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INTRODUCTION TO COCHAINS OF DIFFERENTIAL OPERATORS

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To Professor W. M. Tulczyjew on the occasion of his 70th birthday

I first heard a lecture by Włodek Tulczyjew at Aix en Provence in 1979, and I remember it. I had been struggling for some time to grasp the basics of spectral sequences in order to understand the topological approach to the statistics of identical particles. Here spoke a physicist who was completely on top of the technology—using it to place the Euler-Lagrange operator in an exact sequence, to provide a test of whether or not a given set of differential equations arose from a Lagrangian. And he even said that this was not his main research interest, but something he did in odd moments! So I was impressed. Many happy returns.

Fred Bloore

1. Introduction. The modern treatment of mechanics, initiated and developed by Tulczyjew and others, uses the differential geometry associated with the tangent and cotangent bundles of the configuration space Q of the mechanical system. Its tools are the vector fields and the differential forms dual to them together with their exterior derivatives.

This theory has a rich structure and enjoys great success in describing mechanical systems. So it is not surprising that less attention has been paid to the fact that, since vector fields may be viewed as first order linear differential operators on the set $\mathcal{F} = C^\infty(Q, \mathbf{C})$ of smooth functions on Q, there may be some reward for studying the linear differential operators of higher order by means of their cochains, in the same spirit that we study the vector fields through their differential forms. (We choose $\mathbf{C}$ rather than $\mathbf{R}$ because one application (section 6.1) requires quantum mechanical wave functions to lie in $\mathcal{F}$.)

In this contribution we offer a start on this study. (It's not our first paper on the stuff but we hope it is a gateway into the field.) There is also a well developed jet bundle theory

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of systems of partial differential equations begun by Cartan, Kaehler, Janet, Riquier, Spencer with which the present material is not yet linked.

We shall study the associative algebra $\mathcal{D}$ of linear differential operators on $\mathcal{F}$ by exploring the properties of the Hochschild complex of $\mathcal{D}$-valued cochains on $\mathcal{D}$. This complex has two interesting subcomplexes, the complex of $\mathcal{F}$-tensorial cochains and its normalised subcomplex, the $\mathcal{F}$-relative cochains. These latter are our principal object of concern. When restricted to act on vector fields they reduce to covariant tensor fields. They may be regarded as the natural extension of covariant tensor fields (and of differential forms in particular) to act on differential operators of order higher than one.

Section 2 contains the definitions and basic properties. The following section gives a local characterisation of $\mathcal{F}$-tensorial cochains in terms of their "structure functions". In section 4 we sketch out a useful isomorphism of differential graded algebras which relates an $\mathcal{F}$-tensorial p-cochain on $\mathcal{D}$ to the jet of a smooth function on Q^{p+1} taken on the diagonal. We take the small liberty of calling the complex of these jets the Alexander-Spanier complex—the two notions are so similar. In section 5 we show a rather natural way to enlist the aid of a connection on TQ to extend the domain of a differential form to include all $\mathcal{D}$. Some possible uses of these cochains are suggested in Section 6.

2. Cochains on $\mathcal{D}$. We first define a general $\mathcal{D}$-valued Hochschild cochain, then specialise to an $\mathcal{F}$-tensorial cochain. Our main objects of study, the $\mathcal{F}$-relative cochains, are simply the normalised $\mathcal{F}$-tensorial ones. Each type has a differential algebra using the Hochschild differential δ and the cup product $\cup$. We define also the composition $\circ$ of a 1-cochain with a p-cochain.

When an $\mathcal{F}$-relative p-cochain is restricted to act only on vector fields it reduces to a covariant tensor field whose anti-symmetric part is a de Rham differential p-form. The consequent map a from $\mathcal{F}$-relative cochains on $\mathcal{D}$ to de Rham differential forms respects the differential structure. This motivates the investigation of how much of the existing structure [CP] of differential forms and vector fields can be carried back to Hochschild cochains and linear differential operators.

We define, for any linear differential operator K, the interior product $\iota_K A$ and the Lie derivative $L_K A$ of a general Hochschild cochain A on $\mathcal{D}$. We then show, mainly by direct calculation, that almost all the classical properties of de Rham forms in regard to ι_X and L_X, $X \in \mathcal{X}(Q)$, are mirrored by properties of cochains in regard to ι_K and L_K. These include the Cartan homotopy relation

$$d\iota_X\omega + \iota_X d\omega = L_X\omega.$$

2.1. *Hochschild p-cochains on $\mathcal{D}$*

Definition. A *Hochschild p-cochain* on $\mathcal{D}$ is a $\mathbf{C}$-linear map

$$A : \mathcal{D} \otimes \mathcal{D} \otimes \ldots \otimes \mathcal{D} \to \mathcal{D}$$

with p factors in the tensor product. Here (and throughout) $\otimes$ will denote $\otimes_{\mathbf{C}}$. A 0-cochain is an element of $\mathcal{D}$.

The set of Hochschild p-cochains on $\mathcal{D}$ will be denoted by $C^p(\mathcal{D}, \mathcal{D})$.

Definition [Ho1]. For $A \in C^p(\mathcal{D},\mathcal{D})$ the *Hochschild differential* δA is the $(p+1)$-cochain defined, for $H_i \in \mathcal{D}$, by

$$(\delta A)(H_1,\ldots,H_{p+1}) = H_1 A(H_2,\ldots,H_{p+1}) - A(H_1H_2,\ldots,H_{p+1}) \\ + \ldots + (-1)^p A(H_1,\ldots,H_pH_{p+1}) + (-1)^{p+1} A(H_1,\ldots,H_p)H_{p+1}. \tag{1}$$

So in particular, for $H \in C^0(\mathcal{D},\mathcal{D})$, $K \in \mathcal{D}$ $(= C^0(\mathcal{D},\mathcal{D}))$,

$$\delta H(K) = [K,H] = KH - HK.$$

Definition. For $A \in C^p(\mathcal{D},\mathcal{D}), B \in C^q(\mathcal{D},\mathcal{D})$, the *cup product* $A \cup B \in C^{p+q}(\mathcal{D},\mathcal{D})$ is defined by

$$(A \cup B)(H_1,\ldots,H_{p+q}) = A(H_1,\ldots,H_p) \circ B(H_{p+1},\ldots,H_{p+q}) \tag{2}$$

where $\circ$ is the composition of differential operators.

It follows from the definitions that

$$\delta(A \cup B) = \delta A \cup B + (-1)^p A \cup \delta B.$$

Note that for $H_1, H_2 \in C^0(\mathcal{D},\mathcal{D}) = \mathcal{D}$,

$$H_1 \cup H_2 = H_1 H_2.$$

For the particular case of $A \in C^1(\mathcal{D},\mathcal{D})$ and $B \in C^p(\mathcal{D},\mathcal{D})$ we also have the composition $A \circ B \in C^p(\mathcal{D},\mathcal{D})$:

$$(A \circ B)(H_1,\ldots,H_p) = A(B(H_1,\ldots,H_p)). \tag{3}$$

Definition. For $p \geq 1$, $A \in C^p(\mathcal{D},\mathcal{D})$ is called *normalised* if $A(H_1,\ldots,H_p) = 0$ whenever one or more H_j is the identity operator in $\mathcal{D}$.

If A and B are normalised so are δA, $A \cup B$ and, for $A \in C^1(\mathcal{D},\mathcal{D})$, $A \circ B$ is normalised whenever B is normalised.

From now on we shall simply write the word cochains for Hochschild cochains on $\mathcal{D}$ and assume that all differential operators mentioned are linear.

2.2. *$\mathcal{F}$-tensorial p-cochains*

Definition. A cochain $A \in C^p(\mathcal{D},\mathcal{D})$ is *$\mathcal{F}$-tensorial* if for $H_i \in \mathcal{D}, f_j \in \mathcal{F}$,

$$A(f_1H_1, f_2H_2, f_3H_3,\ldots,f_pH_pf_{p+1}) = f_1 \circ A(H_1 \circ f_2, H_2 \circ f_3,\ldots,H_p) \circ f_{p+1} \tag{4}$$

i.e. the f_j can jump between the H_j but not through them (except when $H_j \in \mathcal{F}$). The set of $\mathcal{F}$-tensorial p-cochains is denoted by $C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$.

The algebra $\mathcal{D}$ is filtered by order; we shall denote by $\mathcal{D}_p$ the subset of operators with order p or less.

