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APPLICATIONS OF LIE SYSTEMS IN QUANTUM MECHANICS AND CONTROL THEORY

JOSÉ F. CARIÑENA and ARTURO RAMOS

Departamento de Física Teórica, Facultad de Ciencias
Universidad de Zaragoza, 50009, Zaragoza, Spain
E-mail: jfc@posta.unizar.es and arrg@wigner.unizar.es

Abstract. Some simple examples from quantum physics and control theory are used to illustrate the application of the theory of Lie systems.

We will show, in particular, that for certain physical models both of the corresponding classical and quantum problems can be treated in a similar way, may be up to the replacement of the Lie group involved by a central extension of it.

The geometric techniques developed for dealing with Lie systems are also used in problems of control theory. Specifically, we will study some examples of control systems on Lie groups and homogeneous spaces.

1. Introduction: Lie systems. There exists a class of systems of time-dependent first order differential equations

$$\frac{dx^i}{dt} = X^i(x^1, \dots, x^n, t), \quad i = 1, \dots, n, \tag{1}$$

for which there is a function $\Phi : \mathbf{R}^{n(m+1)} \to \mathbf{R}^n$ such that the general solution can be written as $x = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_n)$, where $\{x_{(j)} \mid j = 1, \dots, m\}$ is any set of particular but independent solutions of (1), and $k_1, \dots, k_n$ are n constants characterizing each particular solution. These systems, to be called Lie systems, have been characterized by Lie [19] and are receiving much attention in recent years, both in physics and in mathematics [5–9]. From the geometric viewpoint, Lie systems correspond to t-dependent vector fields which can be written as a linear combination, with t-dependent coefficients, of a finite set of true vector fields closing on a finite-dimensional real Lie algebra [5, 7].

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A simple instance of Lie system is the linear system

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j, \quad i = 1, \dots, n, \tag{2}$$

for which the general solution can be written as a linear combination of n independent particular solutions $x_{(1)}, \dots, x_{(n)}$, i.e., $x = \Phi(x_{(1)}, \dots, x_{(n)}, k_1, \dots, k_n) = k_1\, x_{(1)} + \cdots + k_n\, x_{(n)}$, and in a similar way, the general solution for an inhomogeneous linear system can be written as an affine function of $n+1$ independent particular solutions.

Another very remarkable example is the Riccati equation

$$\frac{dx(t)}{dt} = a_2(t)\, x^2(t) + a_1(t)\, x(t) + a_0(t),$$

for which the superposition formula comes from the fact that the cross ratio of four different solutions is a constant, see, e.g., [10].

The main point is that Lie systems are always related with Lie systems on Lie groups defined by right-invariant vector fields. Let G be a Lie group. If $\{a_1, \dots, a_r\}$ is a basis of the tangent space T_eG at the neutral element $e \in G$ and X_α^R denotes the right-invariant vector field in G such that $X_\alpha^R(e) = a_\alpha$, a Lie system on G will be written as

$$\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g(t)). \tag{3}$$

When applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides we obtain

$$(R_{g(t)^{-1}})_{*g(t)}(\dot{g}(t)) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha, \tag{4}$$

which is usually written, with a slight abuse of notation, as

$$(\dot{g}\, g^{-1})(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha.$$

This equation is right-invariant: if $\bar{g}(t)$ is a solution with initial condition $\bar{g}(0) = e$, the solution with initial condition $g(0) = g_0$ is given by $\bar{g}(t) g_0$. Therefore, we only need to find the solution of (4) starting from the neutral element.

Let H be any closed subgroup of G, $M = G/H$ the corresponding homogeneous space, $\tau : G \to G/H$ the natural projection, and $\Phi : G \times M \to M$ the usual left action of G on M. The right-invariant vector fields X_α^R are τ-projectable onto the corresponding fundamental vector fields $-X_\alpha = -X_{a_\alpha}$, i.e.,

$$\tau_{*g} X_\alpha^R(g) = -X_\alpha(gH),$$

and therefore we have a Lie system on $M = G/H$ associated to (3):

$$\dot{x}(t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x(t)),$$

where we denote $x = gH \in M$. Then, the solution of this system starting from x_0 is given by $x(t) = \Phi(g(t), x_0)$, where $g(t)$ is the solution of (3) starting from the identity. In this sense the equation (3) has a universal character, and it will have an associated Lie system on each homogeneous space of G [7].

Lie systems are of interest not only in the theory of differential equations but also in other related fields. For example, they are important in classical or even in quantum physics (for instance, in order to study the non-relativistic dynamics of a spin 1/2 particle, when only the spinorial part is considered [5, 6]). Another field where Lie systems play an important rôle is in geometric control theory.

The aim of this article is to illustrate these applications. Thus, after a brief account of a generalization of the method proposed by Wei and Norman [6, 8, 12, 23, 24], to be used later, we will study the particular case where the Lie systems of interest are Hamiltonian systems as well, both in the classical and quantum frameworks. The theory is illustrated through the particularly interesting example of generic classical and quantum quadratic time-dependent Hamiltonians. In particular, we show that there exist t-dependent quantum systems for which one is able to write in an explicit way the time evolution of any state of the system. The very simple case of both the classical and quantum time-dependent linear potential will be explicitly solved. We will show as well the use of the theory of Lie systems in geometric control theory, when dealing with drift-free systems that are linear in the control functions. In particular, we will study from this new perspective several well-known control systems: the robot unicycle [21], the Brockett nonholonomic integrator [4] and its realization in the model of a hopping robot in flight phase [20], and the kinematic equations of a generalization due to Jurdjevic [15] of the elastic problem of Euler.

2. The Wei and Norman method. Let G be a Lie group as in the previous section. We are interested in finding the curve $g(t) \in G$ such that

$$\dot{g}(t)\, g(t)^{-1} = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha, \tag{5}$$

with $g(0) = e \in G$. We can use a method which is a generalization of the method proposed by Wei and Norman, in order to find the time-evolution operator for linear systems of type

$$\frac{dU(t)}{dt} = H(t)U(t),$$

with $U(0) = I$. The generalized Wei–Norman method consists of writing the previous $g(t)$ in terms of a set of second kind canonical coordinates,

$$g(t) = \prod_{\alpha=1}^{r} \exp(-v_\alpha(t) a_\alpha) = \exp(-v_1(t)a_1) \cdots \exp(-v_r(t)a_r),$$

and transforming the equation (5) into a system of differential equations for the $v_\alpha(t)$, with initial conditions $v_\alpha(0) = 0$, $\alpha = 1, \ldots, r$. Such a system is obtained from the following relation:

$$\sum_{\alpha=1}^{r} \dot{v}_\alpha \Big(\prod_{\beta<\alpha} \exp(-v_\beta(t)\, \mathrm{ad}(a_\beta)) \Big) a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha. \tag{6}$$

If the Lie algebra of G is solvable, the solution of the previous system can be obtained by quadratures. If instead, the Lie algebra of G is semi-simple, then the integrability by quadratures is not assured [6, 8, 12, 23, 24].

3. Hamiltonian systems of Lie type. An interesting and important case occurs when (M,Ω) is a symplectic manifold and the vector fields in M arising in the expression of the t-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a finite-dimensional real Lie algebra $\mathfrak{g}$. These vector fields correspond to a symplectic action of a Lie group G with Lie algebra $\mathfrak{g}$ on the symplectic manifold (M,Ω).

