^k(\mathbf{c}, q_0), v)$, where v is a vector belonging to the orientation o of $(\mathbf{c}, q_0)$ at q_0. We denote by $\mathsf{V}^k Q$ the space of k-jets with volume. For $k < \mathfrak{c}$ it makes sense to associate k-jets with volume with the terminal configuration of a process. Spaces $\mathsf{V}^0 Q$ and V^1 are identified with the bundle $\overset{\circ}{\mathsf{T}} Q \subset \mathsf{T} Q$ of tangent vectors with the zero vectors removed. This is possible since in the pair $(\mathrm{j}^0(\mathbf{c}, q_0), v)$ or $(\mathrm{j}^1(\mathbf{c}, q_0), v)$ the vector v already contains the complete information about the jet. We denote by V^k the k-jets with volume associated with admissible processes. Note that if $(\mathrm{j}^k(\mathbf{c}, q_0), v) \in V^k$, then $(\mathrm{j}^k(\mathbf{c}, q_0), \lambda v) \in V^k$ for each $\lambda > 0$. Sets V^0 and V^1 are interpreted as a subset $V \subset \overset{\circ}{\mathsf{T}} Q$.

The work of a realistic static system is defined in terms of a *work form*

$$\vartheta \colon V^k \to \mathbb{R}. \tag{42}$$

The work form is positive homogeneous in its vector argument: if $\lambda > 0$, then $\vartheta(\mathrm{j}^k(\mathbf{c}, q_0), \lambda v) = \lambda \vartheta(\mathrm{j}^k(\mathbf{c}, q_0), v)$. The work form is used to define the work function $w_{(\mathbf{c}, q_0)}$ along an admissible processes. Let q_1 be the terminal point of the process $(\mathbf{c}, q_0)$. For each configuration $q \in \mathbf{c} \setminus \{q_1\}$ the value of the work function is the integral

$$w_{(\mathbf{c}, q_0)}(q) = \int_{(\mathbf{c}|q, q_0)} \vartheta \tag{43}$$

of the work form defined as the Riemann integral

$$\int_0^{\gamma^{-1}(q)} \vartheta(\mathrm{j}^k(\mathbf{c}_{|\gamma(s)}, \gamma(s)), \mathsf{t}\gamma(s)) \mathrm{d}s \tag{44}$$

in terms of a parameterization $\gamma \colon [0, a] \to Q$. Homogeneity of ϑ makes the integral independent of the parameterization. The work of the process is the limit

$$W(\mathbf{c}, q_0) = \lim_{q \to q_0} w_{(\mathbf{c}, q_0)}(q). \tag{45}$$

The work form of a system of the usually considered type is a positive homogeneous function

$$\vartheta \colon V \to \mathbb{R}. \tag{46}$$

This corresponds to $k = 0$ or $k = 1$.

9. The principle of virtual work. Variational principles of classical physics are versions of what is known in statics as the *principle of virtual work*. This principle consists in applying the first necessary equilibrium condition to a static system. It is usually assumed that the constraints are of order 0 or 1 and that the work is represented by a work form

$$\vartheta \colon V \to \mathbb{R}. \tag{47}$$

Elements of V are the *admissible virtual displacements*. The principle of virtual work states that the inequality

$$\vartheta(v) \geqslant 0 \tag{48}$$

holds for each virtual displacement v tangent to an admissible process initiating at a configuration of equilibrium.

The principle of virtual work can be extended to more general realistic systems. If the system has constraints of order l and the work is defined in terms of a work form

$$\vartheta \colon V^k \to \mathbb{R}, \tag{49}$$

then the principle of virtual work (the first order necessary condition of equilibrium) states that the inequality

$$\vartheta(\mathsf{j}^k(\mathbf{c}, q_0), v) \geqslant 0 \tag{50}$$

holds for each k-jet with volume $(\mathsf{j}^k(\mathbf{c}, q_0), v)$ belonging to an admissible process $(\mathbf{c}, q_0)$ initiating at a configuration of equilibrium.

10. Potential systems and the Legendre transformation. A *potential* order 1 of a *potential static system* is a differentiable function U on the configuration space Q. A potential system is unconstrained. For each process $(\mathbf{c}, q_0) \in \mathsf{C} = \mathsf{P}Q$ with terminal point q_1 the function $w_{(\mathbf{c}, q_0)}$ is defined by

$$w_{(\mathbf{c}, q_0)}(q) = U(q) - U(q_0) \tag{51}$$

for $q \in \mathbf{c} \setminus \{q_1\}$ and the work $W(\mathbf{c}, q_0) = w_{(\mathbf{c}, q_0)}(q_1)$ is the limit

$$W(\mathbf{c}, q_0) = \lim_{q \to q_1} w_{(\mathbf{c}, q_0)}(q). \tag{52}$$

The set V of a potential system is the tangent bundle $\overset{\circ}{\mathsf{T}} Q$ and the function ϑ is the function

$$\vartheta \colon \overset{\circ}{\mathsf{T}} Q \to \mathbb{R}, \quad v \mapsto \langle \mathrm{d}u, v \rangle, \tag{53}$$

constructed from the differential $\mathrm{d}U$ of the potential. The principle of virtual work for a potential system states that the equality

$$\langle \mathrm{d}U, v \rangle = 0 \tag{54}$$

holds for every virtual displacement from a configuration of equilibrium. The appearance of the equality in place of the inequality is due to the *reversibility* of virtual displacements: if v is a admissible virtual displacement from a configuration of equilibrium, then so is $-v$ and $\langle \mathrm{d}U, -v \rangle = -\langle \mathrm{d}U, v \rangle$.

The *Legendre transformation* associates with a static system represented by a set $V \subset \overset{\circ}{\mathsf{T}} Q$ and a function $\vartheta \colon V \to \mathbb{R}$ the *constitutive set* S of the system defined as

$$S = \{f \in \mathsf{T}^*Q; \ \pi_Q(f) \in C^0, \vartheta(v) - \langle f, v \rangle \geqslant 0 \ \text{ for each } \ v \in V$$

$$\text{such that } \ \tau_Q(v) = \pi_Q(f)\}. \quad (55)$$

The set C^0 is obtained by applying the tangent projection τ_Q to virtual displacements. If constraints are holonomic, then $V = \overset{\circ}{\mathsf{T}} C^0$. A virtual displacement is in $\overset{\circ}{\mathsf{T}} C^0$ if there is a curve $\xi \colon \mathbb{R} \to Q$ and a number $\varepsilon > 0$ such that $\xi([0, \varepsilon]) \subset C^0$ and $v = \mathsf{t}\xi(0) \neq 0$. The inequality in the definition of S is replaced by an equality in the case of reversibility.

The physics of the Legendre transformation is that of control of a static system by means of potential external devices. The covector $f = -\mathrm{d}U(\pi_Q(f))$ is the *external force* applied to the controlled system by the potential device with potential U. The constitutive set S characterizes the response of a static system to control by potential devices only. Under certain conditions (convexity) this characterization is complete since the set V and the function ϑ can by reconstructed from the constitutive set by the *inverse Legendre transformation*.

11. Examples of static systems and Legendre transformations. Configuration spaces of most systems considered here are affine spaces. If Q is an affine space modeled on a vector space W, then the tangent bundle $\mathsf{T}Q$ is identified with the product $Q \times W$, the cotangent bundle is identified with $Q \times W^*$ and the canonical pairing is the mapping

$$\langle \ , \ \rangle \colon (Q \times W^*) \times (Q \times W) \colon ((q, f), (q, w)) \mapsto \langle f, w \rangle. \quad (56)$$

We denote by $q_1 - q_0$ the vector associated with the points q_0 and q_1.

For the sake of simplicity we will not exclude zero vectors from the set V of virtual displacements.

EXAMPLE 1. A material point with configuration q in the affine Euclidean space Q of Newtonian mechanics is tied to a fixed point q_0 with a spring of spring constant k. This is an unconstrained potential system. The internal energy is the function

$$U \colon Q \to \mathbb{R}, \quad q \mapsto \frac{k}{2}\|q - q_0\|^2. \quad (57)$$

The set

$$S = \{(q, f) \in Q \times W^*; \ f = kg(q - q_0)\} \quad (58)$$

is the constitutive set. ▲

EXAMPLE 2. Let q_0 be a point in Q and let $\omega \colon W \times W \to \mathbb{R}$ be a bilinear mapping. The function

$$\vartheta \colon Q \times W \to \mathbb{R}, \quad (q, w) \mapsto \omega(q - q_0, w), \quad (59)$$

used in the principle of virtual work produces the constitutive set

$$\begin{aligned} S &= \{(q, f) \in \mathsf{T}^*Q; \ \omega(q - q_0, v) - \langle f, v \rangle = 0 \ \text{ for each } \ v \in W\} \\ &= \{(q, f) \in \mathsf{T}^*Q; \ \omega(q - q_0, \cdot) - f = 0\}. \end{aligned} \quad (60)$$

The function ϑ is obtained from the differential of the quadratic function

$$U\colon Q \to \mathbb{R}, \quad q \mapsto \frac{1}{2}\omega(q - q_0, q - q_0), \tag{61}$$

only if ω is symmetric. ▲

EXAMPLE 3. Let Q be the configuration manifold of a static mechanical system and let $\rho\colon \mathsf{T}Q \to \mathsf{T}^*Q$ be a mapping with the properties of a Euclidean metric. This means that ρ defines an isomorphism of vector bundles

$$\begin{array}{ccc} \mathsf{T}Q & \xrightarrow{\ \rho\ } & \mathsf{T}^*Q \\ {\scriptstyle \tau_Q}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_Q} \\ Q & =\!=\!=\!= & Q \end{array} \tag{62}$$

the symmetry relation

$$\langle \rho(u), v \rangle = \langle \rho(v), u \rangle \tag{63}$$

holds for any pair $(u, v) \in \mathsf{T}Q \underset{(\tau_Q, \tau_Q)}{\times} \mathsf{T}Q$, the inequality

$$\langle \rho(v), v \rangle > 0 \tag{64}$$

holds for each $v \neq 0$. The function

$$\vartheta\colon \mathsf{T}Q \to \mathbb{R}, \quad v \mapsto \sqrt{\langle \rho(v), v \rangle}, \tag{65}$$

is positive homogeneous on fibres of the tangent fibration. It represents the virtual work of a virtual displacement v due to friction. The principle of virtual work is the inequality

$$\sqrt{\langle \rho(v), v \rangle} - \langle f, v \rangle \geqslant 0. \tag{66}$$

It is satisfied if f is in the constitutive set S and $(f, v) \in \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q$.

Let $f \in \mathsf{T}^*Q$ be in the constitutive set. By using $v = \rho^{-1}(f)$ in the principle of virtual work we arrive at the inequality

$$\langle f, \rho^{-1}(f) \rangle \leqslant \sqrt{\langle f, \rho^{-1}(f) \rangle}. \tag{67}$$

Hence,

$$\langle f, \rho^{-1}(f) \rangle \leqslant 1. \tag{68}$$

The inequality

$$\langle f, v \rangle \leqslant \sqrt{\langle f, \rho^{-1}(f) \rangle}\sqrt{\langle \rho(v), v \rangle}. \tag{69}$$

is the result of the Schwarz inequality

$$\langle \rho(u), v \rangle \leqslant \sqrt{\langle \rho(u), u \rangle}\sqrt{\langle \rho(v), v \rangle} \tag{70}$$

applied to the pair of vectors $u = \rho^{-1}(f)$ and v. If $\langle f, \rho^{-1}(f) \rangle \leqslant 1$, then

$$\langle f, v \rangle \leqslant \sqrt{\langle \rho(v), v \rangle}. \tag{71}$$

Hence, $f \in S$.