An $\mathcal{F}$-tensorial 0-cochain is taken to be an element of $\mathcal{F} = \mathcal{D}_0 \subset \mathcal{D}$, the commutative subalgebra of differential operators of zero order. This is to ensure that δ preserves the $\mathcal{F}$-tensorial property: For $H \in C^0(\mathcal{D},\mathcal{D}) = \mathcal{D}$, we note that $\delta H \in C^1_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ only if $\delta H(fK) = f\delta H(K)$ for all $f \in \mathcal{F}$, $K \in \mathcal{D}$. But this implies $[H,f] = 0$, so we need $H \in \mathcal{F}$.

Definition. A p-cochain is said to be *$\mathcal{F}$-relative* [Ho2] if it is both $\mathcal{F}$-tensorial and normalised.

It follows that if A is $\mathcal{F}$-relative then

$$A(H_1, \ldots, H_p) = 0$$

whenever one or more $H_j \in \mathcal{F}$. The set of $\mathcal{F}$-relative cochains is denoted by $C^p(\mathcal{D}, \mathcal{F}; \mathcal{D})$. And $C^0(\mathcal{D}, \mathcal{F}; \mathcal{D}) = C^0_{\mathcal{F}}(\mathcal{D}, \mathcal{D}) = \mathcal{F}$.

If A and B are $\mathcal{F}$-tensorial ($\mathcal{F}$-relative) then so are δA, $A \cup B$ and, when defined, $A \circ B$.

We shall see in section 3 that the composition product of $\mathcal{F}$-tensorial 1-cochains is commutative.

2.3. *The restriction of $\mathcal{F}$-relative cochains to vector fields.* Consider an $\mathcal{F}$-relative 2-cochain A acting on vector fields X_1, X_2:

$$A(X_1, X_2) \in \mathcal{D}.$$

For $f \in \mathcal{D}_0 = \mathcal{F}$,

$$\begin{aligned}
A(X_1, X_2) \circ f &= A(X_1, X_2 \circ f) && \text{by } \mathcal{F}\text{-tensoriality} \\
&= A(X_1, [X_2, f] + fX_2) && \\
&= A(X_1, fX_2) && \text{by normalisation and linearity} \\
&= A(X_1 \circ f, X_2) && \text{by } \mathcal{F}\text{-tensoriality} \\
&= A([X_1, f] + fX_1, X_2) && \\
&= A(fX_1, X_2) && \text{by normalisation} \\
&= f \circ A(X_1, X_2) && \text{by } \mathcal{F}\text{-tensoriality.}
\end{aligned}$$

So

$$A(X_1, X_2) \circ f - f \circ A(X_1, X_2) = [A(X_1, X_2), f] = 0,$$

and therefore

$$A(X_1, X_2) \in \mathcal{F}.$$

A similar argument yields

$$A(f_1 X_1, f_2 X_2) = f_1 f_2 A(X_1, X_2),$$

telling us that $A|_{\mathcal{X}}$ is a $(0, 2)$ tensor field, and indeed for $A \in C^p(\mathcal{D}, \mathcal{F}; \mathcal{D})$, we have that $A|_{\mathcal{X}}$ is a $(0, p)$ tensor field.

Let $\operatorname{ord} H$ denote the order of the differential operator H. One can show [HB] that for $A \in C^p(\mathcal{D}, \mathcal{F}; \mathcal{D})$ and $H_j \in \mathcal{D}$ with $\operatorname{ord} H_j = k_j$ then

$$\operatorname{ord}(A(H_1, \ldots, H_p)) = \begin{cases} \sum_{j=1}^p (k_j - 1) & \text{if all } k_j \neq 0, \\ 0 & \text{if any } k_j = 0. \end{cases}$$

Let us define the map a from $\mathcal{F}$-relative cochains to differential forms by

$$a : C^p(\mathcal{D}, \mathcal{F}; \mathcal{D}) \to \Omega^p(Q, \mathbf{C}),$$
$$(aA)(X_1, \ldots, X_p) = \sum_{\sigma \in \mathcal{S}_p} (\operatorname{sgn} \sigma) A(X_{\sigma_1}, \ldots, X_{\sigma_p}),$$

where $X_j \in \mathcal{X}$ and where $\mathcal{S}_p$ is the symmetric group. That is to say, a is the restriction of $C^p(\mathcal{D}, \mathcal{F}; \mathcal{D})$ to vector fields, followed by anti-symmetrisation. Direct calculation reveals that

$$a \circ \delta = d \circ a \quad \text{and} \quad a(A \cup B) = (aA) \wedge (aB).$$

Thus a is in fact a morphism of the differential graded algebras:

$$a : (C^*(\mathcal{D}, \mathcal{F}; \mathcal{D}), \delta, \cup) \to (\Omega^*(Q, \mathbf{C}), d, \wedge).$$

2.4. *Interior product and Lie derivative.* Vector fields X and differential forms ω possess a rich algebraic structure based on the interior product $\iota_V\omega$, the Lie derivative $L_V\omega$, the coboundary operator d and the wedge product $\wedge$ [CP]. Almost all this structure is inherited from corresponding structure on differential operators and cochains, (general cochains, not just $\mathcal{F}$-tensorial ones). The definitions and calculations for the general cochains make no use of the properties of the linear differential operators. The proofs of equations (5), (6) and of properties [1], [2], [3], [5], [6], [8] are combinatoric and could be applied equally to any cochains on any associative algebra with unit. Only the statements regarding $\mathcal{F}$-tensorial or $\mathcal{F}$-relative cochains are specific to $\mathcal{D}$.

2.4.1. *Interior product*

Definition. For $H_i, K \in \mathcal{D}$, the *interior product*

$$\iota_K : C^{p+1}(\mathcal{D}, \mathcal{D}) \to C^p(\mathcal{D}, \mathcal{D})$$

is given by

$$(\iota_K A)(H_1, \ldots, H_p) = A(K, H_1, \ldots, H_p) + \sum_{k=1}^{p-1} (-1)^k A(H_1, \ldots, H_k, K, H_{k+1}, \ldots, H_p) + (-1)^p A(H_1, \ldots, H_p, K).$$

Then by straightforward calculation

$$\iota_K(A \cup B) = (\iota_K A) \cup B + (-1)^{\deg A} A \cup (\iota_K B) \tag{5}$$

and

$$\iota_K^2 = 0. \tag{6}$$

If $A \in C^p(\mathcal{D}, \mathcal{D})$ is normalised then so also is $\iota_K A$.
If A is $\mathcal{F}$-tensorial then $\iota_K A$ is $\mathcal{F}$-tensorial if and only if $K \in \mathcal{F}$.
If A is $\mathcal{F}$-relative then $\iota_K A$ is $\mathcal{F}$-relative if and only if $\operatorname{ord} K \leq 1$.

2.4.2. *Lie derivative*

Definition. For $K, H \in \mathcal{D}$ we define the *Lie derivative* of H by K to be

$$L_K H = [K, H]. \tag{7}$$

Definition. For $A \in C^p(\mathcal{D}, \mathcal{D})$, $L_K A \in C^p(\mathcal{D}, \mathcal{D})$ is given by

$$(L_K A)(H_1, \ldots, H_p) = [K, A(H_1, \ldots, H_p)] - \sum_{k=1}^{p} A(H_1, \ldots, [K, H_k], \ldots, H_p). \tag{8}$$

Note that for $H \in C^0(\mathcal{D}, \mathcal{D}) = \mathcal{D}$ we have $L_K H = [K, H]$. The action of L_K is the same whether you regard H as a cochain or a linear differential operator.

If $A \in C^p(\mathcal{D}, \mathcal{D})$ is normalised then so is $L_K A$.

If A is $\mathcal{F}$-tensorial ($\mathcal{F}$-relative) then L_K is $\mathcal{F}$-tensorial ($\mathcal{F}$-relative) if and only if $\operatorname{ord} K \leq 1$.

2.4.3. *Properties of Lie derivative and interior product.* Let $K_i, H_j \in \mathcal{D}$, let $f \in \mathcal{F}$, and let $A \in C^p(\mathcal{D},\mathcal{D}),\ B \in C^q(\mathcal{D},\mathcal{D})$. The following properties [1]-[4] are trivial.

[1] $$L_K(H_1H_2) = (L_KH_1)H_2 + H_1(L_KH_2),$$

[2] $$L_{K_1K_2}H = (L_{K_1}H)K_2 + K_1(L_{K_2}H),$$

[3] $$L_K(A\cup B) = (L_KA)\cup B + A\cup(L_KB),$$

[4] $$L_K(fA) = (L_Kf)A + fL_KA.$$

Property [4] is a special case of [3], since $f \in C^0(\mathcal{D},\mathcal{D})$ and $fA = f\cup A$.