The Hamiltonian functions h_α of such vector fields, defined by $i(X_\alpha)\Omega = -dh_\alpha$, in general do not close on the same Lie algebra $\mathfrak{g}$ when the Poisson bracket is considered, since we can only assure that

$$d(\{h_\alpha, h_\beta\} - h_{[\alpha,\beta]}) = 0,$$

and therefore, they span a Lie algebra extension of the original one.

The situation in quantum mechanics is quite similar: the Hilbert space $\mathcal{H}$ can be seen as a real manifold with a global chart. The tangent space $T_\phi\mathcal{H}$ at any point $\phi \in \mathcal{H}$ can be identified with $\mathcal{H}$ itself, where the isomorphism which associates $\psi \in \mathcal{H}$ with the vector $\dot\psi \in T_\phi\mathcal{H}$ is given by:

$$\dot\psi f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0}, \qquad \forall f \in C^\infty(\mathcal{H}).$$

The Hilbert space $\mathcal{H}$ is endowed with a symplectic 2-form Ω defined by

$$\Omega_\phi(\dot\psi, \dot\psi') = 2\,\mathrm{Im}\langle\psi|\psi'\rangle.$$

A vector field is just a map $A{:}\,\mathcal{H} \to \mathcal{H}$; therefore a linear operator A on $\mathcal{H}$ is a special kind of vector field. Given a smooth function $a{:}\,\mathcal{H} \to \mathbf{R}$, its differential da_ϕ at $\phi \in \mathcal{H}$ is an element of the (real) dual $\mathcal{H}'$ given by

$$\langle da_\phi, \psi\rangle := \left(\frac{d}{dt} a(\phi + t\psi)\right)_{|t=0}.$$

Actually, the skew-self-adjoint linear operators $-i\,A$ in $\mathcal{H}$, for a self-adjoint operator A, define Hamiltonian vector fields, the Hamiltonian function of $-i\,A$ being $a(\phi) = \frac{1}{2}\langle\phi, A\phi\rangle$. Therefore, the Schrödinger equation plays the rôle of Hamilton equations, because it determines the integral curves of the vector field $-i\,H$, where H is the Hamiltonian of the system [2].

In particular, the theory of Lie systems applies in the previous framework when we have a t-dependent quantum Hamiltonian that can be written as a linear combination, with t-dependent coefficients, of Hamiltonians H_i closing on a finite-dimensional real Lie algebra under the commutator bracket. However, note that this Lie algebra does not necessarily coincide with that of the corresponding classical problem, but it may be a Lie algebra extension of the latter.

4. Time-dependent quadratic Hamiltonians. For the illustration of the classical and quantum situations described in the previous section, we consider now the important examples provided by the time-dependent classical and quantum quadratic Hamiltonians.

The first one is the mechanical system for which the configuration space is the real line $\mathbf{R}$, the corresponding phase space $T^*\mathbf{R}$, endowed with its canonical symplectic structure

$\omega = dq \wedge dp$, and the time-dependent classical Hamiltonian

$$H = \alpha(t)\,\frac{p^2}{2} + \beta(t)\,\frac{q\,p}{2} + \gamma(t)\,\frac{q^2}{2} + \delta(t)\,p + \epsilon(t)\,q. \tag{7}$$

The dynamical vector field solution of the dynamical equation

$$i(\Gamma_H)\,\omega = dH,$$

is given by

$$\Gamma_H = \left(\alpha(t)\,p + \frac{1}{2}\beta(t)\,q + \delta(t)\right)\frac{\partial}{\partial q} - \left(\frac{1}{2}\beta(t)\,p + \gamma(t)\,q + \epsilon(t)\right)\frac{\partial}{\partial p}, \tag{8}$$

which can be rewritten as

$$\Gamma_H = \alpha(t)\,X_1 + \beta(t)\,X_2 + \gamma(t)\,X_3 - \delta(t)\,X_4 + \epsilon(t)\,X_5,$$

with

$$X_1 = p\,\frac{\partial}{\partial q},\ X_2 = \frac{1}{2}\left(q\,\frac{\partial}{\partial q} - p\,\frac{\partial}{\partial p}\right),\ X_3 = -q\,\frac{\partial}{\partial p},\ X_4 = -\frac{\partial}{\partial q},\ X_5 = -\frac{\partial}{\partial p},$$

being vector fields which satisfy the following commutation relations:

$$\begin{aligned}
&[X_1, X_2] = X_1, \quad [X_1, X_3] = 2\,X_2, \quad [X_1, X_4] = 0, \quad [X_1, X_5] = -X_4,\\
&[X_2, X_3] = X_3, \quad [X_2, X_4] = -\frac{1}{2}\,X_4, \quad [X_2, X_5] = \frac{1}{2}\,X_5,\\
&[X_3, X_4] = X_5, \quad [X_3, X_5] = 0, \quad [X_4, X_5] = 0,
\end{aligned} \tag{9}$$

and therefore they close on a five-dimensional real Lie algebra. Consider the abstract, five-dimensional, Lie algebra $\mathfrak{g}$ such that in a basis $\{a_1,\ a_2,\ a_3,\ a_4,\ a_5\}$, the Lie products are analogous to that of (9). Then, $\mathfrak{g}$ is a semi-direct sum of the Abelian two-dimensional Lie algebra generated by $\{a_4,\ a_5\}$ with the $\mathfrak{sl}(2, \mathbf{R})$ Lie algebra generated by $\{a_1,\ a_2,\ a_3\}$, i.e., $\mathfrak{g} = \mathbf{R}^2 \rtimes \mathfrak{sl}(2,\ \mathbf{R})$. The corresponding Lie group will be the semi-direct product $G = T_2 \odot SL(2, \mathbf{R})$ relative to the linear action of $SL(2, \mathbf{R})$ on the two-dimensional translation algebra. When computing the flows of the previous vector fields X_α, we see that they correspond to the affine action of G on $\mathbf{R}^2$, and therefore, the vector fields X_α can be regarded as fundamental fields with respect to that action, associated to the previous basis of the Lie algebra.

In order to find the time-evolution provided by the Hamiltonian (7), i.e., the integral curves of the time-dependent vector field (8), we can solve first the corresponding equation in the Lie group G and then use the affine action of G on $\mathbf{R}^2$. We focus on the first of these questions: we should find the curve $g(t)$ in G such that

$$\dot g\,g^{-1} = -\sum_{i=1}^{5} b_i(t)\,a_i, \qquad g(0) = e,$$

with $b_1(t) = \alpha(t)$, $b_2(t) = \beta(t)$, $b_3(t) = \gamma(t)$, $b_4(t) = -\delta(t)$, and $b_5(t) = \epsilon(t)$. The explicit calculation can be carried out by using the generalized Wei–Norman method, i.e., writing $g(t)$ in terms of a set of second class canonical coordinates, for instance,

$$g(t) = \exp(-v_4(t)a_4)\exp(-v_5(t)a_5)\exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3),$$

and then, a straightforward application of (6) leads to the system

$$\dot v_1 = b_1 + b_2\, v_1 + b_3\, v_1^2, \quad \dot v_2 = b_2 + 2\, b_3\, v_1, \quad \dot v_3 = e^{v_2}\, b_3,$$
$$\dot v_4 = b_4 + \frac{1}{2}\, b_2\, v_4 + b_1\, v_5, \quad \dot v_5 = b_5 - b_3\, v_4 - \frac{1}{2}\, b_2\, v_5,$$

with initial conditions $v_1(0) = \cdots = v_5(0) = 0$.