The constitutive set of the system is the set

$$S = \{f \in \mathsf{T}^*Q;\ \langle f, \rho^{-1}(f) \rangle \leqslant 1\}. \ \blacktriangle \tag{72}$$

EXAMPLE 4. Let a material point with configuration q in the Euclidean affine space Q be tied with a rigid rod of length a to a point with configuration q_0. The configuration q is constrained to the sphere

$$C^0 = \{q \in Q; \ \|q - q_0\| = a\}. \tag{73}$$

This is a system with holonomic bilateral constraints. The set

$$V = \{(q, \delta q) \in Q \times W; \ \|q - q_0\| = a, \ \langle g(q - q_0), \delta q \rangle = 0\} \tag{74}$$

of admissible virtual displacements is the tangent set $\mathsf{T}C^0$ of the holonomic constraint C^0. With the work form $\vartheta = 0$ the constitutive set is the set

$$S = \{(q, f) \in Q \times W^*; \ \|q - q_0\| = a, \ f = a^{-2}\langle f, q - q_0\rangle g(q - q_0)\}. \ \blacktriangle \tag{75}$$

EXAMPLE 5. Let q_0 be a point in the Euclidean affine space Q of dimension 3 and let u_1, u_2 and u_3 be mutually orthogonal unit vectors in the model space W. Let the configuration $q \in Q$ of a material point be constrained to the set

$$C^0 = \{q \in Q; \ \langle g(u_1), q - q_0\rangle \geqslant 0, \ \langle g(u_2), q - q_0\rangle \geqslant 0\}. \tag{76}$$

This is a system with holonomic one-sided constraints with the tangent set

$$V = \{(q, \delta q) \in Q \times W; \ \langle g(u_1), q - q_0\rangle \geqslant 0, \ \langle g(u_2), q - q_0\rangle \geqslant 0,$$
$$\langle g(u_1), \delta q\rangle \geqslant 0 \ \text{ if } \ \langle g(u_1), q - q_0\rangle = 0, \ \langle g(u_2), \delta q\rangle \geqslant 0 \ \text{ if } \ \langle g(u_2), q - q_0\rangle = 0\}. \tag{77}$$

of C^0 as the set of admissible virtual displacements. With $\vartheta = 0$ the constitutive set is the set

$$S = \{(q, f) \in Q \times W^*; \ \langle g(u_1), q - q_0\rangle \geqslant 0, \ \langle g(u_2), q - q_0\rangle \geqslant 0,$$
$$f = 0 \ \text{ if } \ \langle g(u_1), q - q_0\rangle > 0 \ \text{ and } \ \langle g(u_2), q - q_0\rangle > 0,$$
$$\langle f, u_1\rangle \leqslant 0, \ \langle f, u_2\rangle = 0, \ \text{ and } \ \langle f, u_3\rangle = 0$$
$$\text{if } \ \langle g(u_1), q - q_0\rangle = 0 \ \text{ and } \ \langle g(u_2), q - q_0\rangle > 0,$$
$$\langle f, u_1\rangle = 0, \ \langle f, u_2\rangle \leqslant 0, \ \text{ and } \ \langle f, u_3\rangle = 0$$
$$\text{if } \ \langle g(u_1), q - q_0\rangle > 0 \ \text{ and } \ \langle g(u_2), q - q_0\rangle = 0,$$
$$\langle f, u_1\rangle \leqslant 0, \ \langle f, u_2\rangle \leqslant 0, \ \text{ and } \ \langle f, u_3\rangle = 0$$
$$\text{if } \ \langle g(u_1), q - q_0\rangle = 0 \ \text{ and } \ \langle g(u_2), q - q_0\rangle = 0\}. \ \blacktriangle \tag{78}$$

EXAMPLE 6. Let $Q = X \times D$ be the configuration space of a skate. The space X is an affine plane modeled on a vector space W and D is the projective space of directions in X. We use a Euclidean metric in X represented by a mapping $g \colon W \to W^*$ to identify the space D with the unit circle

$$D = \{\varphi \in W; \ \langle g(\varphi), \varphi\rangle = 1\}. \tag{79}$$

This is the only use we make of the metric. The tangent bundle $\mathsf{T}Q$ is identified with $X \times W \times \mathsf{T}D$, where

$$\mathsf{T}D = \{(\varphi, \delta\varphi) \in D \times W; \ \langle g(\varphi), \delta\varphi\rangle = 0\}. \tag{80}$$

The skate is a system with non-holonomic constraints. The set C^0 is the entire space Q. The constraint consists in restricting virtual displacements in X to those parallel to the

direction specified by an element of D. Thus

$$V = \{(x, \delta x, \varphi, \delta\varphi) \in X \times W \times \mathsf{T}D;\ \exists_{k \in \mathbb{R}} \delta x = k\varphi\}. \tag{81}$$

The cotangent bundle T^*Q is the product $X \times W^* \times \mathsf{T}^*D$. For each $\varphi \in D$ the fibre $\mathsf{T}_\varphi D$ is a subset of W. Hence, the cotangent bundle T^*D can be specified as the set of pairs (φ, τ), where $\varphi \in D$ and τ is in the quotient space $W^*/\mathsf{T}_\varphi^\circ D$. The set

$$\begin{aligned} S &= \{(x, f, \varphi, \tau) \in \mathsf{T}^*Q;\ \langle f, \delta x\rangle + \langle\tau, \delta\varphi\rangle = 0 \ \text{ for each } \ (x, \delta x, \varphi, \delta\varphi) \in V\} \\ &= \{(x, f, \varphi, \tau) \in \mathsf{T}^*Q;\ \langle f, \varphi\rangle = 0, \tau = 0\} \end{aligned} \tag{82}$$

is the constitutive set of the system. ▲

EXAMPLE 7. Let Q be the Euclidean affine space of Newtonian mechanics. The model space for Q is a vector space W of dimension 3. The Euclidean structure is represented by a metric tensor $g\colon W \to W^*$.

The example gives a formal description of experiments performed by Coulomb in his study of static friction. Let a material point be constrained to the set

$$C^0 = \{q \in Q;\ \langle g(k), q - q_0\rangle \geqslant 0\}, \tag{83}$$

where q_0 is a point in Q and $k \in W$ is a unit vector. The boundary

$$\partial C^0 = \{q \in Q;\ \langle g(k), q - q_0\rangle = 0\} \tag{84}$$

is a plane passing through q_0 and orthogonal to k. In its displacements on the boundary the point encounters friction proportional to the component of the external force pressing the point against the boundary. The system is characterized by the virtual work function $\vartheta = 0$ defined on the non-holonomic constraints

$$\begin{aligned} V = \{(q, \delta q) \in \mathsf{T}Q;\ &\langle g(k), q - q_0\rangle \geqslant 0,\ \langle g(k), \delta q\rangle \geq \nu\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2} \\ &\text{if } \ \langle g(k), q - q_0\rangle = 0\}. \end{aligned} \tag{85}$$

The principle of virtual work states that (q, f) is in the constitutive set S if and only if the inequality

$$\langle f, \delta q\rangle \leqslant 0 \tag{86}$$

is satisfied for each $(q, \delta q) \in V$.

If $\langle g(k), q - q_0\rangle > 0$, then a pair $(q, f) \in \mathsf{T}^*Q$ is in the constitutive set S if and only if $f = 0$.

We consider pairs (q, f) with $\langle g(k), q - q_0\rangle = 0$. If $f = -\|f\|k$, then (q, f) is in the constitutive set and $\|f\|^2 - \langle f, k\rangle^2 = 0$. Let (q, f) be in the constitutive set and let $\|f\|^2 - \langle f, k\rangle^2 \neq 0$. The virtual displacement $(q, \delta q)$ with

$$\delta q = g^{-1}(f) - \langle f, k\rangle k + \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}\,k \tag{87}$$

is in V since

$$\langle g(k), \delta q\rangle = \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}. \tag{88}$$

From the principle of virtual work and

$$\langle f, \delta q\rangle = \|f\|^2 - \langle f, k\rangle^2 + \sqrt{\|f\|^2 - \langle f, k\rangle^2}\langle f, k\rangle \tag{89}$$

it follows that

$$\|f\|^2 - \langle f, k\rangle^2 + \sqrt{\|f\|^2 - \langle f, k\rangle^2}\langle f, k\rangle \leqslant 0 \tag{90}$$

and

$$\sqrt{\|f\|^2 - \langle f, k \rangle^2} + \nu \langle f, k \rangle \leqslant 0 \tag{91}$$

since $\|f\|^2 - \langle f, k \rangle^2 > 0$.

The Schwarz inequality

$$\langle g(u), v \rangle - \langle g(k), u \rangle \langle g(k), v \rangle \leqslant \sqrt{\|u\|^2 - \langle g(k), u \rangle^2} \sqrt{\|v\|^2 - \langle g(k), v \rangle^2} \tag{92}$$

for the bilinear symmetric form

$$(u, v) \mapsto \langle g(u), v \rangle - \langle g(k), u \rangle \langle g(k), v \rangle \tag{93}$$

applied to the pair $(g^{-1}(f), \delta q)$ leads to the inequality

$$\langle f, \delta q \rangle - \langle f, \delta q \rangle \langle g(k), \delta q \rangle \leqslant \sqrt{\|f\|^2 - \langle f, k \rangle^2} \sqrt{\|\delta q\|^2 - \langle g(k), \delta q \rangle^2}. \tag{94}$$

If $\langle g(k), q - q_0 \rangle = 0$, $\sqrt{\|f\|^2 - \langle f, k \rangle^2} + \nu \langle f, k \rangle \leqslant 0$, and $\langle g(k), \delta q \rangle \geq \nu \sqrt{\|\delta q\|^2 - \langle g(k), \delta q \rangle^2}$, then $\langle f, \delta q \rangle \leqslant 0$. Hence, (q, f) is in the constitutive set S.

We have shown that the set

$$S = \{(q, f) \in \mathsf{T}^*Q; \ \langle g(k), q - q_0 \rangle \geqslant 0, \ f = 0 \ \text{if} \ \langle g(k), q - q_0 \rangle > 0$$
$$\text{and} \ \sqrt{\|f\|^2 - \langle f, k \rangle^2} + \nu \langle f, k \rangle \leqslant 0 \ \text{if} \ \langle g(k), q - q_0 \rangle = 0\} \tag{95}$$

is the constitutive set of the system. ▲

12. The Legendre transformation for partially controlled systems. Let $\overline{Q}$ be the internal configuration space of a system, let Q be the control space and let $\eta \colon \overline{Q} \to Q$ be a differential fibration relating the two spaces. The work function $\vartheta \colon V \to \mathbb{R}$ is defined on a set $V \subset \mathsf{T}\overline{Q}$. The system is partially controlled by potential systems. If $\widetilde{Q} = \overline{Q}$, then

$$\overline{S} = \{(\overline{q}, f) \in \overline{Q} \underset{(\eta, \pi_Q)}{\times} \mathsf{T}^*Q; \ \overline{q} \in \tau_{\overline{Q}}(V), \ \vartheta(v) - \langle f, \mathsf{T}\eta(v) \rangle \geqslant 0$$

$$\text{for each} \ v \in V \ \text{such that} \ \tau_{\overline{Q}}(v) = \overline{q}\}. \tag{96}$$

If $\widetilde{Q} = Q$, then the constitutive set $S \subset \mathsf{T}^*Q$ is the image $pr_{\mathsf{T}^*Q}(\overline{S})$ of $\overline{S}$ by the canonical projection

$$pr_{\mathsf{T}^*Q} \colon \overline{Q} \underset{(\eta, \pi_Q)}{\times} \mathsf{T}^*Q \to \mathsf{T}^*Q. \tag{97}$$

The concept of the critical set adapted to first order equilibrium criteria is useful. The *critical set* is the set

$$Cr = \{\overline{q} \in \overline{Q}; \ \overline{q} \in \tau_{\overline{Q}}(V), \ \vartheta(v) \geq 0 \ \text{for each} \ v \in V$$

$$\text{such that} \ \tau_{\overline{Q}}(v) = \overline{q} \ \text{and} \ \eta(v) = 0\}. \tag{98}$$

13. Examples of the Legendre transformation for partially controlled systems. We give examples of partially controlled systems in the affine space Q of Newtonian mechanics modeled on a Euclidean vector space W with a metric tensor $g \colon W \to W^*$. The tangent bundle $\mathsf{T}Q$ is identified with the product $Q \times W$ and the cotangent bundle is identified with $Q \times W^*$. The tangent and the cotangent bundle of the product $Q \times Q$ are identified with $Q \times Q \times W \times W$ and $Q \times Q \times W^* \times W^*$ respectively.

EXAMPLE 8. Three material points with configurations q_0, q_1, and q_2 in the affine space Q are interconnected with springs with spring constants k_{10}, k_{20}, and k_{21}. The virtual work of the system is derived from the internal energy

$$\overline{U}: Q \times Q \times Q \to \mathbb{R}, \quad (q_0, q_1, q_2) \mapsto \frac{k_{10}}{2}\|q_1 - q_0\|^2 + \frac{k_{20}}{2}\|q_2 - q_0\|^2 + \frac{k_{21}}{2}\|q_2 - q_1\|^2, \quad (99)$$

of the springs. The point q_0 is fixed and not controlled. The configuration space is the product $Q \times Q$. The two points q_1 and q_2 are not constrained. The constitutive set is the set

$$\overline{S} = \{(q_1, q_2, f_1, f_2) \in \mathsf{T}^*(Q \times Q); \ f_1 = k_{10}g(q_1 - q_0) - k_{21}g(q_2 - q_1),$$
$$f_2 = k_{20}g(q_2 - q_0) + k_{21}g(q_2 - q_1)\}. \quad (100)$$

If the configuration of q_2 is not controlled but observed, then $Q \times Q$ is the internal configuration space, Q is the control configuration space and the canonical projection

$$pr_1: Q \times Q \to Q, \quad (q_1, q_2) \mapsto q_1, \quad (101)$$

is the relation between the two spaces. The critical set

$$Cr = \{(q_1, q_2) \in Q \times Q; \ k_{20}g(q_2 - q_0) + k_{21}g(q_2 - q_1) = 0\} \quad (102)$$

is the image of the section

$$\mu: Q \to Q \times Q, \quad q \mapsto (q, q_0 + k_{21}(k_{20} + k_{21})^{-1}(q - q_0)), \quad (103)$$

of the projection pr_1. The set

$$\widetilde{S} = \left\{ (q_1, q_2, f_1) \in Q \times Q \times W^*; \ q_2 = q_0 + \frac{k_{21}}{k_{20} + k_{21}}(q_1 - q_0), \right.$$
$$\left. f_1 = \left(k_{10} + \frac{k_{20}k_{21}}{k_{20} + k_{21}}\right)g(q_1 - q_0) \right\} \quad (104)$$

is the constitutive set.