A bit less trivial is

[5] $$L_{[K_1,K_2]}A = L_{K_1}L_{K_2}A - L_{K_2}L_{K_1}A \equiv [L_{K_1}, L_{K_2}]A.$$

Proof of [5]: We adopt the notation

$$\underline{H} = H_1\otimes\ldots\otimes H_p$$

and call it a *p-chain.* Denote the space of p-chains as $C_p(\mathcal{D})$. Write also

$$[K,\underline{H}] = -[\underline{H},K] = \sum_{k=1}^{p}(H_1\otimes\ldots\otimes[K,H_k]\otimes\ldots\otimes H_p).$$

Then

$$(L_KA)(\underline{H}) = [K, A(\underline{H})] - A([K,\underline{H}]) \tag{9}$$

and

$$[K_1,[K_2,\underline{H}]] + [K_2,[\underline{H},K_1]] + [\underline{H},[K_1,K_2]] = 0.$$

So

$$\begin{aligned}L_{K_1}L_{K_2}A(\underline{H}) &= [K_1,(L_{K_2}A)(\underline{H})] - (L_{K_2}A)([K_1,\underline{H}])\\ &= [K_1,[K_2,A(\underline{H})]] - [K_1,A([K_2,\underline{H}])] - [K_2,A([K_1,\underline{H}])] + A([K_2,[K_1,\underline{H}]])\end{aligned}$$

whence the result.

For a p-form α and vector fields V, W we have

[6′] $$\iota_{[V,W]}\alpha = L_V\iota_W\alpha - \iota_WL_V\alpha,$$

[7′] $$L_{fV}\alpha = fL_V\alpha + df\wedge(\iota_V\alpha).$$

Direct calculation yields the corresponding cochain version of [6′],

[6] $$\iota_{[K_1,K_2]}A = L_{K_1}\iota_{K_2}A - \iota_{K_2}L_{K_1}A.$$

Identity [7′] does not have a simple generalisation to cochains.

2.4.4. *The Cartan identity.* The Cartan relation in differential geometry,

[8′] $$L_X = d\iota_X + \iota_Xd$$

extends to cochains in the form

[8] $$L_K = \delta\iota_K + \iota_K\delta.$$

Proof. It is helpful first to install some more notation. We have from equation (9)

$$L_K(A(\underline{H})) = (L_KA)(\underline{H}) + A(L_K\underline{H}).$$

The operators ι_K and δ have so far been defined only on cochains. Let us define them on chains also, $\iota_K : C_p \to C_{p+1}, \quad \partial : C_p \to C_{p-1}$, by

$$\iota_K \underline{H} = K \otimes \underline{H} + \sum_{j=1}^{p} (-1)^j H_1 \otimes \ldots \otimes H_j \otimes K \otimes H_{j+1} \otimes \ldots \otimes H_p,$$
$$\partial \underline{H} = H_1 H_2 \otimes H_3 \otimes \ldots \otimes H_p - H_1 \otimes H_2 H_3 \otimes H_4 \otimes \ldots \otimes H_p + \ldots$$
$$+ (-1)^p H_1 \otimes \ldots \otimes H_{p-1} H_p.$$

Then

$$\iota_K A = A \circ \iota_K \text{ and } \delta A + A \circ \partial = 1 \cup A + (-1)^{p+1} A \cup 1$$

where $A \in C^p(\mathcal{D}, \mathcal{D})$ and $1 \in C^1(\mathcal{D}, \mathcal{D})$ denotes the identity 1-cochain whose value on H is H. On products we have

$$\iota_K(\underline{H} \otimes \underline{H}') = (\iota_K \underline{H}) \otimes \underline{H}' + (-1)^{\deg \underline{H}} \underline{H} \otimes \iota_K \underline{H}' - (-1)^{\deg \underline{H}} \underline{H} \otimes K \otimes \underline{H}',$$
$$\partial(\underline{H} \otimes \underline{H}') = (\partial \underline{H}) \otimes \underline{H}' + (-1)^{\deg \underline{H}} \underline{H} \otimes \partial \underline{H}' - (-1)^{\deg \underline{H}} \underline{H}.\underline{H}'$$

where the concatenation product [Coq] of chains is

$$\underline{H}.\underline{H}' = H_1 \otimes \ldots \otimes H_p H'_1 \otimes H'_2 \otimes \ldots \otimes H'_q.$$

LEMMA. $(\partial \iota_K + \iota_K \partial)\underline{H} = L_K(\underline{H})$.

Proof. On the right, the map $\underline{H} \mapsto L_K(\underline{H}) = [K, \underline{H}]$ is a derivation over $\otimes$. It is easy to check that the lemma holds on 1-chains, and also that $\partial \iota_K + \iota_K \partial$ is a derivation over $\otimes$.

Returning to [8], the terms on the right side may be written

$$(\delta(\iota_K A))(\underline{H}) = (1 \cup (\iota_K A) - (\iota_K A) \circ \partial + (-1)^p (\iota_K A) \cup 1)(\underline{H}),$$
$$(\iota_K(\delta A))(\underline{H}) = (\delta A)(\iota_K(\underline{H})) = (1 \cup A - A \circ \partial + (-1)^{p+1} A \cup 1)(\iota_K(\underline{H})).$$

The first terms of the right sides of these two equations add to give $KA(\underline{H})$. The last terms add to give $-A(\underline{H})K$. The middle terms add, by the lemma, to give $-A([K, \underline{H}])$. Hence [8].

3. The structure of $\mathcal{F}$-tensorial cochains

3.1. *Introduction*

NOTATION. For local coordinates x^i on Q let $I = \{i_1, \ldots, i_{|I|}\}$ be an unordered set of indices, and write

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \partial_I = \partial_{i_1} \partial_{i_2} \ldots \partial_{i_{|I|}} = \prod_{i \in I} \partial_i, \quad f_{,I} = \partial_I f. \tag{10}$$

In the case $I = \emptyset$, the empty set, we shall mean by $\partial_\emptyset$ the unit operator. It is a differential operator of zero order, and may be identified with the unit function 1 on Q.

With this notation the Leibniz rule for derivatives of products of functions may be written

$$(fg)_{,I} = \sum_{I_1 \cup I_2 = I} f_{,I_1} g_{,I_2},$$

with the sum taken over all partitions of I including $(\emptyset, I)$ and $(I, \emptyset)$. It is understood that $I_1 \cap I_2 = \emptyset$.

In section 3.2 it is shown that any $\mathcal{F}$-tensorial p-cochain A is uniquely characterised, in local coordinates, by its "structure functions" $A_{(I_1,\dots,I_p)}$ where $A_{(I_1,\dots,I_p)}(x) = A(\partial_{I_1},\dots,\partial_{I_p})1|_x$ is the zero order term of the differential operator $A(\partial_{I_1},\dots,\partial_{I_p})$ evaluated at the point $x \in Q$.

In section 3.3 are computed the structure functions of $\delta A, A \cup B$, the composition product $\theta \circ \theta'$ of two 1-cochains and the structure functions of normalised $\mathcal{F}$-tensorial cochains. Those of $\theta \circ \theta'$ are used to prove that composition of such 1-cochains is commutative.

In section 3.4 the structure functions are used to express A in terms of a Taylor-like series of "basis" cochains $\delta x^{I_1} \cup \ldots \cup \delta x^{I_p}$. This is the cochain version of the expression of a p-form in terms of wedge products of coordinate differentials, $dx^{i_1} \wedge \ldots \wedge dx^{i_p}$.

3.2. *The structure theorem for $\mathcal{F}$-tensorial cochains*

(i) Consider the example of $A \in C^2_{\mathcal{F}}(\mathcal{D},\mathcal{D})$. Let $H, K \in \mathcal{D}$,

$$H = h^{i_1\dots i_{|I|}}\partial_{i_1}\dots\partial_{i_{|I|}} \equiv h^I\partial_I, \qquad K = k^J\partial_J,$$

where any repeated index i_l is summed. Then

$$\begin{aligned} A(h^I\partial_I, k^J\partial_J) &= h^I A(\partial_I \circ k^J, \partial_J) && \text{using } \mathcal{F}\text{-tensoriality,} \\ &= h^I A\Big(\sum_{I_1\cup I_2=I} k^J_{,I_1}\partial_{I_2},\ \partial_J\Big) && \text{using Leibniz' rule,} \\ &= \sum_{I_1\cup I_2=I} h^I k^J_{,I_1} A(\partial_{I_2},\ \partial_J) && \text{using } \mathcal{F}\text{-tensoriality.} \end{aligned}$$

For any $A \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ we can cascade the functions to the front as in the above example. Thus A is completely determined by its action on the ∂_I, the basis elements of $\mathcal{D}$ in the coordinate neighbourhood.