For some specific choices of the functions $\alpha(t), \ldots, \epsilon(t)$, the problem becomes simpler and it may be enough to consider a subgroup, instead of the whole Lie group G, to deal with the arising system. For instance, consider the classical Hamiltonian

$$H = \frac{p^2}{2m} + f(t)\, q,$$

which in the notation of (7) has the only non-vanishing coefficients $\alpha(t) = 1/m$ and $\epsilon(t) = f(t)$. Then, the problem is reduced to one in a three-dimensional subalgebra, generated by $\{X_1,\, X_4,\, X_5\}$. The associated Lie group will be the subgroup of G generated by $\{a_1, a_4, a_5\}$. This example will be used later for illustrating the theory: since such a subgroup is solvable, the problem can be integrated by quadratures.

Another remarkable property is that the Hamiltonian functions h_α corresponding to the Hamiltonian vector fields $X_1, \ldots, X_5$, defined by $i(X_\alpha)\omega = -dh_\alpha$, i.e.,

$$h_1(q,p) = -\frac{p^2}{2}, \quad h_2(q,p) = -\frac{1}{2} q\, p, \quad h_3(q,p) = -\frac{q^2}{2}, \quad h_4(q,p) = p, \quad h_5(q,p) = -q,$$

have almost the same Poisson bracket relations as the vector fields X_α, but they do not coincide because of $\{h_4, h_5\} = 1$, instead of $[X_4, X_5] = 0$. In other words, they close on a Lie algebra which is a central extension of $\mathbf{R}^2 \rtimes \mathfrak{sl}(2, \mathbf{R})$ by a one-dimensional algebra.

Let us now consider the quantum case [25], with applications in a number of physical problems, as for instance, the quantum motion of charged particles subject to time-dependent electromagnetic fields (see, e.g., [14]), and connects with the theory of exact invariants developed by Lewis and Riesenfeld (see [18] and references therein).

A generic time-dependent quadratic quantum Hamiltonian is given by

$$H = \alpha(t)\, \frac{P^2}{2} + \beta(t)\, \frac{Q\,P + P\,Q}{4} + \gamma(t)\, \frac{Q^2}{2} + \delta(t) P + \epsilon(t)\, Q + \phi(t) I. \tag{10}$$

where Q and P are the position and momentum operators satisfying the commutation relation

$$[Q, P] = i\, I.$$

The previous Hamiltonian can be written as a sum with t-dependent coefficients

$$H = \alpha(t)\, H_1 + \beta(t)\, H_2 + \gamma(t)\, H_3 - \delta(t)\, H_4 + \epsilon(t)\, H_5 - \phi(t) H_6,$$

of the Hamiltonians

$$H_1 = \frac{P^2}{2},\ H_2 = \frac{1}{4}(QP + P\,Q),\ H_3 = \frac{Q^2}{2},\ H_4 = -P,\ H_5 = Q,\ H_6 = -I,$$

which satisfy the commutation relations

$$[iH_1, iH_2] = iH_1, \quad [iH_1, iH_3] = 2\, iH_2, \quad [iH_1, iH_5] = -iH_4, \quad [iH_2, iH_3] = iH_3,$$
$$[iH_2, iH_4] = -\frac{i}{2}\, H_4, \quad [iH_2, iH_5] = \frac{i}{2}\, H_5, \quad [iH_3, iH_4] = iH_5, \quad [iH_4, iH_5] = iH_6,$$

and $[iH_1, iH_4] = [iH_3, iH_5] = [iH_\alpha, iH_6] = 0$, $\alpha = 1, \ldots, 5$. That is, the skew-self-adjoint operators iH_α generate a six-dimensional real Lie algebra which is a central extension of the Lie algebra arising in the classical case, $\mathbf{R}^2 \rtimes \mathfrak{sl}(2, \mathbf{R})$, by a one-dimensional Lie algebra. It can be identified as the semi-direct sum of the Heisenberg–Weyl Lie algebra $\mathfrak{h}(3)$, which is an ideal in the total Lie algebra, with the Lie subalgebra $\mathfrak{sl}(2, \mathbf{R})$, i.e., $\mathfrak{h}(3) \rtimes \mathfrak{sl}(2, \mathbf{R})$. Sometimes this Lie algebra is referred to as the extended symplectic Lie algebra $\mathfrak{hsp}(2, \mathbf{R}) = \mathfrak{h}(3) \rtimes \mathfrak{sp}(2, \mathbf{R})$. The corresponding Lie group is the semi-direct product $H(3) \odot SL(2, \mathbf{R})$ of the Heisenberg–Weyl group $H(3)$ with $SL(2, \mathbf{R})$, see also [25].

The time-evolution of a quantum system can be described in terms of the evolution operator $U(t)$ which satisfies the Schrödinger equation (see, e.g., [13])

$$i\frac{dU}{dt} = H(t)U, \qquad U(0) = \mathrm{Id},$$

where $H(t)$ is the Hamiltonian of the system. In our current case, the Hamiltonian is given by (10), and therefore the time-evolution of the system is given by an equation of the type

$$\dot{g}\, g^{-1} = -\sum_{\alpha=1}^{6} b_\alpha(t)\, a_\alpha, \qquad g(0) = e, \tag{11}$$

with the identification of $g(t)$ with $U(t)$, e with Id, iH_α with a_α for $\alpha \in \{1, \ldots, 6\}$ and the time-dependent coefficients $b_\alpha(t)$ are given by

$$\begin{aligned} &b_1(t) = \alpha(t), \quad b_2(t) = \beta(t), \;\; b_3(t) = \gamma(t), \\ &b_4(t) = -\delta(t), \;\; b_5(t) = \epsilon(t), \;\; b_6(t) = -\phi(t). \end{aligned}$$

We would like to remark that time-dependent quantum Hamiltonians are seldom studied, because it is generally difficult to find their time evolution. However, in the case the system could be treated as a Lie system in a certain Lie group, the calculation of the evolution operator is reduced to the problem of integrating the system appearing after the application of the Wei–Norman method. In the case the associated Lie group is solvable, the integration can be made by quadratures, leading to an exact solution of the problem. We will see an example in the next section.

The calculation of the solution of (11) can be carried out by using the generalized Wei–Norman method, i.e., writing $g(t)$ in terms of a set of second class canonical coordinates. We take, for instance, the factorization

$$\begin{aligned} g(t) = {} & \exp(-v_4(t)a_4)\exp(-v_5(t)a_5)\exp(-v_6(t)a_6) \\ & \times \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3), \end{aligned}$$

and therefore, the equation (6) leads in this case to the system

$$\begin{aligned} &\dot{v}_1 = b_1 + b_2\, v_1 + b_3\, v_1^2, \quad \dot{v}_2 = b_2 + 2\, b_3\, v_1, \quad \dot{v}_3 = e^{v_2}\, b_3, \\ &\dot{v}_4 = b_4 + \frac{1}{2}\, b_2\, v_4 + b_1\, v_5, \quad \dot{v}_5 = b_5 - b_3\, v_4 - \frac{1}{2}\, b_2\, v_5, \\ &\dot{v}_6 = b_6 + b_5\, v_4 - \frac{1}{2}\, b_3\, v_4^2 + \frac{1}{2}\, b_1\, v_5^2, \end{aligned}$$

with initial conditions $v_1(0) = \cdots = v_6(0) = 0$.