If the configuration of q_2 is neither controlled nor observed, then the set

$$S = \left\{ (q_1, f_1) \in Q \times W^*; \ f_1 = \left(k_{10} + \frac{k_{20}k_{21}}{k_{20} + k_{21}}\right)g(q_1 - q_0) \right\} \quad (105)$$

is the constitutive set. Note that the presence of the material point with configuration q_2 can be ignored. This is due to the fact that the critical set is the image of a section of the projection (101). ▲

EXAMPLE 9. The present example gives a simplified discrete model of the buckling of a rod. One end of the rod is a point in the affine space Q with configuration q_1 constrained to the line

$$L = \{q \in Q; \ q - q_0 = \langle g(u), q - q_0 \rangle u\} \quad (106)$$

through a point q_0 in the direction of a unit vector u. The other end is a point with configuration q_2 constrained to the plane

$$P = \{q \in Q; \ \langle g(u), q - q_0 \rangle = 0\} \quad (107)$$

through q_0 perpendicular to u. The rod can be compressed or extended in length but not bent. Its relaxed length is a and the elastic constant is k. The buckling of the rod is simulated by displacements of its end point in the plane P tied elastically to the point q_0

with a spring of spring constant k'. The configuration space is the product $Q \times Q$ with holonomic constraints represented by

$$C^0 = \{(q_1, q_2) \in Q \times Q;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,\ \langle g(u), q_2 - q_0 \rangle = 0\}. \tag{108}$$

The set

$$V = \{(q_1, q_2, \delta q_1, \delta q_2) \in \mathsf{T}(Q \times Q);\ q - q_0 = \langle g(u), q_1 - q_0 \rangle u,$$
$$\langle g(u), q_2 - q_0 \rangle = 0,\ \delta q_1 = \langle g(u), \delta q_1 \rangle u,\ \langle g(u), \delta q_2 \rangle = 0\} \tag{109}$$

of admissible virtual displacements is the tangent set of C^0. The internal energy of the system is the function

$$\overline{U}: C^0 \to \mathbb{R}, \quad (q_1, q_2) \mapsto \frac{k}{2}(\|q_1 - q_2\| - a)^2 + \frac{k'}{2}\|q_2 - q_0\|. \tag{110}$$

This expression for the internal energy excludes certain non realistic configurations. The set

$$\overline{S} = \{(q_1, q_2, f_1, f_2) \in \mathsf{T}^*(Q \times Q);\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,\ \langle g(u), q_2 - q_0 \rangle = 0,$$
$$\langle f_1, u \rangle = k(1 - a\|q_2 - q_1\|^{-1})\langle g(q_1 - q_0), u \rangle,$$
$$f_2 - \langle f_2, u \rangle g(u) = (k + k' - ka\|q_2 - q_1\|^{-1})g(q_2 - q_0)\} \tag{111}$$

is the constitutive set of the system.

If the configuration q_2 is not controlled, then the critical set is the union of sets

$$Cr_1 = \{(q_1, q_2) \in Q \times Q;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,\ q_2 = q_0\} \tag{112}$$

and

$$Cr_2 = \{(q_1, q_2) \in Q \times Q;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,$$
$$\langle g(u), q_2 - q_0 \rangle = 0,\ (k + k')\|q_2 - q_1\| = ka\}. \tag{113}$$

If the configuration q_2 is not controlled but observed, then the constitutive set is the union of sets

$$\widetilde{S}_1 = \{(q_1, q_2, f_1) \in Q \times Q \times W^*;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,$$
$$q_2 = q_1,\ \langle f_1, u \rangle = k(1 - a\|q_1 - q_0\|^{-1})\langle g(q_1 - q_0), u \rangle\} \tag{114}$$

and

$$\widetilde{S}_2 = \{(q_1, q_2, f_1) \in Q \times Q \times W^*;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,\ \langle g(u), q_2 - q_0 \rangle = 0,$$
$$(k + k')\|q_2 - q_1\| = ka,\ \langle f_1, u \rangle = -k'\langle g(q_1 - q_0), u \rangle\}. \tag{115}$$

If the configuration q_2 is neither controlled nor observed, then the constitutive set is the union of sets

$$S_1 = \{(q_1, f_1) \in Q \times W^*;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,$$
$$\langle f_1, u \rangle = k(1 - a\|q_1 - q_0\|^{-1})\langle g(q_1 - q_0), u \rangle\} \tag{116}$$

and

$$S_2 = \{(q_1, f_1) \in Q \times W^*;\ q_1 - q_0 = \langle g(u), q_1 - q_0 \rangle u,$$
$$(k + k')\|q_1 - q_0\| \leqslant ka,\ \langle f_1, u \rangle = -k'\langle g(q_1 - q_0), u \rangle\}. \tag{117}$$

The constitutive set $S = S_1 \cup S_2$ is not a submanifold of T^*Q. ▲

EXAMPLE 10. A material point with configuration q_2 in the affine space Q is connected to a fixed point q_0 with a rigid rod of length a. A second material point with configuration q_1 is tied elastically to q_2 with a spring of spring constant k. The configuration space is the product $Q \times Q$ with holonomic constraints represented by

$$C^0 = \{(q_1, q_2) \in Q \times Q; \ \|q_2 - q_0\| = a\}. \tag{118}$$

The set

$$V = \{(q_1, q_2, \delta q_1, \delta q_2) \in \mathsf{T}(Q \times Q); \ \|q_2 - q_0\| = a, \ \langle g(q_2 - q_1), \delta q_2 \rangle = 0\} \tag{119}$$

is the tangent set of C^0. The set

$$\overline{S} = \{(q_1, q_2, f_1, f_2) \in \mathsf{T}^*(Q \times Q); \ \|q_2 - q_0\| = a, \ f_1 = kg(q_1 - q_2),$$
$$f_2 - kg(q_2 - q_1) = a^{-2}\langle f_2 - kg(q_2 - q_1), q_2 - q_0 \rangle g(q_2 - q_0)\}. \tag{120}$$

is the constitutive set and

$$Cr = \{(q_1, q_2) \in Q \times Q; \ \|q_2 - q_0\| = a, \|q_1 - q_0\|(q_2 - q_0) = \pm a(q_1 - q_0)\}. \tag{121}$$

is the critical set.

If the configuration q_2 is not controlled but observed, then

$$\widetilde{S} = \{(q_1, q_2, f_1) \in Q \times Q \times W^*; \ \|q_2 - q_0\| = a,$$
$$\|q_1 - q_0\|(q_2 - q_0) = \pm a(q_1 - q_0), \ f_1 = kg(q_1 - q_2)\}. \tag{122}$$

is the constitutive set.

If q_2 is not observed, then we have the constitutive set

$$S = \{(q_1, f_1) \in \mathsf{T}^*Q; \ \|f_1\| = ka \ \text{ if } \ q_1 = q_0,$$
$$f_1 = k(1 \pm a\|q_1 - q_0\|^{-1})g(q_1 - q_0) \ \text{ if } \ q_1 \neq q_0\}. \tag{123}$$

This set is the image of the injective mapping

$$\varphi: K \times \mathbb{R} \to Q \times W^*, \quad (w, r) \mapsto (q_0 + rw, k(r - a)g(w)), \tag{124}$$

where K is the unit sphere

$$K = \{w \in W; \ \|w\| = 1\}. \tag{125}$$

The space $\mathsf{T}(K \times \mathbb{R})$ is identified with the set

$$\{(w, r, \delta w, \delta r) \in W \times \mathbb{R} \times (W \oplus \mathbb{R}); \ \|w\| = 1, \ \langle g(w), \delta w \rangle = 0\} \tag{126}$$

and the space $\mathsf{T}(Q \times W^*)$ is identified with $Q \times W^* \times Q \times W^*$. The tangent mapping

$$\mathsf{T}\varphi: \mathsf{T}(K \times \mathbb{R}) \to \mathsf{T}(Q \times W^*),$$

$$(w, r, \delta w, \delta r) \mapsto (q_0 + rw, k(r - a)g(w), \delta rw + r\delta w, k\delta rg(w) + k(r - a)g(\delta w)), \tag{127}$$

maps injectively each tangent space of $K \times \mathbb{R}$ at (w, r) into the tangent space of $Q \times W^*$ at $\varphi(w, r)$. It follows that φ is an injective immersion.

The space $\mathsf{T}\mathsf{T}^*Q \underset{\mathsf{T}^*Q}{\times} \mathsf{T}\mathsf{T}^*Q$ is identified with $Q \times W^* \times (W \oplus W^*) \times (W \oplus W^*)$. The symplectic form on $Q \times W^*$ is introduced as the bilinear mapping

$$\omega_Q: Q \times W^* \times (W \oplus W^*) \times (W \oplus W^*) \to \mathbb{R},$$

$$(q, f, \delta_1 q, \delta_1 f, \delta_2 q, \delta_2 f) \mapsto \langle \delta_1 f, \delta_2 q \rangle - \langle \delta_2 f, \delta_1 q \rangle. \tag{128}$$

The product $\mathsf{T}(K \times \mathbb{R}) \underset{K \times \mathbb{R}}{\times} \mathsf{T}(K \times \mathbb{R})$ is the set

$$\{(w, r, \delta_1 w, \delta_1 r, \delta_2 w, \delta_2 r) \in W \times \mathbb{R} \times (W \oplus \mathbb{R}) \times (W \oplus \mathbb{R}); \ \|w\| = 1,$$
$$\langle g(w), \delta_1 w \rangle = 0, \ \langle g(w), \delta_2 w \rangle = 0\} \quad (129)$$

and the pull back form

$$\varphi^* \omega_Q : \mathsf{T}(K \times \mathbb{R}) \underset{K \times \mathbb{R}}{\times} \mathsf{T}(K \times \mathbb{R}) \to \mathbb{R},$$
$$(w, r, \delta_1 w, \delta_1 r, \delta_2 w, \delta_2 r) \mapsto \langle k\delta_1 rg(w) + k(r-a)g(\delta_1 w), \delta_2 rw + r\delta_2 w \rangle$$
$$- \langle k\delta_2 rg(w) + k(r-a)g(\delta_2 w), \delta_1 rw + r\delta_1 w \rangle, \quad (130)$$

is zero. We have shown that the image S of φ is an immersed Lagrangian submanifold of $\mathsf{T}^* Q$. It can be shown to be embedded. The mapping

$$\pi_Q \circ \varphi : K \times \mathbb{R} \to Q, \quad (w, r) \mapsto q_0 + rw, \quad (131)$$

is not injective. The tangent mapping

$$\mathsf{T}(\pi_Q \circ \varphi) : \mathsf{T}(K \times \mathbb{R}) \to \mathsf{T}Q, \quad (w, r, \delta w, \delta r) \mapsto (q_0 + rw, \delta rw + r\delta w), \quad (132)$$

is of rank 3 if $r \neq 0$ and of rank 1 if $r = 0$. This indicates the presence of a *Lagrangian singularity* of S above the point q_0.