(ii) Let $\psi \in \mathcal{F}$ be the target function for the differential operator $A(\partial_I, \partial_J)$. Then, with 1 denoting the unit function,

$$\begin{aligned} A(\partial_I,\partial_J)\psi &= A(\partial_I, \partial_J \circ \psi)1 \\ &= \sum_{J_1\cup J_2=J} A(\partial_I \circ \psi_{,J_1}, \partial_{J_2})1 \\ &= \sum_{I_1\cup I_2=I}\ \sum_{J_1\cup J_2=J} \psi_{,I_1\cup J_1} A(\partial_{I_2},\partial_{J_2})1 \\ &= \sum_{I_1\cup I_2=I}\ \sum_{J_1\cup J_2=J} (A(\partial_{I_2},\partial_{J_2})1)\psi_{,I_1\cup J_1}. \end{aligned}$$

The same argument for p-cochains gives [BR]

$$A(\partial_{I_1},\dots,\partial_{I_p}) = \sum_{I_{11}\cup I_{12}=I_1} \cdots \sum_{I_{p1}\cup I_{p2}=I_p} A_{(I_{12},\dots,I_{p2})}\partial_{I_{11}\cup\dots\cup I_{p1}}$$

where we have written

$$A(\partial_{I_{12}},\dots,\partial_{I_{p2}})1 = A_{(I_{12},\dots,I_{p2})}. \tag{11}$$

The functions $A_{(J_1,\dots,J_p)}$ are called the *structure functions* of the cochain A. They evidently characterise A on the coordinate patch.

Two low order examples are:

If $A \in C^1_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ then

$$\begin{aligned} A(\partial_i) &= A_{(i)} + A_{(\emptyset)}\partial_i, \\ A(\partial_i\partial_j) &= A_{(ij)} + A_{(i)}\partial_j + A_{(j)}\partial_i + A_{(\emptyset)}\partial_i\partial_j, \\ A(\partial_i\partial_j\partial_k) &= A_{(ijk)} + A_{(ij)}\partial_k + A_{(jk)}\partial_i + A_{(ki)}\partial_j + A_{(i)}\partial_j\partial_k \\ &\quad + A_{(j)}\partial_k\partial_i + A_{(k)}\partial_i\partial_j + A_{(\emptyset)}\partial_i\partial_j\partial_k. \end{aligned}$$

If $A \in C^2_{\mathcal{F}}(\mathcal{D},\mathcal{D})$, then

$$\begin{aligned} A(\partial_i, \partial_j\partial_k) &= A_{(i,jk)} + A_{(\emptyset,jk)}\partial_i + A_{(i,k)}\partial_j + A_{(i,j)}\partial_k \\ &\quad + A_{(\emptyset,k)}\partial_i\partial_j + A_{(\emptyset,j)}\partial_i\partial_k + A_{(i,\emptyset)}\partial_j\partial_k + A_{(\emptyset,\emptyset)}\partial_i\partial_j\partial_k. \end{aligned}$$

3.3. *The structure functions for $\delta A, A \cup B, \theta \circ \theta'$ and for normalised $\mathcal{F}$-tensorial cochains.* Let $A \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$, $\quad B \in C^q_{\mathcal{F}}(\mathcal{D},\mathcal{D})$. The following results are easy.

(i) $(\delta A)_{(I_1,\dots,I_{p+1})} = A_{(I_2,\dots,I_{p+1}),I_1} - A_{\partial(I_1,\dots,I_{p+1})} + (-1)^{p+1}A_{(I_1,\dots,I_p)}\delta_{I_{p+1},\emptyset}$ where

$$\begin{aligned} \partial(I_1,\dots,I_{p+1}) \equiv (I_1 \cup I_2,\dots,I_{p+1}) - (I_1, I_2 \cup I_3,\dots,I_{p+1}) \\ + \dots + (-1)^{p+1}(I_1,\dots,I_p \cup I_{p+1}), \end{aligned}$$

(a formal sum of p-tuples of sets of indices),

$$\delta_{I_{p+1},\emptyset} = \begin{cases} 0 & \text{if } I_{p+1} \neq \emptyset \\ 1 & \text{if } I_{p+1} = \emptyset, \end{cases}$$

and $A_{(I_1,\dots,I_p)+(J_1,\dots,J_p)} = A_{(I_1,\dots,I_p)} + A_{(J_1,\dots,J_p)}$.

(ii) $(A \cup B)_{(I_1,\dots,I_{p+q})} = \displaystyle\sum_{I_{11}\cup I_{12}=I_1} \cdots \sum_{I_{p1}\cup I_{p2}=I_p} A_{(I_{11},\dots,I_{p1})}B_{(I_{p+1},\dots,I_{p+q}),I_{12}\cup\dots\cup I_{p2}}$.

(iii) For $\theta, \theta' \in C^1_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ we have

$$(\theta \circ \theta')_{(I)} = \sum_{I_1 \cup I_2 = I} \theta'_{(I_2)}\theta_{(I_1)}.$$

This is the same as $(\theta' \circ \theta)_{(I)}$. Hence $\theta' \circ \theta = \theta \circ \theta'$ and the composition of $\mathcal{F}$-tensorial 1-cochains is commutative.

(iv) An $\mathcal{F}$-tensorial p-cochain A is normalised, i.e. $\mathcal{F}$-relative if and only if $A_{(I_1,\dots,I_p)} = 0$ whenever at least one of the $I_k = \emptyset$.

3.4. *"Taylor series" for $\mathcal{F}$-tensorial cochains.* For a general 0-cochain $H \in \mathcal{D} = C^0(\mathcal{D},\mathcal{D})$, δH is a 1-cochain with $\delta H(K) = [K,H]$. In the case of the $\mathcal{F}$-tensorial 0-cochains consisting of the coordinate functions x^i we have the 1-cochains δx^i with $(\delta x^i)(K) = [K, x^i]$. We can also form composite cochains $\delta x^{i_1} \circ \delta x^{i_2} \circ \dots \circ \delta x^{i_{|I|}}$ which are themselves 1-cochains. By 3.3 this composition is commutative and we may write

$$\delta x^{i_1} \circ \dots \circ \delta x^{i_{|I|}} = \delta x^I \text{ with } I = \{i_1,\dots,i_{|I|}\}.$$

It can be shown that the action of δx^I on ∂_J is given by

$$\delta x^I(\partial_J) = \begin{cases} 0, & \text{if } |J| < |I| \\ \dfrac{1}{(|J|-|I|)!} \displaystyle\sum_{\sigma \in \mathcal{S}_{|J|}} \delta^{i_1}_{j_{\sigma_1}} \dots \delta^{i_{|I|}}_{j_{\sigma_{|I|}}} \partial_{j_{\sigma_{|I|+1}}} \dots \partial_{j_{\sigma_{|J|}}}, & \text{if } |J| \geq |I|. \end{cases} \tag{12}$$

We may now write any $\mathcal{F}$-tensorial 1-cochain as a series,

$$\begin{aligned} A = {} & A_{(\emptyset)}\delta x^{\emptyset} + A_{(i)}\delta x^i + \frac{A_{(ij)}}{2!}\delta x^i \circ \delta x^j + \dots + \frac{A_{(I)}}{|I|!}\delta x^I + \dots \\ = {} & \sum_I \frac{1}{|I|!} A_{(I)} \delta x^I \end{aligned} \tag{13}$$

where $\delta x^{\emptyset} = 1$, the identity cochain on $\mathcal{D}$. This is verified by checking that both sides have the same structure functions, as follows.

The action of the operator (12) on the unit function gives us the structure function of δx^I. We find that

$$(\delta x^I)_{(J)} = 0 \quad \text{if} \quad J \neq I$$

and for $J = I$ (same unordered set of indices)

$$(\delta x^I)_{(J)} = \sum_{\sigma \in \mathcal{S}_{|I|}} \delta^{i_1}_{j_{\sigma_1}} \dots \delta^{i_{|I|}}_{j_{\sigma_{|I|}}}.$$

Hence the structure function of

$$\frac{1}{|I|!} A_{(i_1 \dots i_{|I|})} \delta x^I$$

is

$$\begin{aligned} \frac{1}{|I|!} A_{(i_1 \dots i_{|I|})} (\delta x^I)_{(J)} &= \frac{1}{|I|!} A_{(i_1 \dots i_{|I|})} \sum_{\sigma \in \mathcal{S}_{|I|}} \delta^{i_1}_{(j_{\sigma_1}} \dots \delta^{i_{|I|}}_{j_{\sigma_{|I|}})} \\ &= \frac{1}{|I|!} \sum_{\sigma \in \mathcal{S}_{|I|}} A_{(j_1 \dots j_{|I|})} = A_{(J)} = A_{(I)}. \end{aligned}$$

The series for any $\mathcal{F}$-tensorial p-cochain is

$$A = \sum_{I_1, \dots, I_p} \frac{1}{\prod_{i=1}^p |I_i|!} A_{(I_1, \dots, I_p)} \delta x^{I_1} \cup \dots \cup \delta x^{I_p}. \tag{14}$$

Again each side has the same structure functions.