Analogously to what happened in the classical case, special choices of the time-dependent coefficient functions $\alpha(t), \ldots, \phi(t)$ may lead to problems for which the associated Lie algebra is a subalgebra of that of the complete system, and similarly for the Lie groups involved. For example, we could consider as well the quantum Hamiltonian linear in the positions

$$H = \frac{P^2}{2m} + f(t)\,Q,$$

which in the notation of (10) has the only non-vanishing coefficients $\alpha(t) = 1/m$ and $\epsilon(t) = f(t)$. This problem can be regarded as a Lie system associated to the four-dimensional Lie algebra generated by $\{iH_1,\, iH_4,\, iH_5,\, iH_6\}$, which is also solvable, and hence the problem can be solved by quadratures.

The treatment of this system, as well as that of its classical version, according to the theory of Lie systems, is the subject of the next section.

5. An example: classical and quantum time-dependent linear potential. Let us consider the classical system described by the classical Hamiltonian

$$H_c = \frac{p^2}{2m} + f(t)\,q, \tag{12}$$

and the corresponding quantum Hamiltonian

$$H_q = \frac{P^2}{2m} + f(t)\,Q, \tag{13}$$

describing, for instance when $f(t) = e\,E_0 + e\,E\,\cos\omega t$, the motion of a particle of electric charge e and mass m driven by a monochromatic electric field.

We will study in parallel the classical and the quantum problems by reduction of both of them to similar equations, and solving them by the generalized Wei–Norman method. The only difference between the two cases is that the Lie algebra arising in the quantum problem is a central extension of that of the classical one.

The classical Hamilton equations of motion for the Hamiltonian (12) are

$$\begin{cases} \dot q &= \dfrac{p}{m}, \\ \dot p &= -f(t), \end{cases} \tag{14}$$

and therefore, the motion is given by

$$\begin{aligned} q(t) &= q_0 + \frac{p_0\,t}{m} - \frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\,dt'', \\ p(t) &= p_0 - \int_0^t f(t')\,dt'. \end{aligned} \tag{15}$$

The t-dependent vector field describing the time evolution,

$$X = \frac{p}{m}\,\frac{\partial}{\partial q} - f(t)\,\frac{\partial}{\partial p}.$$

can be written as a linear combination

$$X = \frac{1}{m}\,X_1 - f(t)\,X_2,$$

with $X_1 = p\,\partial/\partial q$ and $X_2 = \partial/\partial p$ being vector fields closing on a 3-dimensional Lie algebra with $X_3 = \partial/\partial q$, isomorphic to the Heisenberg–Weyl algebra, namely,

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = 0.$$

The flow of these vector fields is given, respectively, by

$$\begin{aligned}\phi_1(t,(q_0,p_0)) &= (q_0 + p_0\,t, p_0),\\ \phi_2(t,(q_0,p_0)) &= (q_0, p_0 + t),\\ \phi_3(t,(q_0,p_0)) &= (q_0 + t, p_0).\end{aligned}$$

In other words, $\{X_1,\, X_2,\, X_3\}$ are fundamental vector fields with respect to the action of the Heisenberg–Weyl group $H(3)$, realized as the Lie group of upper triangular 3×3 matrices, on $\mathbf{R}^2$ given by

$$\begin{pmatrix} \bar q \\ \bar p \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \\ 1 \end{pmatrix}.$$

Note that X_1, X_2 and X_3 are Hamiltonian vector fields with respect to the usual symplectic structure, $\Omega = dq \wedge dp$, while the corresponding Hamiltonian functions h_α such that $i(X_\alpha)\Omega = -dh_\alpha$ are

$$h_1 = -\frac{p^2}{2}, \qquad h_2 = q, \qquad h_3 = -p,$$

and therefore

$$\{h_1, h_2\} = -h_3, \quad \{h_1, h_3\} = 0, \quad \{h_2, h_3\} = -1.$$

Then, the functions $\{h_1,\, h_2,\, h_3\}$, jointly with $h_4 = 1$, close on a four-dimensional Lie algebra under the Poisson bracket which is a central extension of that generated by $\{X_1,\, X_2,\, X_3\}$.

If $\{a_1,\, a_2,\, a_3\}$ is a basis of the Lie algebra with non-vanishing defining relations $[a_1, a_2] = -a_3$, the corresponding equation in the group $H(3)$ to the system (14) reads

$$\dot g\, g^{-1} = -\frac{1}{m}\, a_1 + f(t)\, a_2.$$

Using the Wei–Norman formula (6) with $g = \exp(-u_3\, a_3)\exp(-u_2\, a_2)\exp(-u_1\, a_1)$ we arrive at the system of differential equations

$$\dot u_1 = \frac{1}{m}, \qquad \dot u_2 = -f(t), \qquad \dot u_3 - \dot u_1\, u_2 = 0,$$

together with the initial conditions $u_1(0) = u_2(0) = u_3(0) = 0$, with solution

$$u_1 = \frac{t}{m}, \quad u_2 = -\int_0^t f(t')\, dt', \quad u_3 = -\frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\, dt''.$$

Therefore, the motion is given by

$$\begin{pmatrix} q \\ p \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{m} & -\frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\, dt'' \\ 0 & 1 & -\int_0^t f(t')\, dt' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \\ 1 \end{pmatrix},$$

in agreement with (15). We can immediately identify the constants of motion

$$\begin{aligned} I_1 &= p(t) + \int_0^t f(t')\,dt', \\ I_2 &= q(t) - \frac{1}{m}(p(t) + \int_0^t f(t')\,dt')t + \frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\,dt''. \end{aligned}$$

As far as the quantum problem is concerned, notice that the quantum Hamiltonian H_q may be written as a sum

$$H_q = \frac{1}{m}\,H_1 - f(t)\,H_2,$$

with

$$H_1 = \frac{P^2}{2}, \qquad H_2 = -Q.$$

The skew-self-adjoint operators $-i\,H_1$ and $-i\,H_2$ close on a four-dimensional Lie algebra with $-i\,H_3 = -i\,P$, and $-iH_4 = i\,I$, isomorphic to the above mentioned central extension of the Heisenberg–Weyl Lie algebra,

$$[-iH_1, -iH_2] = -iH_3,\ [-iH_1, -iH_3] = 0,\ [-iH_2, -iH_3] = -iH_4.$$

As we have seen in the preceding section, the time-evolution of our current system is described by means of the evolution operator U, which satisfies

$$\frac{dU}{dt} = -iH_qU, \qquad U(0) = \mathrm{Id}.$$

This equation can be identified as that of a Lie system in a Lie group such that its Lie algebra is the one mentioned above. Let $\{a_1, a_2, a_3, a_4\}$ be a basis of the Lie algebra with non-vanishing defining relations $[a_1, a_2] = a_3$ and $[a_2, a_3] = a_4$. The equation in the group to be considered now is

$$\dot g\, g^{-1} = -\frac{1}{m}\,a_1 + f(t)\,a_2.$$

Using $g = \exp(-u_4\,a_4)\exp(-u_3\,a_3)\,\exp(-u_2\,a_2)\,\exp(-u_1\,a_1)$, the Wei–Norman method provides the following equations:

$$\begin{aligned} \dot u_1 &= \frac{1}{m}, & \dot u_2 &= -f(t), \\ \dot u_3 &= -\frac{1}{m}\,u_2, & \dot u_4 &= f(t)\,u_3 + \frac{1}{2m}\,u_2^2, \end{aligned}$$

together with the initial conditions $u_1(0) = u_2(0) = u_3(0) = u_4(0) = 0$, whose solution is

$$u_1(t) = \frac{t}{m}, \quad u_2(t) = -\int_0^t f(t')\,dt', \quad u_3(t) = \frac{1}{m}\int_0^t dt' \int_0^{t'} f(t'')\,dt'',$$

and

$$u_4(t) = \frac{1}{m}\int_0^t dt' f(t') \int_0^{t'} dt'' \int_0^{t''} f(t''')\,dt''' + \frac{1}{2m}\int_0^t dt' \left(\int_0^{t'} dt'' f(t'')\right)^2.$$