Note that the critical set is not the image of a section of η. For each control configuration (x, y) we have two different internal equilibrium configurations if $x^2 + y^2 > 0$ and an infinity of internal equilibrium configurations if $x^2 + y^2 = 0$. The external force necessary to maintain the control configuration (x, y) depends on the internal configuration. Thus even if the internal configuration is not directly observed its presence can not be completely ignored. ▲

14. The Legendre transformation of composed static systems. Let two static systems in a configuration manifold Q be characterized by sets $(C^0{}_1, V_1, \vartheta_1, S_1)$ and $(C^0{}_2, V_2, \vartheta_2, S_2)$ respectively. Let the *composition* of the systems be characterized by (C^0, V, ϑ, S). The constraints C^0 and V are the intersections $C^0 = C^0{}_1 \cap C^0{}_2$ and $V = V_1 \cap V_2$. The form $\vartheta : V \to \mathbb{R}$ is the sum $\vartheta = \vartheta_1 | V + \vartheta_2 | V$. Sets $S_{1q} = S_1 \cap \mathsf{T}^*_q Q$, $S_{2q} = S_2 \cap \mathsf{T}^*_q Q$, and $S_q = S \cap \mathsf{T}^*_q Q$ are defined by

$$S_{1q} = \{f \in \mathsf{T}^*_q Q; \ f|V_{1q} = \vartheta_{1q}\}, \quad (133)$$
$$S_{2q} = \{f \in \mathsf{T}^*_q Q; \ f|V_{2q} = \vartheta_{2q}\}, \quad (134)$$
$$S_q = \{f \in \mathsf{T}^*_q Q; \ f|V_q = \vartheta_q\}. \quad (135)$$

If $f_1 \in S_{1q}$ and $f_2 \in S_{2q}$, then $f_1 + f_2 \in S_q$. This establishes the inclusion $S_{1q} + S_{2q} \subset S_q$. Assuming that sets $V_{1q} = V_1 \cap \mathsf{T}_q Q$, $V_{2q} = V_2 \cap \mathsf{T}_q Q$, and $V_q = V \cap \mathsf{T}_q Q$ are subspaces of $\mathsf{T}_q Q$ we will show that the inclusion $S_q \subset S_{1q} + S_{2q}$ holds as well. We choose complements V'_{1q} of V_q in V_{1q}, V'_{2q} of V_q in V_{2q}, and V_{0q} of $V_{1q} + V_{2q}$ in $\mathsf{T}_q Q$. Then

$$\mathsf{T}_q Q = V_{1q} + V'_{2q} + V_{0q} = V_{2q} + V'_{1q} + V_{0q}, \quad (136)$$
$$V_{1q} \cap V'_{2q} = \{0\}, \quad V'_{2q} \cap V_{0q} = \{0\}, \quad V_{0q} \cap V_{1q} = \{0\}, \quad (137)$$
$$V_{2q} \cap V'_{1q} = \{0\}, \quad V'_{1q} \cap V_{0q} = \{0\}, \quad V_{0q} \cap V_{2q} = \{0\}. \quad (138)$$

If $f \in S_q$, then f is the sum of covectors $f_1 \in S_{1q}$ and $f_2 \in S_{2q}$ characterized by

$$f_1|V_{1q} = \vartheta_{1q}, \quad f_1|V'_{2q} = f|V'_{2q} - \vartheta_{2q}|V'_{2q}, \quad f_1|V_{0q} = \tfrac{1}{2}f|V_{0q}, \tag{139}$$

$$f_2|V_{2q} = \vartheta_{2q}, \quad f_2|V'_{1q} = f|V'_{2q} - \vartheta_{1q}|V'_{1q}, \quad f_2|V_{0q} = \tfrac{1}{2}f|V_{0q}. \tag{140}$$

We have established rules of composition of static systems in sufficiently simple cases. These rules are expressed by

$$C^0 = C^0{}_1 \cap C^0{}_2, \tag{141}$$

$$V = V_1 \cap V_2, \tag{142}$$

$$\vartheta = \vartheta_1|V + \vartheta_2|V, \tag{143}$$

$$S_q = S_{1q} + S_{2q} \quad \text{for each } q \in C^0. \tag{144}$$

Two submanifolds C_1 and C_2 of a manifold Q are said to have *clean intersection* if the intersection $C_1 \cap C_2$ is a submanifold and $\mathsf{T}C_1 \cap \mathsf{T}C_2 = \mathsf{T}(C_1 \cap C_2)$. Examples show that the simple rules listed above do not apply to the composition of systems with holonomoic constraints if the intersection of constraints is not clean.

EXAMPLE 11. We consider the composition of two holonomic systems of the type described in Example 4. The constraints and the constitutive set for the first system are represented by the sets

$$C^0{}_1 = \{q \in Q;\ \|q - q_1\| = a\}, \tag{145}$$

$$V_1 = \{(q, \delta q) \in Q \times W;\ \|q - q_1\| = a,\ \langle g(q - q_1), \delta q \rangle = 0\}, \tag{146}$$

$$S_1 = \{(q, f) \in Q \times W^*;\ \|q - q_2\| = a,\ f = a^{-2}\langle f, q - q_1 \rangle g(q - q_1)\}. \tag{147}$$

For the second system we have

$$C^0{}_2 = \{q \in Q;\ \|q - q_2\| = a\}, \tag{148}$$

$$V_2 = \{(q, \delta q) \in Q \times W;\ \|q - q_2\| = a,\ \langle g(q - q_2), \delta q \rangle = 0\}, \tag{149}$$

$$S_2 = \{(q, f) \in Q \times W^*;\ \|q - q_2\| = a,\ f = a^{-2}\langle f, q - q_2 \rangle g(q - q_2)\}. \tag{150}$$

If the distance $\|q_2 - q_1\|$ between the centres of the spheres $C^0{}_1$ and $C^0{}_2$ is less than $2a$, then the composed system is a system with holonomic constraints. The intersection

$$C^0 = C^0{}_1 \cap C^0{}_2 = \{q \in Q;\ \|q - q_1\| = a,\ \|q - q_2\| = a\} \tag{151}$$

is clean. The intersection

$$V = V_1 \cap V_2 = \{(q, \delta q) \in Q \times W;\ \|q - q_1\| = a,\ \|q - q_2\| = a,$$
$$\langle g(q - q_1), \delta q \rangle = 0,\ \langle g(q - q_2), \delta q \rangle = 0\} \tag{152}$$

is the tangent set $\mathsf{T}C^0$. The constitutive set

$$S = \{(q, f) \in Q \times W^*;\ \|q - q_1\| = a,\ \|q - q_2\| = a,\ \langle f, \delta q \rangle = 0 \ \text{for each}$$
$$\delta q \in W \ \text{such that}\ \langle g(q - q_1), \delta q \rangle = 0 \ \text{and}\ \langle g(q - q_2), \delta q \rangle = 0\} \tag{153}$$

is obtained from the principle of virtual work. At each $q \in C^0$ the set

$$S_q = \{f \in W^*;\ (q, f) \in S\} \tag{154}$$

is the sum

$$\{f \in W^*;\ (q, f) \in S_1\} + \{f \in W^*;\ (q, f) \in S_2\}. \tag{155}$$

The simple rules are followed in this case.

If $\|q_2 - q_1\| = 2a$, then the set

$$C^0 = C^0{}_1 \cap C^0{}_2 = \{q \in Q; \ \|q - q_1\| = a, \ \|q - q_2\| = a\} \tag{156}$$

has only one element $q = q_1 + \frac{1}{2}(q_2 - q_1)$. The intersection $V_1 \cap V_2$ is the set

$$\{(q, \delta q) \in Q \times W; \ q = q_1 + \tfrac{1}{2}(q_2 - q_1), \ \langle g(q_2 - q_1), \delta q \rangle = 0\}. \tag{157}$$

This is not the tangent set of C^0. The intersection of $C^0{}_1$ with $C^0{}_2$ is not clean. With

$$V = \mathsf{T}C^0 = \{(q, f) \in Q \times W^*; \ q = q_1 + \tfrac{1}{2}(q_2 - q_1), \ \delta q = 0\} \tag{158}$$

the principle of virtual work produces the constitutive set

$$S = \{(q, f) \in Q \times W^*; \ q = q_1 + \tfrac{1}{2}(q_2 - q_1)\}. \tag{159}$$

This is not the correct constitutive set for the composed system. The reason of this failure is that the approximative assumption of perfect rigidity of the separate constraints is no longer realistic in the case of composition which is not clean. To obtain a complete description of the composed system the precise elastic properties of the constraints must be known. A partial characterization of the system is provided by the constitutive set

$$S = \{(q, f) \in Q \times W^*; \ q = q_1 + \tfrac{1}{2}(q_2 - q_1), \ f = a^{-2}\langle f, q - q_1 \rangle g(q - q_1)\} \tag{160}$$

generated by the non-holonomic constraints $V = V_1 \cap V_2$. Note that these constraints are not integrable since the inclusion $V \subset \mathsf{T}C^0$ does not hold. The simple composition rules cannot be followed. ▲

15. Configuration spaces for the dynamics of autonomous mechanical systems. We consider the space $\mathsf{Q}(M|\mathbb{R}; [t_0, t_1])$ of differentiable mappings $\gamma \colon [t_0, t_1] \to M$ of an interval $[t_0, t_1] \subset \mathbb{R}$ into a manifold M. This space serves as the configuration space for dynamics in the time interval $[t_0, t_1]$. It is not a differential manifold. Basic constructions of differential geometry are nevertheless possible. Tangent vectors and covectors can be defined if a class of functions and a class of curves are chosen to act as differentiable mappings.

From a function

$$\varphi \colon \mathsf{T}M \to \mathbb{R} \tag{161}$$

we derive the function

$$F \colon \mathsf{Q}(M|\mathbb{R}; [t_0, t_1]) \to \mathbb{R}, \qquad \gamma \mapsto \int_{t_0}^{t_1} \varphi \circ \mathsf{t}\gamma. \tag{162}$$

Functions on $\mathsf{Q}(M|\mathbb{R}; [t_0, t_1])$ obtained by this construction are declared differentiable.

Starting with any differentiable mapping

$$\chi \colon \mathbb{R} \times [t_0, t_1] \to M \tag{163}$$

we introduce the curve

$$\Gamma \colon \mathbb{R} \to \mathsf{Q}(M|\mathbb{R}; [t_0, t_1]), \qquad s \mapsto \chi(s, \, \cdot \,), \tag{164}$$

in $\mathsf{Q}(M|\mathbb{R}; [t_0, t_1])$ considered differentiable.

The restriction of a differentiable curve to an interval $[0, a]$ is a parameterization of a process. Virtual displacements are tangent vectors defined as equivalence classes of

differentiable curves. Curves Γ and Γ' are equivalent if

$$(F \circ \Gamma')(0) = (F \circ \Gamma)(0) \tag{165}$$

and

$$D(F \circ \Gamma')(0) = D(F \circ \Gamma)(0) \tag{166}$$

for each differentiable function F. Let Γ and Γ' be derived from mappings $\chi\colon \mathbb{R} \times [t_0, t_1] \to M$ and $\chi'\colon \mathbb{R} \times [t_0, t_1] \to M$ and let F be constructed from a mapping $\varphi\colon \mathsf{T}M \to \mathbb{R}$. Equivalence conditions involve the function

$$F \circ \Gamma\colon \mathbb{R} \to \mathbb{R}, \qquad s \mapsto \int_{t_0}^{t_1} \varphi \circ \mathsf{t}\chi(s, \cdot) \tag{167}$$

and a similar function with χ'. Equivalence conditions will be satisfied if χ and χ' produce the same mapping

$$w\colon [t_0, t_1] \to \mathsf{T}M\colon t \mapsto \mathsf{t}\chi(\cdot, t)(0). \tag{168}$$

This mapping is a natural choice of a representative of the equivalence class of Γ.

A covector is an equivalence class of pairs (q, F) of a configuration $q \in \mathsf{Q}(M|\mathbb{R}; [t_0, t_1])$ and a differentiable function F on $\mathsf{Q}(M|\mathbb{R}; [t_0, t_1])$. Pairs (q, F) and (q', F') are equivalent if $q' = q$ and

$$D(F' \circ \Gamma)(0) = D(F \circ \Gamma)(0) \tag{169}$$

for each differentiable curve Γ.