In (13) and (14) the coefficient functions $A_{(\dots)}$ may be chosen arbitrarily. For example, in (13) one might take $A_{(I)} = 0$ for $|I| \leq p-1$ and $A_{(I)} \neq 0$ for $|I| = p$. Then $A(H) = 0$ for ord $H < p$; we say then that ord $A = p$ and find that the leading term in the Taylor series for A is $A_{(i_1, \dots i_p)} \delta x^{i_1} \circ \dots \circ \delta x^{i_p}$, where $A_{(i_1, \dots, i_p)}$ is a symmetric covariant tensor field. This is the dual result to the well known fact that the coefficients of the leading terms of a linear differential operator make up a symmetric contravariant tensor field.

4. The isomorphism between $C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D})$ and $J_\Delta C^*_{AS}$

4.1. *Introduction.* We introduce a complex (C^*_{AS}, δ_{AS}) called the Alexander-Spanier (AS)-complex, [Sp], [Mas]. The elements of C^p_{AS} are smooth, complex-valued functions each defined on some open neighbourhood of the diagonal subset Δ_{p+1} of Q^{p+1}, the

Cartesian product of $p+1$ copies of Q. C^*_{AS} has a product, called concatenation, [Coq], and written as a dot

$$C^p_{AS} \times C^q_{AS} \to C^{p+q}_{AS}, \qquad (F,G) \mapsto F.G$$

and a differential, δ_{AS} :

$$\delta_{AS} : C^p_{AS} \to C^{p+1}_{AS}.$$

Together these make $(C^*_{AS}, \delta_{AS}, .)$ into an associative differential graded algebra (DG-algebra).

We define a map

$$\Phi : C^*_{AS} \to C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D})$$

which depends only on the partial derivatives of the AS-cochains on the diagonal subset $\Delta_{p+1} = \{(x, x, \ldots, x) : x \in Q\}$ of Q^{p+1} and not elsewhere. With this in mind we use the AS-algebra to construct an associated DG-algebra $(J_\Delta C^*_{AS}, \delta_{AS}, .)$ whose elements of degree p are the jets of AS-p-cochains which are evaluated on Δ_{p+1}.

The main result of this chapter is that the map Φ gives an isomorphism between the DG-algebras $(J_\Delta C^*_{AS}, \delta_{AS}, .)$ and $(C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D}), \delta, \cup)$.

Section 4.2 describes the AS-complex. Particular elements, called [BR] separable cochains, are introduced. In section 4.3 we define the map $\Phi : C^p_{AS} \to C^p_{\mathcal{F}}(\mathcal{D}, \mathcal{D})$, and show that Φ is a morphism of DG-algebras. It is not however injective. In section 4.4 we show that, for $F \in C^p_{AS}$, the differential operator $\Phi^F(H_1, \ldots, H_p)$ depends on F only through the jet of F on the diagonal subset $\Delta_{p+1} \subset M^{p+1}$. Two functions $F_1, F_2 \in C^p_{AS}$ having the same jet on Δ_{p+1} but differing elsewhere will satisfy $\Phi^{F_1} = \Phi^{F_2}$. In section 4.5 we prove that $\Phi : J_\Delta C^*_{AS} \to C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D})$ is an isomorphism of DG-algebras.

The isomorphism gives a cross-fertilisation of the properties of the two types of cochain, $\mathcal{F}$-tensorial and Alexander-Spanier. Thus AS-cochains acquire a Lie derivative with respect to vector fields and an interior product with respect to functions, whilst $\mathcal{F}$-tensorial cochains gain a commutative product. Some details are given in section 4.6.

In section 4.7 the isomorphism is refined slightly to one between a subalgebra of $J_\Delta C^*_{AS}$ consisting of jets of what we call normalised AS cochains and the subalgebra $C^*(\mathcal{D}, \mathcal{F}, \mathcal{D}) \subset C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D})$ of normalised $\mathcal{F}$-tensorial cochains.

4.2. *The Alexander-Spanier complex*

Definition. An *AS-p-cochain* is an element of $C^\infty(U_{p+1}, \mathbf{C})$ where U_{p+1} is an open neighbourhood of the diagonal $\Delta_{p+1} \subset Q^{p+1}$, $\Delta_{p+1} = \{(x, x, \ldots, x) : x \in Q\}$. The vector space

$$C^*_{AS} = \sum_{p=0}^{\infty} C^p_{AS}$$

is made into an algebra by defining the *concatenation product*:

For $F \in C^p_{AS}$, $G \in C^q_{AS}$ the product $F.G \in C^{p+q}_{AS}$ is defined by

$$(F.G)(x_0, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) = F(x_0, \ldots, x_p) G(x_p, \ldots, x_{p+q}).$$

The *Alexander-Spanier differential* $\delta_{AS} : C^p_{AS} \to C^{p+1}_{AS}$ is defined by

$$(\delta_{AS}F)(x_0,..,x_{p+1}) = \sum_{j=0}^{p+1}(-1)^j F(x_0,\ldots,\hat{x}_j,\ldots,x_{p+1})$$

where $\hat{x}_j$ denotes omission of x_j.

There are particular sorts of AS-cochains called *separable* cochains which are of the form

$$F = f_0 \otimes f_1 \otimes \ldots \otimes f_p \in \mathcal{F}^{\otimes p+1}.$$

That is,

$$F(x_0,\ldots,x_p) = f_0(x_0)f_1(x_1)\ldots f_p(x_p).$$

For separable $F = f_0 \otimes \ldots \otimes f_p \in C^p_{AS}$ and $G = g_0 \otimes \ldots \otimes g_q \in C^q_{AS}$ we have

$$F.G = f_0 \otimes \ldots \otimes f_p g_0 \otimes \ldots \otimes g_q$$

and

$$\delta_{AS}F = (1 \otimes f_0 \otimes \ldots \otimes f_p) - (f_0 \otimes 1 \otimes f_1 \otimes \ldots \otimes f_p)$$
$$+ \ldots + (-1)^{p+1}(f_0 \otimes \ldots \otimes f_p \otimes 1).$$

Every $F \in C^p_{AS}$ is the limit of a sequence of sums of separable functions. If a conjecture holds for separable functions in C^p_{AS} then it follows from linearity and continuity that the conjecture will hold true for all elements of C^p_{AS}.

4.3. *The map* $\Phi : C^p_{AS} \to C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$. For $F \in C^p_{AS}$, $H_j \in \mathcal{D}$, $\psi \in \mathcal{F}$ we define $\Phi^F \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ by [HB, BR]

$$(\Phi^F(H_1,\ldots,H_p)\psi)(x_0)$$
$$= [H_1(x_1)[H_2(x_2)[\ldots[H_p(x_p)[F(x_0,\ldots,x_p).\psi(x_p)]]_{x_p=x_{p-1}}]\cdots]_{x_2=x_1}]_{x_1=x_0}. \tag{15}$$

The cochain Φ^F is evidently $\mathcal{F}$-tensorial.

For separable F the equation (15) simplifies to

$$\Phi^F(H_1,\ldots,H_p) = \Phi^{f_0\otimes\ldots\otimes f_p}(H_1,\ldots,H_p) = f_0H_1f_1\ldots H_pf_p \in \mathcal{D}.$$

Again, for separable F,

$$\begin{aligned}
&\Phi^{\delta_{AS}F}(H_1,\ldots,H_{p+1})\\
&= ((1\otimes f_0\otimes\ldots\otimes f_p) - (f_0\otimes 1\otimes\ldots f_p) + \ldots + (-1)^{p+1}(f_0\otimes\ldots\otimes f_p\otimes 1))(H_1,\ldots,H_p)\\
&= 1H_1f_0H_2\ldots H_{p+1}f_p - f_0H_11H_2\ldots H_{p+1}f_p + \ldots + (-1)^{p+1}f_0H_1\ldots f_pH_{p+1}\\
&= H_1f_0H_2\ldots H_{p+1}f_p - f_0H_1H_2f_1\ldots H_{p+1}f_p + \ldots + (-1)^{p+1}f_0H_1\ldots f_pH_{p+1}\\
&= \delta\Phi^F(H_1,\ldots,H_{p+1}).
\end{aligned}$$

So $\Phi^{\delta_{AS}F} = \delta\Phi^F$ for separable functions F and hence by linearity and continuity for all $F \in C^p_{AS}$. In a similar way, for $F \in C^p_{AS}$ and $G \in C^q_{AS}$, both separable, one may verify that

$$\Phi^{F.G}(H_1,\ldots,H_{p+q}) = (\Phi^F \cup \Phi^G)(H_1,\ldots,H_{p+q}).$$

Hence $\Phi^{F.G} = \Phi^F \cup \Phi^G$ for separable functions and therefore for all C^*_{AS}.