These functions provide the explicit form of the time-evolution operator:

$$U(t, 0) = \exp(-iu_4(t))\exp(iu_3(t)P)\exp(-iu_2(t)Q)\exp(iu_1(t)P^2/2).$$

However, in order to find the expression of the wave-function in a simple way, it is advantageous to use instead the factorization

$$g = \exp(-v_4\, a_4) \exp(-v_2\, a_2)\, \exp(-v_3\, a_3)\, \exp(-v_1\, a_1).$$

In such a case, the Wei–Norman method gives the system

$$\begin{aligned} \dot v_1 &= \frac{1}{m}, & \dot v_2 &= -f(t), \\ \dot v_3 &= -\frac{1}{m}\, v_2, & \dot v_4 &= -\frac{1}{2m} v_2^2, \end{aligned}$$

jointly with the initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0$. The solution is

$$\begin{aligned} v_1(t) &= \frac{t}{m}, \quad v_2(t) = -\int_0^t dt'\, f(t'), \\ v_3(t) &= \frac{1}{m}\int_0^t dt' \int_0^{t'} dt'' f(t''), \\ v_4(t) &= -\frac{1}{2m}\int_0^t dt' \left(\int_0^{t'} dt'' f(t'')\right)^2 . \end{aligned}$$

Then, applying the evolution operator on the initial wave-function $\phi(p, 0)$, which is assumed to be written in momentum representation, we have

$$\begin{aligned} \phi(p,t) &= U(t,0)\phi(p,0) \\ &= \exp(-iv_4(t)) \exp(-iv_2(t)Q) \exp(iv_3(t)P) \exp(iv_1(t)P^2/2)\phi(p,0) \\ &= \exp(-iv_4(t)) \exp(-iv_2(t)Q) e^{i(v_3(t)p+v_1(t)p^2/2)}\phi(p,0) \\ &= \exp(-iv_4(t)) e^{i(v_3(t)(p+v_2(t))+v_1(t)(p+v_2(t))^2/2)}\phi(p+v_2(t),0), \end{aligned}$$

where the functions $v_i(t)$ are given by the preceding equations.

6. Applications in control theory. Control systems are described by systems of differential equations

$$\frac{dx^i}{dt} = F(x^i, u^\alpha), \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m, \tag{16}$$

where u^α are the so-called control functions or simply controls, which are to be determined in such a way that, e.g., the trajectory passes through one or two specific points in the configuration space, or maybe gives some cost functional a stationary value.

A control system is said to be controllable if for any given initial point p there exists an integral curve of the corresponding vector field along $\pi : \mathbf{R}^{n+m} \to \mathbf{R}^n$ such that $(\pi \circ \gamma)(0) = p$, and a value t_1 of the parameter of the curve γ such that $(\pi \circ \gamma)(t_1) = q$ for any final point q.

Consider the case of drift-free systems, linear in the control functions $u^\alpha(t)$, for which the time-dependent vector field, whose integral curves are the solutions of (16), is

$$X(t, x) = u^1(t)X_1(x) + \cdots + u^r(t)X_r(x).$$

Lie systems arise when the vector fields X_α close on a finite-dimensional real Lie algebra. The cases in which the X_α are either right-invariant vector fields in a certain Lie group G, or vector fields in a homogeneous space of G, can be dealt with according to the theory of Lie systems: reducing the problem to solving an equation on the Lie group G, of the form

$$\dot{g}(t) = \sum_{\alpha=1}^{r} u_\alpha(t) X_\alpha(g(t)).$$

Controllability of control systems on Lie groups has been analyzed by Brockett [3] and Jurdjevic and Sussmann [16]. It can be determined by studying algebraic properties of the corresponding Lie algebra $\mathfrak{g}$.

THEOREM. *A drift-free right invariant system on a connected Lie group G is controllable if and only if the Lie algebra generated by $\{X_1, \ldots, X_r\}$ is $\mathfrak{g}$.*

Proof. If $\mathfrak{h}$ is the Lie algebra generated by $\{X_1, \ldots, X_r\}$, then the Lie algebra of the Lie system is not $\mathfrak{g}$ but the subalgebra $\mathfrak{h}$. The orbit of the neutral element $e \in G$ is the subgroup H of G with Lie subalgebra $\mathfrak{h}$. It is then clear that if $\mathfrak{h}$ is a proper subalgebra of G, the system is not controllable, while it is so when $\mathfrak{h} = \mathfrak{g}$.

As examples of application of the theory of Lie systems to specific systems treated in control theory, we will study several well-known systems: the robot unicycle [17, 21], the Brockett system termed sometimes as Brockett nonholonomic integrator [4], and a system which under certain approximation can be reduced to the former, i.e., the model of a hopping robot in flight phase [20]. Afterwards, we will study the kinematic equations of a generalization due to Jurdjevic [15] of the elastic problem of Euler, and finally we will briefly show how the reduction theory of Lie systems can be applied to two of these examples.

6.1. *Robot unicycle or model of an automobile as a Lie system.* Our first example corresponds to the robot unicycle (see, e.g., [17]). Essentially, the same control system arises in a very simplified model of maneuvering an automobile [21].

The configuration space is $\mathbf{R}^2 \times S^1$, with coordinates (x_1, x_2, x_3). The control system can be written as

$$\dot{x}_1 = b_2(t) \sin x_3, \quad \dot{x}_2 = b_2(t) \cos x_3, \quad \dot{x}_3 = b_1(t), \tag{17}$$

where $b_1(t)$ and $b_2(t)$ are the control functions. Its solutions are the integral curves of $b_1(t)\, X_1(x) + b_2(t)\, X_2(x)$, where

$$X_1 = \frac{\partial}{\partial x_3}, \qquad X_2 = \sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2}.$$

The Lie bracket of both vector fields,

$$X_3 = [X_1, X_2] = \cos x_3 \frac{\partial}{\partial x_1} - \sin x_3 \frac{\partial}{\partial x_2},$$

is linearly independent of X_1, X_2. They satisfy

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = -X_2,$$

therefore closing on a Lie algebra isomorphic to $\mathfrak{se}(2)$. This Lie algebra has a basis $\{a_1, a_2, a_3\}$ for which

$$[a_1, a_2] = a_3, \quad [a_2, a_3] = 0, \quad [a_1, a_3] = -a_2.$$

Writing the solution of the associated problem in $SE(2)$ as

$$g(t) = \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3),$$

the Wei–Norman method leads to the system

$$\dot v_1 = b_1, \quad \dot v_2 = b_2\,\cos v_1, \quad \dot v_3 = b_2\,\sin v_1,$$

with $v_1(0) = v_2(0) = v_3(0) = 0$. Denoting $B_1(t) = \int_0^t b_1(s)\,ds$, the solution is

$$v_1(t) = B_1(t), \quad v_2(t) = \int_0^t b_2(s)\cos B_1(s)\,ds, \quad v_3(t) = \int_0^t b_2(s)\sin B_1(s)\,ds.$$