Let F, F' and Γ be derived from mappings $\varphi\colon \mathsf{T}M \to \mathbb{R}$, $\varphi\colon \mathsf{T}M \to \mathbb{R}$ and $\chi\colon \mathbb{R} \times [t_0, t_1] \to M$ respectively. It follows from the formula

$$D(F \circ \Gamma)(0) = \int_{t_0}^{t_1} \langle \mathrm{d}\varphi, \mathsf{t}w \rangle \tag{170}$$

that the mapping $\mathrm{d}\varphi \circ \mathsf{t}\chi(0, \cdot)$ could be used to represent the equivalence class of (q, F). This is not a convenient representation since $\mathsf{t}w$ is not a generic mapping from $[t_0, t_1]$ to $\mathsf{T}\mathsf{T}M$. A better representation is derived from the equality

$$D(F \circ \Gamma)(0) = -\int_{t_0}^{t_1} \langle \mathcal{E}\varphi \circ \mathsf{t}^2\chi(0, \cdot), w \rangle + \langle (\mathcal{P}\varphi \circ \mathsf{t}\chi(0, \cdot)), w(t_1) \rangle - \langle (\mathcal{P}\varphi \circ \mathsf{t}\chi(0, \cdot)), w(t_0) \rangle, \tag{171}$$

where

$$\mathcal{E}\varphi\colon \mathsf{T}^2M \to \mathsf{T}^*M \tag{172}$$

and

$$\mathcal{P}\varphi\colon \mathsf{T}M \to \mathsf{T}^*M \tag{173}$$

are mappings defined in [DA]. The equality suggests that a covector should be represented by a mapping

$$f\colon [t_0, t_1] \to \mathsf{T}^*M \tag{174}$$

and two covectors p_0 and p_1 in T^*M. The pairing between a covector represented by (f, p_0, p_1) and a vector represented by w is expressed by

$$\langle (f, p_0, p_1), w \rangle = -\int_{t_0}^{t_1} \langle f, w \rangle + \langle p_1, w(t_1) \rangle - \langle p_0, w(t_0) \rangle. \tag{175}$$

In dynamics the term *action* replaces the term *work* in statics. For simple autonomous systems usually considered in analytical mechanics all processes are admissible and the action of a process is the difference of values at the end points of the process of a function

$$L \colon \mathsf{Q}(M|\mathbb{R}; [t_0, t_1]) \to \mathbb{R}, \qquad \gamma \mapsto \int_{t_0}^{t_1} \lambda \circ \mathsf{t}\gamma, \tag{176}$$

derived from a *Lagrangian*

$$\lambda \colon \mathsf{T}M \to \mathbb{R}. \tag{177}$$

The variational principle

$$\int_{t_0}^{t_1} \langle \mathrm{d}\lambda, \mathsf{t}w \rangle = -\int_{t_0}^{t_1} \langle f, w \rangle + \langle p_1, w(t_1) \rangle - \langle p_0, w(t_0) \rangle \tag{178}$$

produces the *Euler-Lagrange equations*

$$\mathcal{E}\lambda \circ \mathsf{t}^2\gamma = f \tag{179}$$

in $[t_0, t_1]$ and the *momentum-velocity relations*

$$(\mathcal{P}\lambda \circ \mathsf{t}\gamma)(t_0) = p_0, \tag{180}$$

$$(\mathcal{P}\lambda \circ \dot{\gamma})(t_1) = p_1. \tag{181}$$

The formulation of analytical mechanics in a finite time interval has its infinitesimal limit with the interval shrinking to a point. The passage to the infinitesimal limit is greatly simplified by observing that in a slightly modified version of the formulation the interval $[t_0, t_1]$ is used exclusively as a domain of integration and can be treated as a current or distribution. Replacing the interval by the Dirac delta δ_t at $t \in \mathbb{R}$ produces the infinitesimal limit. We will describe the modified version of the formulation using currents in $\mathbb{R}$ in place of intervals.

We introduce an equivalence relation in the set of pairs $(\mathbf{c}, \gamma)$, where $\mathbf{c}$ is a current in $\mathbb{R}$ and γ is a curve in M. Pairs $(\mathbf{c}, \gamma)$ and $(\mathbf{c}', \gamma')$ are equivalent if $\mathbf{c}' = \mathbf{c}$ and

$$\int_{\mathbf{c}} \varphi \circ \mathsf{t}\gamma' = \int_{\mathbf{c}} \varphi \circ \mathsf{t}\gamma \tag{182}$$

for each differentiable function $\varphi \colon \mathsf{T}M \to \mathbb{R}$. The space $\mathsf{Q}(M|\mathbb{R}; \mathbf{c})$ of equivalence classes is the *configuration space* corresponding to the current $\mathbf{c}$. The equivalence class of a pair $(\mathbf{c}, \gamma)$ will be denoted by $\mathsf{q}(\mathbf{c}, \gamma)$.

Each differentiable function $\varphi \colon \mathsf{T}M \to \mathbb{R}$ defines a function

$$F \colon \mathsf{Q}(M|\mathbb{R}; \mathbf{c}) \to \mathbb{R}, \qquad \mathsf{q}(\mathbf{c}, \gamma) \mapsto \int_{\mathbf{c}} \varphi \circ \mathsf{t}\gamma, \tag{183}$$

considered differentiable.

From a mapping

$$\chi \colon \mathbb{R} \times \mathbb{R} \to M \tag{184}$$

we construct the curve

$$\Gamma \colon \mathbb{R} \to \mathsf{Q}(M|\mathbb{R}; \mathbf{c}), \qquad s \mapsto \mathsf{q}(\mathbf{c}, \chi(s, \cdot)), \tag{185}$$

in $\mathsf{Q}(M|\mathbb{R}; \mathbf{c})$ differentiable by declaration.

Tangent vectors are defined as equivalence classes of differentiable curves. Equalities (165) and (166) define equivalence of curves. The space of tangent vectors is denoted by $\mathsf{TQ}(M|\mathbb{R};\mathbf{c})$. A natural representative of a tangent vector is an equivalence class of pairs $(\mathbf{c}, w)$, where $\mathbf{c}$ is a current in $\mathbb{R}$ and w is a curve in $\mathsf{T}M$. Pairs $(\mathbf{c}, w)$ and $(\mathbf{c}', w')$ are equivalent if $\mathbf{c}' = \mathbf{c}$ and

$$\int_{\mathbf{c}} \langle \mathrm{d}\varphi, \mathsf{t}w' \rangle = \int_{\mathbf{c}} \langle \mathrm{d}\varphi, \mathsf{t}w \rangle. \tag{186}$$

The space of equivalence classes will be denoted by $\mathsf{Q}(\mathsf{T}M|\mathbb{R};\mathbf{c})$.

It is convenient to consider the construction of spaces $\mathsf{Q}(M|\mathbb{R};\mathbf{c})$ and $\mathsf{Q}(\mathsf{T}M|\mathbb{R};\mathbf{c})$ as the results of applications of a covariant functor $\mathsf{Q}(\,\cdot\,|\mathbb{R};\mathbf{c})$ to manifolds M and $\mathsf{T}M$. This functor applied to the morphism $\tau_M \colon \mathsf{T}M \to M$ results in the morphism

$$\mathsf{Q}(\tau_M|\mathbb{R};\mathbf{c}) \colon \mathsf{Q}(\mathsf{T}M|\mathbb{R};\mathbf{c}) \to \mathsf{Q}(M|\mathbb{R};\mathbf{c}), \qquad \mathsf{q}(\mathbf{c}, w) \mapsto \mathsf{q}(\mathbf{c}, \tau_M \circ w), \tag{187}$$

which can be treated as a vector fibration. The morphism

$$\tau_{\mathsf{Q}(M|\mathbb{R};\mathbf{c})} \colon \mathsf{TQ}(M|\mathbb{R};\mathbf{c}) \to \mathsf{Q}(M|\mathbb{R};\mathbf{c}), \qquad \mathsf{t}\Gamma(0) \mapsto \Gamma(0), \tag{188}$$

can be turned into a vector fibration by requiring that the diagram

$$\begin{array}{ccc}
\mathsf{TQ}(M|\mathbb{R};\mathbf{c}) & \xrightarrow{\ \kappa_{\mathsf{Q}(M|\mathbb{R};\mathbf{c})}\ } & \mathsf{Q}(\mathsf{T}M|\mathbb{R};\mathbf{c}) \\
{\scriptstyle \tau_{\mathsf{Q}(M|\mathbb{R};\mathbf{c})}}\downarrow & \quad {\scriptstyle \mathsf{Q}(\tau_M|\mathbb{R};\mathbf{c})}\downarrow & \\
\mathsf{Q}(M|\mathbb{R};\mathbf{c}) & =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & \mathsf{Q}(M|\mathbb{R};\mathbf{c})
\end{array} \tag{189}$$

be a vector fibration morphism with the morphism $\kappa_{\mathsf{Q}(M|\mathbb{R};\mathbf{c})}$ defined by

$$\kappa_{\mathsf{Q}(M|\mathbb{R};\mathbf{c})} \colon \mathsf{TQ}(M|\mathbb{R};\mathbf{c}) \to \mathsf{Q}(\mathsf{T}M|\mathbb{R};\mathbf{c}), \qquad \mathsf{t}\Gamma(0) \mapsto \mathsf{q}(\mathbf{c}, w), \tag{190}$$

in terms of a curve

$$\Gamma \colon \mathbb{R} \to \mathsf{Q}(M|\mathbb{R};[t_0, t_1]), \qquad s \mapsto \chi(s, \,\cdot\,), \tag{191}$$

and a curve

$$w \colon \mathbb{R} \to \mathsf{T}M, \qquad t \mapsto \mathsf{t}\chi(\,\cdot\,, t)(0), \tag{192}$$

both derived from the same mapping $\chi \colon \mathbb{R} \times \mathbb{R} \to M$. Note that $\kappa_{\mathsf{Q}(\,\cdot\,|\mathbb{R};\mathbf{c})}$ is a natural transformation from the composition $\mathsf{TQ}(\,\cdot\,|\mathbb{R};\mathbf{c})$ to the composition $\mathsf{Q}(\mathsf{T}\,\cdot\,|\mathbb{R};\mathbf{c})$ of functors T and $\mathsf{Q}(\,\cdot\,|\mathbb{R};\mathbf{c})$ composed in reverse order.

Covectors are equivalence classes of pairs (q, F) of a configuration $q \in \mathsf{Q}(M|\mathbb{R};\mathbf{c})$ and a differentiable function F on $\mathsf{Q}(M|\mathbb{R};\mathbf{c})$. Pairs (q, F) and (q', F') are equivalent if $q' = q$ and

$$\mathrm{D}(F' \circ \Gamma)(0) = \mathrm{D}(F \circ \Gamma)(0) \tag{193}$$

for each differentiable curve Γ. The space of cotangent vectors will be denoted by $\mathsf{T}^*\mathsf{Q}(M|\mathbb{R};\mathbf{c})$. The equivalence class of (q, F) is denoted by $\mathrm{d}F(q)$. The mapping $\pi_{\mathsf{Q}(M|\mathbb{R};\mathbf{c})} \colon \mathsf{T}^*\mathsf{Q}(M|\mathbb{R};\mathbf{c}) \to \mathsf{Q}(M|\mathbb{R};\mathbf{c})$ is a vector fibration. There is an alternative representation of covectors. A covector can be represented as an equivalence class of triples $(\mathbf{c}, p, f)$, where $\mathbf{c}$ is a current in $\mathbb{R}$ and p and f are curves $p \colon \mathbb{R} \to \mathsf{T}^*M$ and $f \colon \mathbb{R} \to \mathsf{T}^*M$ such that $\pi_M \circ f = \pi_M \circ p$. Triples $(\mathbf{c}, p, f)$ and $(\mathbf{c}', p', f')$ are equivalent if $\mathbf{c}' = \mathbf{c}$ and

$$-\int_{\mathbf{c}} \langle f', w \rangle + \int_{\partial \mathbf{c}} \langle p', w \rangle = -\int_{\mathbf{c}} \langle f, w \rangle + \int_{\partial \mathbf{c}} \langle p, w \rangle \tag{194}$$

for each mapping $w \colon \mathbb{R} \to \mathsf{T}M$ such that $\tau_M \circ w = \pi_M \circ f$. The space of equivalence classes will be denoted by $\mathsf{P}(\mathsf{T}^*M|\mathbb{R}; \mathbf{c})$ and $\mathsf{p}(\mathbf{c}, p, f)$ will denote the class of $(\mathbf{c}, p, f)$. The mapping

$$\mathsf{P}(\pi_M|\mathbb{R}; \mathbf{c}) \colon \mathsf{P}(\mathsf{T}^*M|\mathbb{R}; \mathbf{c}) \to \mathsf{Q}(M|\mathbb{R}; \mathbf{c}), \quad \mathsf{p}(\mathbf{c}, p, f) \mapsto \mathsf{q}(\mathbf{c}, \pi_M \circ f), \tag{195}$$

is a vector fibration dual to the fibration $\mathsf{Q}(\tau_M|\mathbb{R}; \mathbf{c})$ with the pairing

$$\langle \ , \ \rangle \colon \mathsf{P}(\mathsf{T}^*M|\mathbb{R}; \mathbf{c}) \times_{\mathsf{Q}(M|\mathbb{R}; \mathbf{c})} \mathsf{Q}(\mathsf{T}M|\mathbb{R}; \mathbf{c}) \to \mathbb{R},$$

$$(\mathsf{p}(\mathbf{c}, p, f), \mathsf{q}(\mathbf{c}, w)) \mapsto - \int_{\mathbf{c}} \langle f, w \rangle + \int_{\partial \mathbf{c}} \langle p, w \rangle. \tag{196}$$

The diagram

$$\begin{array}{ccc} \mathsf{P}(\mathsf{T}^*M|\mathbb{R}; \mathbf{c}) & \xrightarrow{\ \alpha_{\mathsf{Q}(M|\mathbb{R}; \mathbf{c})}\ } & \mathsf{T}^*\mathsf{Q}(M|\mathbb{R}; \mathbf{c}) \\ {\scriptstyle \mathsf{P}(\pi_M|\mathbb{R}; \mathbf{c})} \Big\downarrow & & \Big\downarrow {\scriptstyle \pi_{\mathsf{Q}(M|\mathbb{R}; \mathbf{c})}} \\ \mathsf{Q}(M|\mathbb{R}; \mathbf{c}) & =\!=\!=\!=\!=\!=\!= & \mathsf{Q}(M|\mathbb{R}; \mathbf{c}) \end{array} \tag{197}$$

is a vector fibration morphism dual to (189). The mapping $\alpha_{\mathsf{Q}(M|\mathbb{R}; \mathbf{c})}$ is defined by

$$\alpha_{\mathsf{Q}(M|\mathbb{R}; \mathbf{c})}(\mathsf{p}(\mathbf{c}, p, f)) = \mathrm{d}F(\mathsf{q}(\mathbf{c}, \gamma)), \tag{198}$$

where the function F and the curves p and f are obtained from the constructions

$$F \colon \mathsf{Q}(M|\mathbb{R}; \mathbf{c}) \to \mathbb{R}, \quad \mathsf{q}(\mathbf{c}, \gamma) \mapsto \int_{\mathbf{c}} \varphi \circ \mathsf{t}\gamma, \tag{199}$$

$$p = \mathcal{P}\varphi \circ \mathsf{t}^2\gamma, \tag{200}$$

$$f = \mathcal{E}\varphi \circ \mathsf{t}\gamma \tag{201}$$

in terms of a mapping $\varphi \colon \mathsf{T}M \to \mathbb{R}$ and a curve $\gamma \colon \mathbb{R} \to M$.