4.4. Φ *factors through* $J_\Delta C^*_{AS}$. In this section we prove that for $F \in C^p_{AS}$, $\Phi^F(H_1, \dots H_p)$ depends on F only through the jet of F at the diagonal $\Delta_{p+1} \subset Q^{p+1}$. We denote by $J_\Delta C^*_{AS}$ the set of jets on Δ_{p+1} of elements $F \in C^p_{AS}$.

NOTATION. The point $x_k \in Q$ has coordinates x^i_k. Denote

$$\frac{\partial}{\partial x^i_k} F(x_0, \dots, x_k, \dots, x_p) \text{ by } F_{,k^i}$$

and

$$\frac{\partial}{\partial x^{i_1}_k} \frac{\partial}{\partial x^{i_2}_k} \cdots \frac{\partial}{\partial x^{i_{|I|}}_k} F(x_0, \dots, x_k, \dots, x_p) = \left(\frac{\partial}{\partial x_k}\right)^I F(x_0, \dots, x_k, \dots, x_p)$$

by

$$F_{,k^I}(x_0, \dots, x_k, \dots, x_p)$$

where $I = \{i_1, \dots, i_{|I|}\}$.

We shall often need to set several of the arguments x_k of F as equal and then to differentiate F with respect to the coordinate x^i_k. For example if $F \in C^2_{AS}$, then by the chain rule

$$\frac{\partial}{\partial x^i_1} F(x_0, x_1, x_1) = F_{,1^i}(x_0, x_1, x_1) + F_{,2^i}(x_0, x_1, x_1) \equiv F_{,(1+2)^i}(x_0, x_1, x_1),$$

$$\left(\frac{\partial}{\partial x_1}\right)^I F(x_0, x_1, x_1) = F_{,(1+2)^{i_1}(1+2)^{i_2}\dots(1+2)^{i_{|I|}}}(x_0, x_1, x_1)$$
$$\equiv F_{,(1+2)^I}(x_0, x_1, x_1).$$

With this notation we may use (15) with $\psi = 1$ to write the structure function of Φ^F as

$$\Phi^F_{(I_1, I_2)}(x_0) = \left[\frac{\partial}{\partial x^{I_1}_1}\left(\left[\frac{\partial}{\partial x^{I_2}_2} F(x_0, x_1, x_2)\right]_{x_2 = x_1}\right)\right]_{x_1 = x_0}$$
$$= F_{,(1+2)^{I_1} 2^{I_2}}(x_0, x_0, x_0).$$

Similarly one may show that for $F \in C^p_{AS}$,

$$\Phi^F_{(I_1, \dots I_p)}(x_0) = F_{,(1+\dots+p)^{I_1}(2+\dots+p)^{I_2}\dots p^{I_p}}(x_0, \dots x_0) \tag{16}$$

which is a combination of jets of F on the diagonal Δ.

4.5. $\Phi : (J_\Delta C^*_{AS}, \delta_{AS}, \,.) \to (C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D}), \delta, \cup)$ *is an isomorphism*

4.5.1. *Injectivity of* $\Phi : J_\Delta C^*_{AS} \to C^*_{\mathcal{F}}(\mathcal{D}, \mathcal{D})$. We have seen that the $\mathcal{F}$-tensorial cochain Φ^F depends only on the jet of F at Δ. The jets on the diagonal inherit from C^*_{AS} the differential, grading and concatenation product structures, so they themselves form a DG-algebra, $(J_\Delta C^*_{AS}, \delta_{AS}, .)$ We next show that if we restrict Φ to these (infinite-order) jets, then Φ becomes injective.

For this, we must show that if $\Phi^F = \Phi^G$ then F and G have the same jets on Δ. Now if $\Phi^F = \Phi^G$ they must have the same structure functions, which implies [BR] that their jets on Δ are equal.

4.5.2. *Surjectivity of* $\Phi : J_\triangle C^*_{AS} \to C^*_{\mathcal{F}}(\mathcal{D},\mathcal{D})$. We have shown that Φ is an injective homomorphism. To show that Φ is an isomorphism we must produce

$$\Phi^{-1} : C^*_{\mathcal{F}}(\mathcal{D},\mathcal{D}) \to J_\triangle C^*_{AS}, \qquad A \mapsto F^A.$$

That is, given $A \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ we must construct the jets on Δ^{p+1} of a function $F^A \in C^p_{AS}$ so that $\Phi^{F^A} = A$.

For $p = 0$ we have $C^0_{\mathcal{F}}(\mathcal{D},\mathcal{D}) = \mathcal{F}$ and $F^A = A$.

For $p > 0$ the cochain $A \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ has structure function $A_{(I_1,\dots,I_p)}(x)$. Now $F \in C^p_{AS}$ has Taylor series

$$F(x_0,\dots x_p) = \sum_{I_1}\dots\sum_{I_p}\Big(\prod_{k=1}^{p}|I_k|!\Big)^{-1} F_{,1^{I_1}\dots p^{I_p}}(x_0,\dots,x_0)(x_1-x_0)^{I_1}\dots(x_p-x_0)^{I_p}.$$

One can show [BR] that by setting

$$\begin{aligned} &F_{,1^{I_1}\dots p^{I_p}}(x_0,x_0,x_0) \\ &\qquad = (-1)^{|I_{12}|+\dots+|I_{p2}|} \sum_{I_{11}\cup I_{12}=I_1}\dots\sum_{I_{p-1,1}\cup I_{p-1,2}=I_{p-1}} A_{(I_{11},I_{12}\cup I_{21},\dots,I_{p-1,2}\cup I_p)}(x_0)\end{aligned}$$

we obtain $\Phi^{F^A} = A$. Thus Φ is an isomorphism. The convergence of the above formal series is not discussed but we invoke Borel's Lemma [Gib] that a C^∞ function on Q^{p+1} which has the above jets does exist.

4.6. *Additional structures on* $C^*_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ *and* C^*_{AS}. Since $C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ and $J_\triangle C^p_{AS}$ are isomorphic DG-algebras, any structure admitted by $C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ will give a corresponding structure to C^p_{AS} (or at least to $J_\triangle C^p_{AS}$) and vice versa.

4.6.1. *Composition of cochains in* C^*_{AS}. For $\theta \in C^1_{\mathcal{F}}(\mathcal{D},\mathcal{D})$, $A \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ the composition $\theta \circ A \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ induces the composition of AS-cochains

$$(F \circ G)(x_0,\dots,x_p) = F(x_0,x_p)G(x_0,\dots,x_p) \tag{17}$$

which might be called the "encircle" product of an AS-p-cochain by an AS-1-cochain. In particular if $F, G \in C^1_{AS}$ then $(F\circ G)(x_0,x_1) = F(x_0,x_1)G(x_0,x_1)$.

4.6.2. *Commutative product of two* $\mathcal{F}$*-tensorial* p*-cochains.* AS-cochains of the same order are real functions on the same space so can be multiplied to produce another AS-cochain of the same order. So therefore can $\mathcal{F}$-tensorial cochains.

Let A, $B \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D})$ have corresponding AS-cochains F^A, F^B,

$$A = \Phi^{F^A}, \quad B = \Phi^{F^B}.$$

Define the (commutative) product

$$AB = BA = \Phi^{F^A F^B} \in C^p_{\mathcal{F}}(\mathcal{D},\mathcal{D}).$$

For 1-cochains θ, θ' we have $\theta\theta' = \theta \circ \theta'$.

4.6.3. *Interior product* ι_f *and Lie derivative* L_V *of* AS*-cochains.* We found in sections 2.5 and 2.6 that the interior product map $\iota_K : C^{p+1}(\mathcal{D},\mathcal{D}) \to C^p(\mathcal{D},\mathcal{D})$ preserves $\mathcal{F}$-tensoriality if and only if $K \in \mathcal{F}$, and the Lie derivative L_K does so only if ord $K \le 1$.