The action of $SE(2)$ on $\mathbf{R}^2 \times S^1$ such that X_1, X_2, X_3 are the associated fundamental vector fields turns out to be $\Phi((\theta,\, a,\, b), (x_1,\, x_2,\, x_3)) = (\bar x_1, \bar x_2, \bar x_3)$ with

$$\bar x_1 = x_1 - b\cos x_3 - a\sin x_3, \quad \bar x_2 = x_2 + b\sin x_3 - a\cos x_3, \quad \bar x_3 = x_3 - \theta,$$

where $(\theta,\, a,\, b)$ are the second kind canonical coordinates determined by the factorization $g = \exp(\theta a_1)\exp(a a_2)\exp(b a_3)$. The composition law is

$$(\theta, a, b)(\theta', a', b') = (\theta + \theta', a' + a\cos\theta' + b\sin\theta', b' - a\sin\theta' + b\cos\theta').$$

Then, the general solution of (17) is

$$\begin{aligned}\Phi((-v_1, -v_2, -v_3), (x_{10}, x_{20}, x_{30})) = (&x_{10} + v_3\cos x_{30} + v_2\sin x_{30},\\ &x_{20} - v_3\sin x_{30} + v_2\cos x_{30}, x_{30} + v_1),\end{aligned}$$

where $v_1 = v_1(t)$, $v_2 = v_2(t)$ and $v_3 = v_3(t)$ are given above.

In an alternative way, as the vector fields X_2 and X_3 commute, there exist coordinates (y_1, y_2, y_3) such that $X_2 = \partial/\partial y_2$ and $X_3 = \partial/\partial y_3$. For instance,

$$y_2 = x_1\,\sin x_3 + x_2\,\cos x_3, \quad y_3 = x_1\,\cos x_3 - x_2\,\sin x_3,$$

which can be completed with $y_1 = x_3$. Then,

$$X_1 = \frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_3}.$$

The control system of interest, whose solutions are again the integral curves of the time-dependent vector field $b_1(t)\,X_1 + b_2(t)\,X_2$, reads

$$\dot y_1 = b_1(t), \quad \dot y_2 = b_1(t)y_3 + b_2(t), \quad \dot y_3 = -b_1(t)y_2. \tag{18}$$

Now, the expression of the previous action in terms of the coordinates $(y_1,\, y_2,\, y_3)$ is

$$\begin{aligned}\Phi((\theta,\, a,\, b),\, (y_1,\, y_2,\, y_3)) = (&y_1 - \theta,\, y_2\cos\theta - y_3\sin\theta - a\cos\theta + b\sin\theta,\\ &y_2\sin\theta + y_3\cos\theta - a\sin\theta - b\cos\theta),\end{aligned}$$

and therefore, the general solution of (18) is $\Phi((-v_1, -v_2, -v_3), (y_{10}, y_{20}, y_{30}))$, i.e.,

$$\begin{aligned} y_1 &= y_{10} + v_1,\\ y_2 &= y_{20}\cos v_1 + y_{30}\sin v_1 + v_2\cos v_1 + v_3\sin v_1,\\ y_3 &= y_{30}\cos v_1 - y_{20}\sin v_1 + v_3\cos v_1 - v_2\sin v_1,\end{aligned}$$

where the v_i's are those given above.

6.2. *Brockett nonholonomic control system.* Another interesting example introduced by Brockett, when dealing with problems of optimal control and its relation with singular Riemannian geometry, is related with the three-dimensional Heisenberg–Weyl group $H(3)$, which is the lowest-dimensional non-Abelian nilpotent Lie group. Such a system is very often considered as one of the prototypical examples relating control theory and extremal problems in sub-Riemannian geometry,

It is the control system in $\mathbf{R}^3$, with coordinates (x, y, z)

$$\dot{x} = b_1(t), \quad \dot{y} = b_2(t), \quad \dot{z} = b_2(t)x - b_1(t)y, \tag{19}$$

where the functions $b_1(t)$ and $b_2(t)$ are regarded as the controls. The solutions of this system are the integral curves of the time-dependent vector field $b_1(t)\,X_1 + b_2(t)\,X_2$, with

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

The Lie bracket

$$X_3 = [X_1, X_2] = 2\frac{\partial}{\partial z}$$

is linearly independent from X_1, X_2, and the set $\{X_1, X_2, X_3\}$ close on the Lie algebra defined by

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = 0, \qquad [X_2, X_3] = 0, \tag{20}$$

isomorphic to the Lie algebra $\mathfrak{h}(3)$ of the Heisenberg–Weyl group $H(3)$.

The Lie algebra $\mathfrak{h}(3)$ has a basis $\{a_1, a_2, a_3\}$ for which the Lie products are

$$[a_1, a_2] = a_3, \quad [a_1, a_3] = 0, \quad [a_2, a_3] = 0.$$

A generic Lie system for the particular case of $H(3)$ takes the form

$$R_{g(t)^{-1}*g(t)}(\dot{g}(t)) = -b_1(t)a_1 - b_2(t)a_2 - b_3(t)a_3,$$

and we are now interested in the one with $b_3(t) = 0$ for all t, i.e.,

$$R_{g(t)^{-1}*g(t)}(\dot{g}(t)) = -b_1(t)a_1 - b_2(t)a_2. \tag{21}$$

Writing the solution starting from the identity of (21) as the product of exponentials

$$g(t) = \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3),$$

and applying the Wei–Norman formula (6) we find the system of differential equations

$$\dot{v}_1 = b_1, \quad \dot{v}_2 = b_2, \quad \dot{v}_3 = b_2\,v_1,$$

with initial conditions $v_1(0) = v_2(0) = v_3(0) = 0$. The solution can be found immediately:

$$v_1(t) = \int_0^t b_1(s)\,ds, \quad v_2(t) = \int_0^t b_2(s)\,ds, \quad v_3(t) = \int_0^t b_2(s)\int_0^s b_1(r)\,dr\,ds. \tag{22}$$

The preceding solution can be used in order to find the general solution of the given system (19). We only need to find a suitable parametrization of the Lie group $H(3)$, and the expression of the group action with respect to which the original vector fields are the fundamental vector fields. If we take the canonical coordinates of second class defined by $g = \exp(aa_1)\exp(ba_2)\exp(ca_3)$, when $g \in H(3)$, it can be shown that such an action reads

$$\Phi : H(3) \times \mathbf{R}^3 \to \mathbf{R}^3$$
$$((a,\, b,\, c),\, (x,\, y,\, z)) \mapsto (x - a,\, y - b,\, z + ay - bx - ab - 2c),$$

the group law being expressed as

$$(a,\, b,\, c)(a',\, b',\, c') = (a + a',\, b + b',\, c + c' - ba').$$

Then, the general solution of (19) is

$$\Phi((-v_1,\, -v_2,\, -v_3), (x_0,\, y_0,\, z_0)) = (x_0 + v_1,\, y_0 + v_2,\, z_0 + x_0 v_2 - y_0 v_1 - v_1 v_2 + 2v_3),$$

where $v_1 = v_1(t)$, $v_2 = v_2(t)$, and $v_3 = v_3(t)$ are given by (22).