For a Lagrangian mechanical system the dynamics is contained in the variational principle

$$\int_{\mathbf{c}} \langle \mathrm{d}\lambda, w \rangle = - \int_{\mathbf{c}} \langle f, w \rangle + \int_{\partial \mathbf{c}} \langle p, w \rangle. \tag{202}$$

There are two important types of currents in $\mathbb{R}$: an interval $\mathbf{c} = [t_0, t_1]$ and a Dirac current $\mathbf{c} = \delta_t$. If $\mathbf{c} = [t_0, t_1]$, then the configuration space $\mathsf{Q}(M|\mathbb{R}; [t_0, t_1])$ is the space introduced earlier. If $\mathbf{c} = \delta_t$ then $\mathsf{Q}(M|\mathbb{R}; \mathbf{c}) = \mathsf{T}M$. The equivalence class of (δ_t, γ) is the velocity $\mathsf{t}\gamma(t)$. Since the space $\mathsf{Q}(M|\mathbb{R}; \delta_t)$ is a differential manifold differentiable functions and curves are well defined. The spaces $\mathsf{T}\mathsf{Q}(M|\mathbb{R}; \mathbf{c})$ and $\mathsf{Q}(\mathsf{T}M|\mathbb{R}; \mathbf{c})$ are both equal to $\mathsf{T}\mathsf{T}M$. The fibrations $\tau_{\mathsf{Q}(M|\mathbb{R}; \mathbf{c})}$ and $\mathsf{Q}(\tau_M\mathbb{R}; \mathbf{c})$ assume the form $\tau_{\mathsf{T}M} \colon \mathsf{T}\mathsf{T}M \to \mathsf{T}M$ and $\mathsf{T}\tau_M \colon \mathsf{T}\mathsf{T}M \to \mathsf{T}M$ respectively. The diagram (189) is in this case the familiar isomorphism

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}M & \xrightarrow{\ \kappa_M\ } & \mathsf{T}\mathsf{T}M \\ {\scriptstyle \tau_{\mathsf{T}M}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathsf{T}\tau_M} \\ \mathsf{T}M & =\!=\!=\!=\!=\!= & \mathsf{T}M \end{array} \tag{203}$$

of vector fibrations [DA]. The space is the cotangent bundle $\mathsf{T}^*\mathsf{T}M$. The space $\mathsf{T}^*\mathsf{Q}(M|\mathbb{R}; \mathbf{c})$ is the fibre product

$$\mathsf{T}\mathsf{T}^*M \times_M \mathsf{T}^*M = \{(\dot{p}, f) \in \mathsf{T}\mathsf{T}^*M \times \mathsf{T}^*M; \ \pi_M(f) = \pi_M(\tau_{\mathsf{T}^*M}(p))\}. \tag{204}$$

The equivalence class of (δ_t, p, f) is the pair $(\mathsf{t}p(t), f(t))$. The pairing (196) assumes the form

$$\langle\ ,\ \rangle \colon (\mathsf{TT}^*M \times_M \mathsf{T}^*M) \times_{\mathsf{T}M} \mathsf{TT}M \to \mathbb{R}, \quad ((\dot{p}, f), w) \mapsto \langle \dot{p}, w\rangle^{\mathsf{T}} - \langle f, \tau_{\mathsf{T}M}(w)\rangle. \quad (205)$$

The domain of this pairing is the set

$$\{((\dot{p}, f), w) \in (\mathsf{TT}^*M \times_M \mathsf{T}^*M) \times \mathsf{TT}M;\ \mathsf{T}\pi_M(\dot{p}) = \mathsf{T}\tau_M(w)\} \quad (206)$$

and the pairing $\langle\ ,\ \rangle^{\mathsf{T}}$ is the tangent pairing defined in [DA]. The diagram

$$
\begin{array}{ccc}
\mathsf{TT}^*M \times_M \mathsf{T}^*M & \xrightarrow{\ \alpha_M\ } & \mathsf{T}^*\mathsf{T}M \\
{\scriptstyle \mathsf{T}\pi_M}\big\downarrow & & \big\downarrow {\scriptstyle \pi_{\mathsf{T}M}} \\
\mathsf{T}M & =\!=\!=\!= & \mathsf{T}M
\end{array}
\quad (207)
$$

is a morphism dual to the vector fibration isomorphism (203). The better known version

$$
\begin{array}{ccc}
\mathsf{TT}^*M & \xrightarrow{\ \alpha_M\ } & \mathsf{T}^*\mathsf{T}M \\
{\scriptstyle \mathsf{T}\pi_M}\big\downarrow & & \big\downarrow {\scriptstyle \pi_{\mathsf{T}M}} \\
\mathsf{T}M & =\!=\!=\!= & \mathsf{T}M
\end{array}
\quad (208)
$$

of this morphism is obtained by setting $f = 0$ and using the tangent pairing

$$\langle\ ,\ \rangle \colon \mathsf{TT}^*M \times_{\mathsf{T}M} \mathsf{TT}M \to \mathbb{R}, \quad (\dot{p}, w) \mapsto \langle \dot{p}, w\rangle^{\mathsf{T}}, \quad (209)$$

in place of (205).

The infinitesimal limit of dynamics is formulated in [DA].

16. Configuration spaces for the statics of continuous media

16.1. *The configuration of a body, strain.* Let M be the *material space* and N the *physical space* of a medium. The physical space is a Euclidean affine space of dimension 3. By treating this space as a Riemannian differential manifold we simplify the presentation of the geometrical framework for statics. The material space is a differential manifold. The set of local diffeomorphisms $\gamma \colon M \to N$ each defined on an open subset $\mathrm{Dom}(\gamma) \subset M$ will be denoted by $\mathfrak{Q}(N|M)$. Elements of this set are called *placement maps*. The 1-jet prolongation $\mathsf{j}^1\gamma \colon M \to \mathsf{J}^1(N|M)$ of a placement map is an important ingredient in the definition of *strain*. This jet prolongation is equivalentlently represented by the tangent map $\mathsf{T}\gamma$. A *body* is represented by an odd current $\mathbf{c}$ of dimension $m = \dim(M)$ and compact support $\mathrm{Sup}(\mathbf{c})$. The vector space of such currents will be denoted by $\mathsf{C}M$. We consider commutative diagrams

$$
\begin{array}{ccc}
\mathsf{J}^1(N|M) & \xrightarrow{\ \kappa\ } & \tilde{\wedge}^m\mathsf{T}^*M \\
{\scriptstyle \sigma_{1(N|M)}}\big\downarrow & & \big\downarrow {\scriptstyle \tilde{\pi}_{mM}} \\
M & =\!=\!=\!= & M
\end{array}
\quad (210)
$$

where $\tilde{\pi}_{mM} \colon \tilde{\wedge}^m\mathsf{T}^*M \to M$ is the fibration of odd m-covectors in M and $\sigma_{1(N|M)} \colon \mathsf{J}^1(N|M) \to M$ is the jet-source projection. The space of such diagrams will be denoted by $\mathcal{K}(N|M)$. Each diagram is well represented by the mapping $\kappa \colon \mathsf{J}^1(N|M) \to \tilde{\wedge}^m\mathsf{T}^*M$.

Let $\mathsf{C}M \times \mathcal{Q}(N|M)$ denote the set

$$\{(\mathbf{c}, \gamma) \in \mathsf{C}M \times \mathcal{Q}(N|M); \ \mathrm{Sup}(\mathbf{c}) \subset \mathrm{Dom}(\gamma)\}. \tag{211}$$

In this *restricted product* set we introduce an equivalence relation. Pairs $(\mathbf{c}, \gamma)$ and $(\mathbf{c}', \gamma')$ are equivalent if $\mathbf{c}' = \mathbf{c}$ and

$$\int_{\mathbf{c}} \kappa \circ \mathrm{j}^1 \gamma' = \int_{\mathbf{c}} \kappa \circ \mathrm{j}^1 \gamma \tag{212}$$

for each $\kappa \in \mathcal{K}(N|M)$. The composition $\kappa \circ \mathrm{j}^1 \gamma$ can be integrated since as a section of the fibration $\tilde{\pi}_{mM} \colon \tilde{\wedge}^m \mathsf{T}^* M \to M$ it is an odd m-form on M also called a *scalar density*. The equivalence class of $(\mathbf{c}, \gamma)$ denoted by $\mathbf{q}(\mathbf{c}, \gamma)$ is a *configuration* of the body $\mathbf{c}$. The set of all equivalence classes is denoted by $\mathbf{Q}(N|M)$. There is the natural projection $\boldsymbol{\gamma}_{(N|M)} \colon \mathbf{Q}(N|M) \to \mathsf{C}M, \mathbf{q}(\mathbf{c}, \gamma) \mapsto \mathbf{c}$. A fibre $\mathbf{Q}(N|M, \mathbf{c}) = \boldsymbol{\gamma}_{(N|M)}^{-1}(\mathbf{c})$ is the *configuration space* of the body $\mathbf{c}$ and the set $\mathbf{Q}(N|M)$ is the *configuration bundle*.

16.2. *Vectors and covectors.* Each element $\kappa \in \mathcal{K}(N|M)$ induces a function

$$\boldsymbol{\kappa} \colon \mathbf{Q}(N|M) \to \mathbb{R}, \quad \mathbf{q}(\mathbf{c}, \gamma) \mapsto \int_{\mathbf{c}} \kappa \circ \mathrm{j}^1 \gamma. \tag{213}$$

Functions constructed in this way are considered *differentiable*. The space of differentiable functions on $\mathbf{Q}(N|M)$ will be denoted by $\mathbf{K}(N|M)$.

The tangent bundle of the set $\mathsf{C}M$ can be identified with the product $\mathsf{C}M \times \mathsf{C}M$ since this set is a vector space.

We consider the set $\mathcal{G}(N|M)$ of differentiable mappings $\chi \colon \mathbb{R} \times M \to N$ such that for each $s \in \mathbb{R}$ the mapping $\chi(s, \cdot)$ is a local diffeomorphism defined on an open subset $\mathrm{Dom}(\chi(s, \cdot)) \subset M$. With each element $(\mathbf{c}, \delta\mathbf{c}, \chi)$ of the *restricted product*

$$\{(\mathbf{c}, \delta\mathbf{c}, \chi) \in \mathsf{C}M \times \mathsf{C}M \times \mathcal{Q}(N|M); \ \forall_{s \in \mathbb{R}} \mathrm{Sup}(\mathbf{c} + s\delta\mathbf{c}) \subset \mathrm{Dom}(\chi(s, \cdot))\} \tag{214}$$

we associate the mapping

$$\mathbf{g}(\mathbf{c}, \delta\mathbf{c}, \chi) \colon \mathbb{R} \to \mathbf{Q}(N|M), \quad s \mapsto \mathbf{q}(\mathbf{c} + s\delta\mathbf{c}, \chi(s, \cdot)). \tag{215}$$

Mappings constructed in this way will be considered *differentiable curves*. The set of differentiable curves will be denoted by $\mathbf{G}(N|M)$.