The AS-cochains inherit these structures as follows:

$$(\iota_f F)(x_0, \ldots, x_p) = \sum_{j=0}^{p} (-1)^j f(x_j) F(x_0, \ldots x_{j-1}, x_j, x_j, x_{j+1}, \ldots, x_p),$$

$$(L_V F)(x_0, \ldots, x_p) = \sum_{j=0}^{p} V^i(x_j) \partial_{x_j^i} F(x_0, \ldots, x_p).$$

4.7. *Normalisation.* An AS-cochain F is said to be *normalised* if it vanishes when two contiguous variables are identified, i.e.

$$F(x_0, \ldots, x_j, x_j, x_{j+2} \ldots, x_p) = 0.$$

It can be shown that the jets of the normalised AS-cochains correspond isomorphically to the normalised $\mathcal{F}$-tensorial cochains.

5. From de Rham forms to Alexander-Spanier cochains. For a given $\omega \in \Omega^p(M)$, [HB] construct a function $F^\omega : \mathcal{V}_{p+1} \to \mathbf{R}$ where $\mathcal{V}_{p+1} \subset Q^{p+1}$ is a certain neighbourhood of the diagonal subset $\Delta_{p+1} = \{(x, \ldots, x) : x \in Q\} \subset Q^{p+1}$ which is specified below. So, $\mathcal{V}_{p+1}$ consists of $(p+1)$-tuples $(x_0, \ldots, x_p) \in Q^{p+1}$ which are "close" to Δ_{p+1}. This means that the x_j in Q are all reasonably close to each other.

The construction uses the geodesic paths of a connection Γ on TQ, so we suppose that we are given this path structure.

At each point $x_0 \in Q$ we take a convex normal neighbourhood $\mathcal{V}_{x_0}$ of x_0 ([CP], chap. 11). Then $\mathcal{V}_{x_0}$ is such that there is one and only one segment of a geodesic path joining any two points in $\mathcal{V}_{x_0}$. We define

$$\mathcal{V}_{p+1} = \{(x_0, \ldots, x_p) \in Q^{p+1} : x_1, \ldots, x_p \in \mathcal{V}_{x_0}\}.$$

For $(x_0, \ldots x_p) \in \mathcal{V}_{p+1}$, we construct a p-simplex $S_p^\Gamma(x_0, \ldots, x_p)$ having vertices $x_0, \ldots, x_p$ as follows.

The edge $S_1^\Gamma(x_i, x_j),\ i < j$, of the p-simplex $S_p^\Gamma(x_0, \ldots, x_p)$ is the oriented geodesic path from x_i to x_j. The 2-face $S_2^\Gamma(x_i, x_j, x_k), i < j < k$, is built by joining x_i to points on $S_1^\Gamma(x_j, x_k)$ by geodesic paths.

3-faces $S_3^\Gamma(x_i, x_j, x_k, x_l),\ i < j < k < l$, are formed by joining x_i by geodesic paths to the points of $S_2^\Gamma(x_j, x_k, x_l)$ and so on.

It follows that

$$\partial S_p(x_0, \ldots, x_p) = \sum_{j=0}^{p} (-1)^j S_{p-1}(x_0, \ldots, \hat{x}_j, \ldots, x_p) \equiv S_{p-1}(\partial(x_0, \ldots, x_p)).$$

Thus the map $F_\Gamma : \Omega^* \to NC^*_{AS}$ defined by

$$F_\Gamma^\omega(x_0, \ldots, x_p) = \int\limits_{S_p^\Gamma(x_0, \ldots, x_p)} \omega$$

satisfies

$$F_\Gamma^{d\omega} = \delta_{AS} F_\Gamma^\omega.$$

One can then show [HB] that, for Φ as defined in Chapter 4,

$$\Phi_\Gamma^{F^\omega}(X_1,\ldots,X_p)=\frac{1}{p!}\omega(X_1,\ldots,X_p), \tag{18}$$

i.e. the action of $\Phi_\Gamma^{F^\omega}$ on vector fields is independent of Γ and agrees up to scale with ω. In particular for $\theta\in\Omega^1$,

$$\Phi_\Gamma^{F^\theta}|_{\mathcal{X}}=\theta.$$

In order to evaluate the 1-cochain $\Phi_\Gamma^{F^\theta}$ on differential operators of order higher than 1 we need to know the structure functions of the cochain, or equivalently, the jets of the function $F_\Gamma^\theta(x_0,x_1)$ at $x_1=x_0$. The first few are

$$\begin{aligned}
F^\theta_{\Gamma,1^a}(x_0,x_0)&=\theta_a(x_0),\\
F^\theta_{\Gamma,1^a1^b}(x_0,x_0)&=\theta_{(a,b)}(x_0),\\
F^\theta_{\Gamma,1^a1^b1^c}&=\theta_{(a,bc)}+\tfrac12 d\theta_{i,(a}\Gamma^i_{bc)},\\
F^\theta_{\Gamma,1^a1^b1^c1^d}&=\theta_{(a,bcd)}+[d\theta_{i,(a}\Gamma^i_{bc}]_{,d)}.
\end{aligned}$$

where curved brackets ( ) around subscripts indicate that they are symmetrised. A general formula exists for the covariant derivatives of F_Γ^θ. It is [Ha]

$$F^\theta_{\Gamma;(1^{a_1}\ldots1^{a_p})}(x_0,x_0)=\theta_{(a_1;a_2\ldots a_p)}(x_0).$$

It is evident from its construction that the p-simplex $S_p^\Gamma(x_0,\ldots,x_p)$ collapses to the $(p-1)$-simplex $S^\Gamma_{p-1}(x_0,\ldots\hat{x}_j,x_{j+1},\ldots,x_p)$ whenever x_{j+1} is set equal to x_j. The functions $F_\Gamma^\omega(x_0,\ldots,x_p)$ defined on $\mathcal{V}_{p+1}$ are thus normalised AS-cochains, and the cochains $\Phi^{F_\Gamma^\omega}$ are $\mathcal{F}$-relative.

6. Applications. Having forged our small Nothung let us bully some small dragons with it.

6.1. *The probability current in quantum mechanics.* We shall use the framework of the standard quantum mechanics of a system having configuration space Q with half density wave functions ψ such that $\int_Q|\psi|^2<\infty$. We suppose that we are given a Hamiltonian. This is a symmetric operator $H\in\mathcal{D}$ which possesses a self adjoint extension. The time development of ψ is then specified by the Schrödinger equation

$$i\partial_t\psi=H\psi.$$

The amplitude density $\rho_{(\phi,\psi)}=\bar\phi\psi$ for two wave functions ϕ,ψ satisfies

$$\int_Q\partial_t\rho_{(\phi,\psi)}=0$$

which implies that we may write

$$\partial_t\rho_{(\phi,\psi)}=-\mathrm{div}J_{(\phi,\psi)}$$

for some transition probability flux vector field, or current. This fixes div$J$, not J itself, though some textbooks claim that for $Q=\mathbf{R^3}$ and $H=-\frac12\nabla^2+V(\mathbf{r})$, the "correct" J among those with the right divergence is

$$J_{(\phi,\psi)}=\Im(\bar\phi\nabla\psi).$$

We investigate what structure is required to select the "correct" J when H is an arbitrary Hamiltonian operator on an arbitrary configuration manifold Q.