6.3. *Hopping robot in flight phase.* Next we consider another example coming from a physical model: a hopping robot in flight phase, which has been studied, e.g., in [20]. The system consists of a body with an actuated leg that can rotate and extend. The coordinates are $(\psi,\, l,\, \theta)$, describing the body angle, leg extension and leg angle of the robot, respectively. The constant m_l is the mass of the leg, and the mass of the body is taken to be one. The interest is focused on the behaviour of the system for small elongation, that is, near $l = 0$. Precisely, the approximation of the system in the neighbourhood of $l = 0$ will lead to a Lie system related to the Heisenberg–Weyl group, and therefore related also to the previous example.

The controls of the system are the leg angle and extension velocities. The control system takes the form [20]

$$\dot{\psi} = b_1(t), \quad \dot{l} = b_2(t), \quad \dot{\theta} = -\frac{m_l(l+1)^2}{1 + m_l(l+1)^2} b_1(t), \tag{23}$$

whose solutions are the integral curves of the time-dependent vector field $b_1(t)\, Y_1 + b_2(t)\, Y_2$, where now

$$Y_1 = \frac{\partial}{\partial \psi} - \frac{m_l(l+1)^2}{1 + m_l(l+1)^2} \frac{\partial}{\partial \theta}, \qquad Y_2 = \frac{\partial}{\partial l}.$$

However, the system (23) cannot be considered as a Lie system, since the iterated Lie brackets

$$[Y_2, [Y_2, \ldots [Y_2, Y_1] \cdots]]$$

generate at each step vector fields linearly independent from those obtained at the previous stage. Notwithstanding, in order to steer the original system by sinusoids, it was proposed in [20] to take the Taylor approximation, linear in l, of the system, that is,

$$\dot{\psi} = b_1(t), \quad \dot{l} = b_2(t), \quad \dot{\theta} = -(k_1 + k_2 l) b_1(t), \tag{24}$$

where the constants k_1 and k_2 are defined as

$$k_1 = \frac{m_l}{1 + m_l}, \quad k_2 = \frac{2m_l}{(1 + m_l)^2},$$

and then the vector fields become

$$X_1 = \frac{\partial}{\partial \psi} - (k_1 + k_2 l) \frac{\partial}{\partial \theta}, \qquad X_2 = \frac{\partial}{\partial l}. \tag{25}$$

Now, the new vector field

$$X_3 = [X_1,\, X_2] = k_2 \frac{\partial}{\partial\theta}$$

closes, jointly with X_1, X_2, the Lie algebra (20), so that (24) can be regarded as a Lie system with associated Lie algebra $\mathfrak{h}(3)$.

If we use the previously defined canonical coordinates $(a,\, b,\, c)$ of second kind for parametrizing the group $g \in H(3)$, the corresponding (local) action to our Lie system reads

$$\begin{aligned}\Phi : H(3) \times M &\to M\\ ((a,\, b,\, c),\, (\psi,\, l,\, \theta)) &\mapsto (\psi - a,\, l - b,\, \theta + k_2(al - c - ab) + ak_1),\end{aligned}$$

where M is a suitable open set of $\mathbf{R}^3$. Then, the general solution of the system (24) can be written, for t small enough, as

$$\Phi((-v_1,\, -v_2,\, -v_3), (\psi_0,\, l_0,\, \theta_0)) = (\psi_0 + v_1,\, l_0 + v_2,\, \theta_0 + k_2(v_3 - v_1 l_0 - v_1 v_2) - k_1 v_1),$$

where $v_1 = v_1(t)$, $v_2 = v_2(t)$, and $v_3 = v_3(t)$ are given by (22). This result can be checked by direct integration.

6.4. *Kinematics of the generalization of the elastic problem of Euler as a Lie system.* Recently, Jurdjevic has generalized the so-called elastic problem of Euler to homogeneous spaces of constant curvature embedded in a three dimensional Euclidean space, in order to study certain integrable Hamiltonian systems from the point of view of optimal control theory [15]. We will only deal with the kinematic equations of such systems, which turn out to be Lie systems.

The system of interest is the control system with configuration space $\mathbf{R}^3$, and coordinates $(x_1,\, x_2,\, x_3)$, given by

$$\dot{x}_1 = -b_1(t)x_2 - b_2(t)x_3, \quad \dot{x}_2 = b_1(t)x_1 + b_3(t)x_3, \quad \dot{x}_3 = \epsilon\,(b_2(t)x_1 - b_3(t)x_2), \tag{26}$$

where $\epsilon = \pm 1, 0$. Its solutions are the integral curves of the time-dependent vector field $b_1(t)\, X_1(x) + b_2(t)\, X_2(x) + b_3(t)\, X_3(x)$, where

$$X_1 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}, \quad X_2 = \epsilon\, x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad X_3 = x_3 \frac{\partial}{\partial x_2} - \epsilon\, x_2 \frac{\partial}{\partial x_3}. \tag{27}$$

These vector fields satisfy the commutation relations

$$[X_1,\, X_2] = X_3, \qquad [X_2,\, X_3] = \epsilon\, X_1, \qquad [X_3,\, X_1] = X_2,$$

and hence they generate a Lie algebra isomorphic to $\mathfrak{g}_\epsilon$ of the Lie group G_ϵ, given by $G_0 = SE(2)$, $G_1 = SO(3)$ and $G_{-1} = SO(2,1)$. Therefore, the case $\epsilon = 0$ essentially reduces to the first example studied in this section. We take a basis $\{a_1,\, a_2,\, a_3\}$ of $\mathfrak{g}_\epsilon$ in which the Lie products read

$$[a_1,\, a_2] = a_3, \qquad [a_2,\, a_3] = \epsilon\, a_1, \quad [a_3,\, a_1] = a_2,$$

and define the signature-dependent trigonometric functions (see, e.g., [1]):

$$C_\epsilon(x) = \begin{cases} \cos x & \epsilon = 1 \\ 1 & \epsilon = 0 \\ \cosh x & \epsilon = -1 \end{cases} \qquad S_\epsilon(x) = \begin{cases} \sin x & \epsilon = 1 \\ x & \epsilon = 0 \\ \sinh x & \epsilon = -1 \end{cases} \qquad T_\epsilon(x) = \frac{S_\epsilon(x)}{C_\epsilon(x)},$$

which satisfy

$$C_\epsilon(x+y) = C_\epsilon(x)C_\epsilon(y) - \epsilon\, S_\epsilon(x)S_\epsilon(y), \quad S_\epsilon(x+y) = C_\epsilon(x)S_\epsilon(y) + S_\epsilon(x)C_\epsilon(y),$$

as well as $C_\epsilon^2(x) + \epsilon\, S_\epsilon^2(x) = 1$, and

$$\frac{dC_\epsilon(x)}{dx} = -\epsilon\, S_\epsilon(x), \quad \frac{dS_\epsilon(x)}{dx} = C_\epsilon(x), \quad \frac{dT_\epsilon(x)}{dx} = 1 + \epsilon\, T_\epsilon^2(x) = \frac{1}{C_\epsilon^2(x)}.$$

Writing the solution of the problem associated to (26) in the group G_ϵ as the product $g(t) = \exp(-v_1(t)a_1)\exp(-v_2(t)a_2)\exp(-v_3(t)a_3)$, and using the Wei–Norman formula (6), we obtain the system of differential equations for $v_1(t)$, $v_2(t)$ and $v_3(t)$:

$$\begin{aligned}
\dot v_1 &= b_1 + \epsilon\, T_\epsilon(v_2)(b_3\,\cos v_1 + b_2\,\sin v_1),\\
\dot v_2 &= b_2\,\cos v_1 - b_3\,\sin v_1,\\
\dot v_3 &= \frac{b_3\,\cos v_1 + b_2\,\sin v_1}{C_\epsilon(v_2)},
\end{aligned}$$

with $v_1(0) = v_2(0) = v_3(0) = 0$. Other possible reorderings of the factorization in exponentials will give rise to similar systems of equations. For $\epsilon = \pm 1$ the group G_ϵ is simple and none of the Wei–Norman systems can be integrated by quadratures in a general case. For the particular case treated by Jurdjevic [16] we must put (with our notation) $b_1(t) = 1$, $b_2(t) = 0$ and $b_3(t) = k(t)$.