Tangent vectors in $\mathbf{Q}(N|M)$ are defined as equivalence classes of differentiable curves. Curves $\mathbf{g}$ and $\mathbf{g}'$ are equivalent if

$$\mathbf{g}'(0) = \mathbf{g}(0) \tag{216}$$

and

$$\mathrm{D}(\boldsymbol{\kappa} \circ \mathbf{g}')(0) = \mathrm{D}(\boldsymbol{\kappa} \circ \mathbf{g})(0) \tag{217}$$

for each differentiable function $\boldsymbol{\kappa}$. The equivalence class of $\mathbf{g}$ will be denoted by $\mathbf{tg}(0)$. The space of tangent vectors will be denoted by $\mathbf{TQ}(N|M)$. It will be shown that the fibres of the projection

$$\boldsymbol{\tau}_{(N|M)} \colon \mathbf{TQ}(N|M) \to \mathbf{Q}(N|M), \quad \mathbf{tg}(0) \mapsto \mathbf{g}(0), \tag{218}$$

are vector spaces.

Covectors in $\mathbf{Q}(N|M)$ are equivalence classes of pairs $(\mathbf{q}, \boldsymbol{\kappa}) \in \mathbf{Q}(N|M) \times \mathbf{K}(N|M)$. Two pairs $(\mathbf{q}, \boldsymbol{\kappa})$ and $(\mathbf{q}', \boldsymbol{\kappa}')$ are equivalent if $\mathbf{q}' = \mathbf{q}$ and

$$\mathrm{D}(\boldsymbol{\kappa}' \circ \mathbf{g}(\mathbf{c}, \delta\mathbf{c}, \chi))(0) = \mathrm{D}(\boldsymbol{\kappa} \circ \mathbf{g}(\mathbf{c}, \delta\mathbf{c}, \chi))(0) \tag{219}$$

for each differentiable curve $\mathbf{g}\colon \mathbb{R} \to \mathbf{Q}(N|M)$ such that $\mathbf{g}(0) = \mathbf{q}$. The equivalence class of $(\mathbf{q}, \boldsymbol{\kappa})$ will be denoted by $\mathrm{d}\boldsymbol{\kappa}(\mathbf{q})$. The bundle of covectors will be denoted by $\mathbf{T}^*\mathbf{Q}(N|M)$. Fibres of the fibration

$$\boldsymbol{\pi}_{(N|M)}\colon \mathbf{T}^*\mathbf{Q}(N|M) \to \mathbf{Q}(N|M), \quad \mathrm{d}\boldsymbol{\kappa}(\mathbf{q}) \mapsto \mathbf{q}, \tag{220}$$

are vector spaces.

16.3. *Vertical vectors and covectors.* A *vertical differentiable curve* is a mapping

$$\mathbf{g}(\mathbf{c}, \chi)\colon \mathbb{R} \to \mathbf{Q}(N|M), \quad s \mapsto \mathbf{q}(\mathbf{c}, \chi(s, \cdot)), \tag{221}$$

associated with an element $(\mathbf{c}, \chi)$ of the *restricted product*

$$\{(\mathbf{c}, \chi) \in \mathsf{C}M \times \mathfrak{Q}(N|M);\ \forall_{s \in \mathbb{R}} \operatorname{Sup}(\mathbf{c}) \subset \operatorname{Dom}(\chi(s, \cdot))\}. \tag{222}$$

Vertical curves $\mathbf{g}$ and $\mathbf{g}'$ are equivalent if

$$\mathbf{g}'(0) = \mathbf{g}(0) \tag{223}$$

and

$$\mathrm{D}(\boldsymbol{\kappa} \circ \mathbf{g}')(0) = \mathrm{D}(\boldsymbol{\kappa} \circ \mathbf{g})(0) \tag{224}$$

for each differentiable function $\boldsymbol{\kappa}$. Equivalence classes are *vertical vectors*. The equivalence class of a vertical curve $\mathbf{g}$ will be denoted by $\mathbf{tg}(0)$ and the space of vertical vectors will be denoted by $\mathbf{VQ}(N|M)$. Fibres of the fibration

$$\boldsymbol{\tau}^{\mathsf{v}}_{(N|M)}\colon \mathbf{VQ}(N|M) \to \mathbf{Q}(N|M), \quad \mathbf{tg}(0) \mapsto \mathbf{g}(0), \tag{225}$$

are vector spaces.

The dual bundle is the set $\mathbf{V}^*\mathbf{Q}(N|M)$ is the set of equivalence classes of pairs $(\mathbf{q}, \boldsymbol{\kappa}) \in \mathbf{Q}(N|M) \times \mathbf{K}(N|M)$. Two pairs $(\mathbf{q}, \boldsymbol{\kappa})$ and $(\mathbf{q}', \boldsymbol{\kappa}')$ are equivalent if $\mathbf{q}' = \mathbf{q}$ and

$$\mathrm{D}(\boldsymbol{\kappa}' \circ \mathbf{g}(\mathbf{c}, \delta\mathbf{c}, \chi))(0) = \mathrm{D}(\boldsymbol{\kappa} \circ \mathbf{g}(\mathbf{c}, \delta\mathbf{c}, \chi))(0) \tag{226}$$

for each vertical differentiable curve $\mathbf{g}\colon \mathbb{R} \to \mathbf{Q}(N|M)$ such that $\mathbf{g}(0) = \mathbf{q}$. This set is isomorphic to the quotient of the bundle $\mathbf{T}^*\mathbf{Q}(N|M)$ of covectors by the polar $\mathbf{V}^\circ\mathbf{Q}(N|M)$ of the subbundle of vertical vectors. We have the fibration

$$\boldsymbol{\pi}^{\mathsf{v}}_{(N|M)}\colon \mathbf{V}^*\mathbf{Q}(N|M) \to \mathbf{Q}(N|M), \quad \mathbf{tg}(0) \mapsto \mathbf{g}(0). \tag{227}$$

16.4. *Virtual displacements, force and stress.* Let $\gamma\colon M \to N$ be a placement map. A commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\ \delta\gamma\ } & \mathsf{T}N \\
& \searrow{\scriptstyle\gamma} \quad \downarrow{\scriptstyle\tau_N} & \\
& & M
\end{array} \tag{228}$$

is called a *deformation* of γ. A deformation is well represented by the mapping $\delta\gamma$. Let $\mathcal{D}(N|M)$ be the space of deformations and let $\mathsf{C}M \times \mathcal{D}(N|M)$ denote the *restricted product* set

$$\{(\mathbf{c}, \delta\gamma) \in \mathsf{C}M \times \mathfrak{Q}(N|M);\ \operatorname{Sup}(\mathbf{c}) \subset \operatorname{Dom}(\delta\gamma)\}. \tag{229}$$

We consider commutative diagrams

$$
\begin{array}{ccc}
\mathsf{T}N & \xrightarrow{\ \varphi\ } & \tilde{\wedge}^m \mathsf{T}^* M \\
{\scriptstyle \tau_N}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}_{mM}} \\
N & \xrightarrow[\ \gamma^{-1}\ ]{} & M
\end{array}
\tag{230}
$$

and

$$
\begin{array}{ccc}
\mathsf{T}N & \xrightarrow{\ \tau\ } & \tilde{\wedge}^{m-1} \mathsf{T}^* M \\
{\scriptstyle \tau_N}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}_{m-1 M}} \\
N & \xrightarrow[\ \gamma^{-1}\ ]{} & M
\end{array}
\tag{231}
$$

where γ is a placement map and $\tilde{\pi}_{m-1 M} \colon \tilde{\wedge}^{m-1}\mathsf{T}^* M \to M$ is the fibration of odd $(m-1)$-covectors in M. The diagram (230) represented by the mapping φ is called a *force field* associated with the placement map γ. The diagram (231) is represented by the mapping τ and is called a *stress field* associated with the placement map γ. The space of force fields will be denoted by $\mathcal{F}(N|M)$ and the space of stress fields will be denoted by $\mathcal{T}(N|M)$. We will use the symbol $\mathsf{C}M \times (\mathcal{F}(N|M) \times \mathcal{T}(N|M))$ to denote the *restricted product* composed of triples $(\mathbf{c}, \varphi, \tau) \in \mathsf{C}M \times \mathcal{F}(N|M) \times \mathcal{T}(N|M)$. The force field φ and the stress field τ in a triple $(\mathbf{c}, \varphi, \tau) \in \mathsf{C}M \times (\mathcal{F}(N|M) \times \mathcal{T}(N|M))$ are associated with the same placement map γ, and $\mathrm{Sup}(\mathbf{c}) \subset \mathrm{Dom}(\gamma)$.

In the set $\mathsf{C}M \times \mathcal{D}(N|M)$ we introduce an equivalence relation. Two pairs $(\mathbf{c}, \delta\gamma)$ and $(\mathbf{c}', \delta\gamma')$ are equivalent if $\mathbf{c}' = \mathbf{c}$ and

$$
\int_{\mathbf{c}} \varphi \circ \delta\gamma' + \int_{\partial\mathbf{c}} \tau \circ \delta\gamma' = \int_{\mathbf{c}} \varphi \circ \delta\gamma + \int_{\partial\mathbf{c}} \tau \circ \delta\gamma
\tag{232}
$$

for each $(\mathbf{c}, \varphi, \tau) \in \mathsf{C}M \times \mathcal{F}(N|M) \times \mathcal{T}(N|M)$. The equivalence class of $(\mathbf{c}, \delta\gamma)$ will be called a *displacement* of the body $\mathbf{c}$ and will be denoted by $\mathbf{d}(\mathbf{c}, \delta\gamma)$. The space of all displacements will be denoted by $\mathsf{D}(N|M)$. There is a dual equivalence relation in the set $\mathsf{C}M \times (\mathcal{F}(N|M) \times \mathcal{T}(N|M))$. Triples $(\mathbf{c}, \varphi, \tau)$ and $(\mathbf{c}', \varphi', \tau')$ are equivalent if $\mathbf{c}' = \mathbf{c}$ and

$$
\int_{\mathbf{c}} \varphi' \circ \delta\gamma + \int_{\partial\mathbf{c}} \tau' \circ \delta\gamma = \int_{\mathbf{c}} \varphi \circ \delta\gamma + \int_{\partial\mathbf{c}} \tau \circ \delta\gamma
\tag{233}
$$

for each $(\mathbf{c}, \delta\gamma) \in \mathsf{C}M \times \mathcal{Q}(N|M)$. The equivalence class of $(\mathbf{c}, \varphi, \tau)$ will be called a *load* applied to the body $\mathbf{c}$ and will be denoted by $\mathsf{l}(\mathbf{c}, \varphi, \tau)$. The space of loads will be denoted by $\mathsf{L}(N|M)$.

16.5. *The Euler-Lagrange mapping.* The fibrations

$$
\begin{array}{c}
\mathsf{J}^1(N|M) \underset{(\tau_{1(N|M)}, \tau_N)}{\times} \mathsf{T}N \\[4pt]
{\scriptstyle \tau_{1(N|M)}{}^*(\tau_N)}\big\downarrow \\[4pt]
\mathsf{J}^1(N|M)
\end{array}
\tag{234}
$$

and

$$\begin{array}{c} \mathsf{J}^2(N|M) \underset{(\tau_{2(N|M)},\tau_N)}{\times} \mathsf{T}N \\ \\ {\scriptstyle \tau_{2(N|M)}{}^*(\tau_N)} \Big\downarrow \\ \\ \mathsf{J}^2(N|M) \end{array} \tag{235}$$

are the inverse images of

$$\begin{array}{c} \mathsf{T}N \\ \\ {\scriptstyle \tau_N} \Big\downarrow \\ \\ N \end{array} \tag{236}$$

under the jet-target projections $\tau_{1(N|M)}\colon \mathsf{J}^1(N|M) \to N$ and $\tau_{2(N|M)}\colon \mathsf{J}^2(N|M) \to N$ respectively. The projections $\tau_{1(N|M)}{}^*(\tau_N)$ and $\tau_{2(N|M)}{}^*(\tau_N)$ are obtained by restricting the projections

$$pr_{\mathsf{J}^1(N|M)}\colon \mathsf{J}^1(N|M) \times \mathsf{T}N \to \mathsf{J}^1(N|M) \tag{237}$$

and

$$pr_{\mathsf{J}^2(N|M)}\colon \mathsf{J}^2(N|M) \times \mathsf{T}N \to \mathsf{J}^2(N|M) \tag{238}$$

to $\mathsf{J}^1(N|M) \underset{(\tau_{1(N|M)},\tau_N)}{\times} \mathsf{T}N$ and $\mathsf{J}^2(N|M) \underset{(\tau_{2(N|M)},\tau_N)}{\times} \mathsf{T}N$ respectively. Both inverse images are vector fibrations. The fibres

$$(\tau_{1(N|M)}{}^*(\tau_N))^{-1}(\mathsf{j}^1\gamma(x)) \quad \text{and} \quad (\tau_{2(N|M)}{}^*(\tau_N))^{-1}(\mathsf{j}^2\gamma(x))$$

are both isomorphic to the fibre $\tau_N{}^{-1}(\gamma(x))$.