We first recall the properties of de Rham p-currents—linear complex-valued functions on "test" p-forms on Q, [deR]. A 0-current ρ is a scalar density,

$$\rho(f) = \int_Q f\rho, \quad f \in \mathcal{F}.$$

A 1-current J is a vector density. It acts on test 1-forms

$$J(\theta) = \int_Q \theta_i J^i, \quad \theta \in \Omega^1.$$

There is a natural map div from p-currents to $(p-1)$-currents,

$$(\mathrm{div} J)(\omega) = -J(d\omega), \quad \omega \in \Omega^{p-1}.$$

We shall need to extend the notion of p-currents to be linear complex-valued functions on test $\mathcal{F}$-relative p-cochains; let $\mathcal{J}_p$ be the space of these p-currents. Copy the definition of div:

$$\mathrm{div} : \mathcal{J}_p \to \mathcal{J}_{p-1}, \quad (\mathrm{div}\ J)(A) = -J(\delta A)$$

where δ is the Hochschild differential. Then for $H \in \mathcal{D},\ f \in \mathcal{F} = C^0(\mathcal{D}, \mathcal{F}; \mathcal{D})$ and wave functions $\phi,\ \psi$ we have a 1-current

$$H_{(\phi,\psi)}(A) = \int_Q \bar{\phi}\, A(H)\psi, \quad A \in C^1(\mathcal{D}, \mathcal{F}; \mathcal{D}),$$

$$(\mathrm{div} H_{(\phi,\psi)})(f) = -H_{(\phi,\psi)}(\delta f) = -\int \bar{\phi}\ \delta f(H)\psi = -\int \bar{\phi}[H, f]\psi.$$

Then treating $\rho_{(\phi,\psi)}$ as an element of $\mathcal{J}_0$, we obtain

$$(\partial_t \rho_{(\phi,\psi)})(f) = \partial_t \int_Q \bar{\phi} f \psi = i \int_Q \bar{\phi}[H, f]\psi = -(\mathrm{div}\ iH_{(\phi,\psi)})(f).$$

So

$$\partial_t \rho_{(\phi,\psi)} = -\mathrm{div}\ iH_{(\phi,\psi)}$$

as an equation of two Hochschild 0-currents. This suggests that $iH_{(\phi,\psi)}$ be regarded as the probability 1-current density. It is a Hochschild 1-current, not yet a de Rham 1-current.

To obtain a de Rham 1-current, which maps 1-forms into $\mathbf{C}$, we need a way to extend the domain of 1-forms from vector fields to all of $\mathcal{D}$. But this map is to hand:

$$D_\Gamma = \Phi \circ F_\Gamma : \Omega^* \to C^*(\mathcal{D}, \mathcal{F}; \mathcal{D}), \quad \omega \mapsto \Phi^{F_\Gamma^\omega} = D_\Gamma^\omega$$

satisfies $D_\Gamma^{d\omega} = \delta D_\Gamma^\omega$ and $D_\Gamma^\omega|_{\mathcal{X}} = (p!)^{-1}\omega$. So we take the de Rham current to be

$$J_{(\phi,\psi)}(\theta) = iH_{(\phi,\psi)}(D_\Gamma^\theta) = i \int_Q \bar{\phi} D_\Gamma^\theta(H)\psi.$$

One can check that for $Q = \mathbf{R}^3,\ H = -\frac{1}{2}\nabla^2 + V(\mathbf{r}),\ \Gamma =$ Euclidean connection, and Cartesian coordinates x^i, we obtain the usual formula

$$J_{(\phi,\psi)}(\theta) = -\frac{i}{2} \int_Q \theta_i (\bar{\phi}\partial_i \psi - (\partial_i \bar{\phi})\psi).$$

6.2. *An $\mathcal{F}$-relative metric 2-cochain for a Riemannian manifold.* We now suppose that Q possesses a Riemannian metric g. There are many $\mathcal{F}$-relative 2-cochains on $\mathcal{D}$ whose restrictions to $\mathcal{X}$ coincide with the symmetric covariant tensor field g. If there were a natural choice for such a cochain then we could define an "inner product" $g(H_1, H_2) \in \mathcal{D}$ of order $\operatorname{ord} H_1 + \operatorname{ord} H_2 - 2$, perhaps extend the Levi-Civita covariant derivative from $\nabla_{X_1} X_2$ to $\nabla_{X_1} H_2$, and explore torsion and curvature in this wider context. In section 5.1 we showed how to obtain an $\mathcal{F}$-relative p-cochain from an antisymmetric covariant tensor field (a differential form) by integrating the form over a certain p-simplex. This trick is not available for symmetric covariant tensor fields, and we must try something else.

Instead of working with the partial differential operators ∂_I of (10) it is convenient to use the symmetrised covariant differential operators

$$\nabla_I = \nabla_{(i_1,\dots i_p)},\ \nabla_I \psi = \psi_{;(i_1 \dots i_p)}.$$

So

$$\nabla_i = \partial_i,\ \nabla_{(i_1,i_2)} = \partial_{i_1}\partial_{i_2} - \Gamma^k_{\ i_1 i_2}\partial_k,$$

and so forth. Note that $\nabla_{ijk} \neq \nabla_{ij}\nabla_k$.

Consider an $\mathcal{F}$-relative 1-cochain θ of order 2; this kills differential operators of order 1 but spares operators of order 2 or more. The Taylor series (13) starts

$$\theta = \theta_{(ij)} \partial x^i \circ \partial x^j + \theta_{(ijk)} \partial x^i \circ \partial x^j \circ \partial x^k + \dots$$

where the $\theta_{(ij)} = \theta(\partial_i \partial_j)$ are the components of a symmetric covariant tensor field. Let us choose one such cochain θ^g such that $\theta^g_{(ij)} = g_{ij}$. Postponing the choice of the higher coefficients in the Taylor series, we note that for $X, Y \in \mathcal{X}$,

$$\begin{aligned}\delta\theta^g(X,Y) &= X \circ \theta^g(Y) - \theta^g(XY) + \theta^g(X) \circ Y \\ &= -\theta^g(X^i \partial_i \circ Y^j \partial_j) = -X^i Y^j \theta^g(\partial_i \partial_j) = -g(X,Y).\end{aligned}$$

For $\operatorname{ord} H = 2$, $H = h^{ij}\partial_i\partial_j + h^i \partial_i + h^{\emptyset}$, we have

$$\theta^g(H) = g_{ij}\partial x^i \circ \partial x^j(H) = h^{ij} g_{ij} = h^i_{\ i}.$$

So θ^g is a sort of trace; not a true trace because usually $\theta^g([H_1, H_2]) \neq 0$. We now fix the higher coefficients of the Taylor series. Instead of the structure functions $\theta^g_{(I)} = \theta^g(\partial_I)1$ used earlier, we replace ∂_I by $\nabla_{(I)}$ and consider the "symmetrised covariant structure functions" $\theta^g(\nabla_{(i_1 \dots i_p)})1$. For fixed p these functions make up the components of a symmetrical covariant tensor field, since the functions $\theta^g(h^{i_1 \dots i_p}\nabla_{(i_1 \dots i_p)})1 = h^{i_1 \dots i_p}\theta^g(\nabla_{(i_1 \dots i_p)})1$ are scalar. We may therefore impose the condition that the tensor components $\theta^g(\nabla_{(I)})1$ vanish for all $|I| > 2$. This determines the higher Taylor coefficients of θ^g; for example $\theta^g_{(ijk)} = \theta^g(\partial_i\partial_j\partial_k)1 = 3\Gamma_{(i.jk)}$, the symmetrised Christoffel symbols. It follows that

$$\theta^g(h^{i_1 \dots i_p}\nabla_{(i_1 \dots i_p)}) = \binom{p}{2} h_a^{\ a i_1 \dots i_{p-2}} \nabla_{(i_1 \dots i_{p-2})}.$$

We may now define the metric 2-cochain g to be $-\delta\theta^g$ on all $\mathcal{D}$. It is natural to enquire whether we may define the covariant derivative of a differential operator. For

$W \in \mathcal{X},\ H \in \mathcal{D}$, can we define $\nabla_W H$ so that

$$(P1) \qquad \nabla_W : \mathcal{D}_p \to \mathcal{D}_p,$$

$$(P2) \qquad \nabla_{fW} = f\nabla_W,$$

$$(P3) \qquad \nabla_W(fH) = W(f) \circ H + f\nabla_W H,$$

$$(P4') \qquad \nabla_W(g(H_1, H_2)) = g(\nabla_W H_1, H_2) + g(H_1, \nabla_W H_2) \ ?$$

The fourth condition is clearly impossible since the first two terms are $\mathcal{F}$-linear in W, but the third is not, when ord $H_1 \geq 2$. So let us replace $(P4')$ by the analogous condition for the Hochschild potential θ^g of the metric g,

$$(P4) \qquad \nabla_W(\theta^g(H)) = \theta^g(\nabla_W H).$$

One may deduce from $(P4)$ that

$$\nabla_i(\nabla_{(j_1 \dots j_p)}) = p\mathcal{S}_{j_1 \dots j_p} \Gamma^a_{ij_1} \nabla_{(j_2 \dots j_p a)}.$$

Here $\mathcal{S}_{j_1 \dots j_p}$ denotes symmetrisation over the indices $j_1, \dots, j_p$. This in turn implies that

$$\nabla_W(h^{j_1 \dots j_p} \nabla_{(j_1 \dots j_p)}) = W^i h^{j_1 \dots j_p}{}_{;i} \nabla_{(j_1 \dots j_p)} \qquad (19)$$

as one might naively expect. So we adopt (19) as our definition of ∇_W on $\mathcal{D}$.

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