6.5. *Reduction of Lie systems in control theory.* Finally, we would like to point out that there exist a technique for reducing the problem of solving a given Lie system in a Lie group G to solving a similar Lie system but in a subgroup H, provided a particular solution of the problem corresponding to the former in an associated homogeneous space G/H is known (see, e.g., [7]). This reduction procedure can be shown to be useful as well in the study of the particular kind of drift-free control systems, linear in the control functions, which in addition are Lie systems.

Take for instance, in the simplified model of maneuvering an automobile discussed before, the subgroup $H = \{(0,\, 0,\, b)\}$. In these coordinates, $\tau : SE(2) \to SE(2)/H$ is $\tau(\theta,\, a,\, b) = (\theta,\, a)$. Taking coordinates $(z_1,\, z_2)$ in $M = SE(2)/H$, we have that the left action of $SE(2)$ on M is given by

$$\Phi((\theta,\, a,\, b), (z_1,\, z_2)) = (z_1 + \theta,\, z_2 + a\cos z_1 + b\sin z_1).$$

The fundamental vector fields with respect to this action are

$$X_1^H = -\frac{\partial}{\partial z_1}, \quad X_2^H = -\cos z_1 \frac{\partial}{\partial z_2}, \quad X_3^H = -\sin z_1 \frac{\partial}{\partial z_2},$$

which satisfy

$$[X_1^H,\, X_2^H] = X_3^H, \quad [X_2^H,\, X_3^H] = 0, \quad [X_1^H,\, X_3^H] = -X_2^H,$$

and the equations on the homogeneous space M to be solved are

$$\dot z_1 = -b_1(t), \quad \dot z_2 = -b_2(t)\cos z_1.$$

Assume we have a curve on $SE(2)$, $g_1(t)$, such that the coordinates of its projection $\tau(g_1(t)) = (z_1(t),\, z_2(t))$ satisfy the previous equations. For example, we take $g_1(t) =$

$(z_1(t),\, z_2(t),\, 0)$. Then, we can reduce the problem to solving an equation on H, which takes the form

$$\dot{b}(t) = b_2(t)\sin z_1(t),$$

which is just a Lie system for the additive group of the real line.

As a second and last example, consider again the kinematic equations of the generalized elastic problem of Euler treated previously. In this case, however, it is advantageous to consider instead of the Lie group G_ϵ, its universal covering $\bar{G}_\epsilon$, in order to perform the reduction. We have that $\bar{G}_1 = SU(2)$, $\bar{G}_{-1} = SU(1,1)$, and $\bar{G}_0 = SE(2)$. The elements of the group $\bar{G}_\epsilon$ can be parametrized by four real numbers $(a,\, b,\, c,\, d)$ such that $a^2 + b^2 + \epsilon(c^2 + d^2) = 1$, the group law

$$(a,\, b,\, c,\, d)(a',\, b',\, c',\, d') = (a'',\, b'',\, c'',\, d'')$$

being given by

$$\begin{aligned} a'' &= aa' - bb' - \epsilon\,(cc' + dd'), \quad & b'' &= ba' + ab' - \epsilon\,(dc' - cd'),\\ c'' &= ca' + db' + ac' - bd', \quad & d'' &= da' - cb' + bc' + ad', \end{aligned}$$

see also [3, 22]. To perform the reduction we take the subgroup H generated by a_1. The projection $\tau : \bar{G}_\epsilon \to \bar{G}_\epsilon/H$ is defined by

$$\tau(a,\, b,\, c,\, d) = \left(\frac{ac - bd}{a^2 + b^2},\, \frac{bc + ad}{a^2 + b^2}\right).$$

Taking coordinates $(z_1,\, z_2)$ in $M = \bar{G}_\epsilon/H$, the left action of $\bar{G}_\epsilon$ on M reads

$$\Phi((a,\, b,\, c,\, d), (z_1,\, z_2)) = \left(\frac{N_1}{D},\, \frac{N_2}{D}\right),$$

where

$$\begin{aligned} N_1 &= (a^2 - b^2 - \epsilon\,(c^2 - d^2))z_1 - 2(ab + \epsilon\,cd)z_2 + (ac - bd)(1 - \epsilon\,(z_1^2 + z_2^2)),\\ N_2 &= 2(ab - \epsilon\,cd)z_1 + (a^2 - b^2 + \epsilon\,(c^2 - d^2))z_2 + (ad + bc)(1 - \epsilon\,(z_1^2 + z_2^2)),\\ D &= a^2 + b^2 - 2\epsilon\,((ac + bd)z_1 + (ad - bc)z_2) + \epsilon^2(c^2 + d^2)(z_1^2 + z_2^2). \end{aligned}$$

The fundamental vector fields with respect to this action are

$$\begin{aligned} X_1^H &= z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2}, \quad X_2^H = -\frac{1}{2}(1 + \epsilon\,(z_1^2 - z_2^2))\frac{\partial}{\partial z_1} - \epsilon\,z_1z_2\frac{\partial}{\partial z_2},\\ X_3^H &= -\epsilon\,z_1z_2\frac{\partial}{\partial z_1} - \frac{1}{2}(1 - \epsilon\,(z_1^2 - z_2^2))\frac{\partial}{\partial z_2}, \end{aligned}$$

which satisfy

$$[X_1^H,\, X_2^H] = X_3^H, \quad [X_2^H,\, X_3^H] = \epsilon\,X_1^H, \quad [X_1^H,\, X_3^H] = -X_2^H,$$

and the equations on the homogeneous space M to be solved are

$$\begin{aligned} \dot{z}_1 &= b_1(t)z_2 - \frac{1}{2}b_2(t)(1 + \epsilon\,(z_1^2 - z_2^2)) - b_3(t)\,\epsilon\,z_1z_2,\\ \dot{z}_2 &= -b_1(t)z_1 - b_2(t)\,\epsilon\,z_1z_2 - \frac{1}{2}b_3(t)(1 - \epsilon\,(z_1^2 - z_2^2)). \end{aligned}$$

Assume we have a curve $g_1(t)$ on $\bar{G}_\epsilon$ such that its projection $\tau(g_1(t)) = (z_1(t), z_2(t))$ satisfies the previous equations. We can take, for example,

$$g_1(t) = \frac{(1,\, 0,\, z_1(t),\, z_2(t))}{\sqrt{1+\epsilon\,(z_1^2(t)+z_2^2(t))}}.$$

Then, we can reduce the problem to an equation on H: if its solution is of the form $h(t) = (\cos(v(t)/2), \sin(v(t)/2), 0, 0)$, then $v(t)$ satisfies

$$\dot{v}(t) = -b_1(t) + \epsilon\,(b_3(t)z_1(t) - b_2(t)z_2(t)).$$

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