The *Euler-Lagrange mapping* is a morphism

$$\begin{array}{ccc} \mathsf{J}^2(N|M) \underset{(\tau_{2(N|M)},\tau_N)}{\times} \mathsf{T}N & \xrightarrow{\ \mathcal{E}\mathrm{d}_V u\ } & \tilde{\wedge}^m \mathsf{T}^*M \\ {\scriptstyle \tau_{2(N|M)}{}^*(\tau_N)} \Big\downarrow & & \Big\downarrow {\scriptstyle \tilde{\pi}_{mM}} \\ \mathsf{J}^2(N|M) & \xrightarrow{\ \sigma_{2(N|M)}\ } & M \end{array} \tag{239}$$

of vector fibrations. The *stress-strain mapping* is a morphism

$$\begin{array}{ccc} \mathsf{J}^1(N|M) \underset{(\tau_{1(N|M)},\tau_N)}{\times} \mathsf{T}N & \xrightarrow{\ \mathcal{P}\mathrm{d}_V u\ } & \tilde{\wedge}^{m-1} \mathsf{T}^*M \\ {\scriptstyle \tau_{1(N|M)}{}^*(\tau_N)} \Big\downarrow & & \Big\downarrow {\scriptstyle \tilde{\pi}_{m-1M}} \\ \mathsf{J}^1(N|M) & \xrightarrow{\ \sigma_{1(N|M)}\ } & M \end{array} \tag{240}$$

of vector fibrations.

A *vertical curve* in $\mathsf{J}^1(N|M)$ is a mapping $\gamma\colon \mathbb{R} \to \mathsf{J}^1(N|M)$ such that the composition $\sigma_{1(N|M)} \circ \gamma$ is a constant mapping. The image of the composition $\kappa \circ \gamma$ of a vertical curve γ with a mapping κ in the commutative diagram (210) is included in the vector space $(\tilde{\pi}_{mM})^{-1}(\sigma_{1(N|M)}(\gamma(0)) \subset R^m M$. The derivative $\mathrm{D}(\kappa \circ \gamma)(0)$ is well defined. The *vertical differential* $\mathrm{d}_V \kappa\colon \mathsf{V}\mathsf{J}^1(N|M) \to R^m M$ of the mapping κ is defined in terms of the

directional derivative $\mathrm{d}_V \kappa(\mathsf{t}\gamma(0)) = \mathrm{D}(\kappa \circ \gamma)(0)$. The diagram

$$
\begin{array}{ccc}
\mathsf{V}\mathsf{J}^1(N|M) & \xrightarrow{\ \ \mathrm{d}_V\kappa\ \ } & \tilde{\wedge}^m\mathsf{T}^*M \\
{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^1(N|M)}}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_{mM}} \\
\mathsf{J}^1(N|M) & \xrightarrow{\ \sigma_{1(N|M)}\ } & M
\end{array}
\tag{241}
$$

is a morphism of vector fibrations.

With a function $a\colon M \to \mathbb{R}$ we associate the diagram

$$
\begin{array}{ccc}
\mathsf{V}\mathsf{J}^k(N|M) & \xrightarrow{\ \ F(a)\ \ } & \mathsf{V}\mathsf{J}^k(N|M) \\
{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^k(N|M)}}\downarrow & & \downarrow{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^k(N|M)}} \\
\mathsf{J}^k(N|M) & =\!=\!=\!=\!= & \mathsf{J}^k(N|M)
\end{array}
\tag{242}
$$

The first row is the mapping

$$
F(a)\colon \mathsf{V}\mathsf{J}^k(N|M) \to \mathsf{V}\mathsf{J}^k(N|M), \quad \mathsf{j}^{1,k}\chi(0,x_0) \mapsto \mathsf{j}^{1,k}\chi^a(0,x_0),
\tag{243}
$$

with the mapping $\chi^a\colon \mathbb{R} \times M \to N$ defined by $\chi^a(s,x) = \chi(s(a(x)-a(x_0)),x)$.

The diagram (242) acts on a diagram

$$
\begin{array}{ccc}
\mathsf{V}\mathsf{J}^k(N|M) & \xrightarrow{\ \ \mu\ \ } & \tilde{\wedge}^m\mathsf{T}^*M \\
{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^k(N|M)}}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_{mM}} \\
\mathsf{J}^1(N|M) & \xrightarrow{\ \sigma_{1(N|M)}\ } & M
\end{array}
\tag{244}
$$

producing the diagram

$$
\begin{array}{ccc}
\mathsf{V}\mathsf{J}^k(N|M) & \xrightarrow{\ \ \mathrm{i}_{F(a)}\mu\ \ } & \tilde{\wedge}^m\mathsf{T}^*M \\
{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^k(N|M)}}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_{mM}} \\
\mathsf{J}^1(N|M) & \xrightarrow{\ \sigma_{1(N|M)}\ } & M
\end{array}
\tag{245}
$$

where the mapping $\mathrm{i}_{F(a)}\mu$ is defined by

$$
\mathrm{i}_{F(a)}\mu(w) = \mu(F(a)(w)).
\tag{246}
$$

The inverse image

$$
\begin{array}{c}
\mathsf{J}^1(N|M) \underset{(\sigma_{1(N|M)},\pi_M)}{\times} \mathsf{T}^*M \\
{\scriptstyle \sigma_{1(N|M)}{}^*(\pi_M)}\Big\downarrow \\
\mathsf{J}^1(N|M)
\end{array}
\tag{247}
$$

diagram is a vector fibration and the mapping

$$
\mathrm{i}_{F(\cdot)}\mu\colon \Big(\mathsf{J}^1(N|M) \underset{(\sigma_{1(N|M)},\pi_M)}{\times} \mathsf{T}^*M\Big) \times_{\mathsf{J}^1(N|M)} \mathsf{V}\mathsf{J}^1(N|M) \to \tilde{\wedge}^m\mathsf{T}^*M,
$$

$$
(\mathrm{d}a(x),w) \mapsto \mathrm{i}_{F(a)}\mu(w),
\tag{248}
$$

is bilinear. It induces a morphism

$$
\begin{array}{ccc}
\mathsf{V}\mathsf{J}^1(N|M) & \xrightarrow{\;\;\widetilde{\mathrm{i}_F\mu}\;\;} & \mathsf{T}M \otimes \tilde{\wedge}^m\mathsf{T}^*M \\
{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^1(N|M)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\
\mathsf{J}^1(N|M) & \xrightarrow{\;\;\sigma_{1(N|M)}\;\;} & M
\end{array}
\tag{249}
$$

of vector fibrations. The mapping $\widetilde{\mathrm{i}_F\mu}$ in the upper row is characterized by

$$
\langle \mathrm{d}a(x) \otimes \tilde{c}, \widetilde{\mathrm{i}_F\mu}(w)\rangle = \langle \tilde{c}, \mathrm{i}_{F(a)}\mu(w)\rangle
\tag{250}
$$

with $a\colon M \to \mathbb{R}$, $x \in M$, $w \in \mathsf{T}\mathsf{J}^1(N|M,x) \subset \mathsf{V}\mathsf{J}^1(N|M)$, and $\tilde{c} \in \tilde{\wedge}^m\mathsf{T}_xM$. We are using in (248) the symbol $\times_{\mathsf{J}^1(N|M)}$ for the fibre product of two vector fibrations. The symbol π in (249) has an obvious meaning. The diagram

$$
\begin{array}{ccc}
\mathsf{V}\mathsf{J}^1(N|M) & \xrightarrow{\;\;\mathrm{i}_F\mu\;\;} & \tilde{\wedge}^{m-1}\mathsf{T}^*M \\
{\scriptstyle \tau^{\mathsf{v}}_{\mathsf{J}^1(N|M)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\
\mathsf{J}^1(N|M) & \xrightarrow{\;\;\sigma_{1(N|M)}\;\;} & M
\end{array}
\tag{251}
$$

is obtained from the diagram (249) with the use of the isomorphism of $\mathsf{T}M \otimes \tilde{\wedge}^m\mathsf{T}^*M$ with $\tilde{\wedge}^{m-1}\mathsf{T}^*M$.

The mapping $\mathrm{i}_F\mu$ is vertical with respect to the projection $\tau_{1(N|M)}$ in the sense that if $\mathsf{T}\tau_{1(N|M)}(w) = 0$, then $\mathrm{i}_F\mu(w) = 0$. This property makes it possible to construct a mapping

$$
\mathcal{P}\mu\colon \mathsf{J}^1(N|M) \underset{(\tau_{1(N|M)},\tau_N)}{\times} \mathsf{T}N \to \tilde{\wedge}^{m-1}\mathsf{T}^*M
\tag{252}
$$

characterized by $\mathcal{P}\mu(j, v) = \mathrm{i}_F\mu(w)$, where w is a vertical vector such that $\tau^{\mathsf{v}}_{\mathsf{J}^1(N|M)}(w) = j$ and $\mathsf{T}\tau_{1(N|M)}(w) = v$. The diagram

$$
\begin{array}{ccc}
\mathsf{J}^1(N|M) \underset{(\tau_{1(N|M)},\tau_N)}{\times} \mathsf{T}N & \xrightarrow{\;\;\mathcal{P}\mu\;\;} & \tilde{\wedge}^{m-1}\mathsf{T}^*M \\
{\scriptstyle \tau_{1(N|M)}{}^*(\tau_N)}\Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{\pi}_{m-1M}} \\
\mathsf{J}^1(N|M) & \xrightarrow{\;\;\sigma_{1(N|M)}\;\;} & M
\end{array}
\tag{253}
$$

is a morphism of vector fibrations.

Let $\delta\gamma\colon M \to \mathsf{T}N$ be a deformation of a placement map $\gamma\colon M \to N$. The jet prolongations $\mathrm{j}^1\delta\gamma\colon M \to \mathsf{J}^1(\mathsf{T}N|M)$ and $\mathrm{j}^2\delta\gamma\colon M \to \mathsf{J}^2(\mathsf{T}N|M)$ are transformed in deformations $\delta\mathrm{j}^1\gamma = \kappa^{1,1}{}_{(N|M)} \circ \mathrm{j}^1\delta\gamma$ and $\delta\mathrm{j}^2\gamma = \kappa^{2,1}{}_{(N|M)} \circ \mathrm{j}^2\delta\gamma$ of the jet prolongations $\mathrm{j}^1\gamma$ and $\mathrm{j}^2\gamma$ respectively. Mappings γ, $\delta\gamma$, $\delta\mathrm{j}^1\gamma$, and $\delta\mathrm{j}^2\gamma$ can be constructed from a mapping $\chi\colon \mathbb{R} \times M \to N$ by the following definitions:

$$
\gamma\colon M \to N, \qquad x \mapsto \chi(0, x),
\tag{254}
$$

$$
\delta\gamma\colon M \to \mathsf{T}N, \qquad x \mapsto \mathsf{t}\chi(\cdot, x)(0),
\tag{255}
$$

$$
\delta\mathrm{j}^1\gamma\colon M \to \mathsf{V}\mathsf{J}^1(N|M), \qquad x \mapsto \mathrm{j}^{1,1}\chi(0, x),
\tag{256}
$$

$$
\delta\mathrm{j}^2\gamma\colon M \to \mathsf{V}\mathsf{J}^2(N|M), \qquad x \mapsto \mathrm{j}^{1,2}\chi(0, x),
\tag{257}
$$

The *total differential* is a differential operator applied to linear morphisms from $\mathsf{V}\mathsf{J}^1(N|M)$ to $\tilde{\wedge}^{m-1}\mathsf{T}^*M$. The total differential of a morphism $\nu\colon\mathsf{V}\mathsf{J}^1(N|M)\to\tilde{\wedge}^{m-1}\mathsf{T}^*M$ is a linear morphism $\mathrm{d}_T\nu\colon\mathsf{V}\mathsf{J}^1(N|M)\to\tilde{\wedge}^{m-1}\mathsf{T}^*M$ characterized by $\mathrm{d}_T\nu\circ\delta\mathsf{j}^2\gamma=\mathrm{d}(\nu\circ\delta\mathsf{j}^1\gamma)$ for each deformation $\delta\gamma$ of a placement γ.

16.6. *The internal energy.* The *internal energy density* of a material is represented by a mapping $u\colon\mathsf{J}^1(N|M)\to\tilde{\wedge}^m\mathsf{T}^*M$. The *internal energy* of a configuration $\mathbf{q}(\mathbf{c},\gamma)$ of a body $\mathbf{c}$ is the integral

$$U(\mathbf{q}(\mathbf{c},\gamma))=\int_{\mathbf{c}}u\circ\mathsf{j}^1\gamma.\tag{258}$$

The internal energy is a function on $\mathbf{Q}(N|M)$.

Variational principles of statics require the definition of the internal energy of a configuration. The form of the internal energy varies with the type of the medium. The study of the variational principles for different types of media is under way.

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