

BINARY OPERATIONS IN CLASSICAL AND QUANTUM MECHANICS

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Abstract. Binary operations on algebras of observables are studied in the quantum as well as in the classical case. It is shown that certain natural compatibility conditions with the associative product imply properties which are usually additionally required.

1. Introduction. It was an observation of P. A. M. Dirac, when considering the quantum Poisson bracket of observables in foundations of Quantum Mechanics, that the Leibniz rule

$$\begin{aligned}[A, B_1 B_2] &= B_1 [A, B_2] + [A, B_1] B_2, \\ [B_1 B_2, A] &= B_1 [B_2, A] + [B_1, A] B_2,\end{aligned}$$

is sufficient to determine the bracket. The Leibniz rule simply tells us that fixing an argument in the bracket we get a derivation, i.e. an infinitesimal automorphism, of the algebra of observables. This shows that, in fact, most of the axioms of the Lie bracket for the corresponding operator algebra are superfluous and one can insist only on the proper behaviour with respect to the composition (Leibniz rule).

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In the classical case, however, the Leibniz rule is not enough to determine proper classical brackets, since every operation on the algebra $C^\infty(M)$ of smooth functions on a manifold M , associated with the contraction with any contravariant 2-tensor, will do. We have to impose a version of the Jacobi identity:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

However, as observed in [GM], the Jacobi identity and the Leibniz rule for a bracket on $C^\infty(M)$ imply the skew-symmetry, i.e. we are dealing with a Poisson bracket.

Note that the Jacobi identity alone (without the skew-symmetry assumption) is the axiom of a generalization of a Lie algebra proposed by J.-L. Loday [Lo]. Such non-skew-symmetric Lie algebras do exist and give rise to a well-defined (co)homology theory. The result of [GM] just shows that such Loday algebras are not possible (except for the skew-symmetric ones, of course) on algebras of functions when the Leibniz rule is required. The natural question arises if we can break the skew-symmetry by passing to more general structures like Jacobi brackets or local Lie algebras in the sense of A. A. Kirillov [Ki].

Our main result shows that this is impossible. More precisely, we show that assuming the differentiability of the bracket and the Jacobi identity, we get the skew-symmetry automatically, so that no local Loday brackets exist except for the well-known skew-symmetric ones. The differentiability condition may be viewed as a compatibility condition with the associative product in the algebra. These observations seem interesting to us in the context of finding a minimal set of requirements for the structures of classical and quantum mechanics.

2. Commutator brackets in associative rings. It was already observed by Dirac ([Di], p. 86) that the Leibniz rule defines the commutator bracket in the algebra of quantum observables uniquely up to a constant factor. In fact, the following holds for any associative ring (cf. [Di]).

PROPOSITION 1. *If $[\cdot, \cdot]_0$ is a bilinear operation in an associative ring \mathcal{A} for which the Leibniz rule is satisfied (with respect to both arguments):*

$$(1) \quad \begin{aligned} [A, B_1 B_2]_0 &= B_1 [A, B_2]_0 + [A, B_1]_0 B_2, \\ [B_1 B_2, A]_0 &= B_1 [B_2, A]_0 + [B_1, A]_0 B_2, \end{aligned}$$

then

$$(2) \quad [A_1, B_1][A_2, B_2]_0 = [A_1, B_1]_0 [A_2, B_2],$$

for all $A_i, B_i \in \mathcal{A}$, $i = 1, 2$, where $[A, B] = AB - BA$ is the commutator bracket.

Proof. Using the Leibniz rule first for the second argument and then for the first one, we get

$$\begin{aligned} [A_1 A_2, B_1 B_2]_0 &= B_1 [A_1 A_2, B_2]_0 + [A_1 A_2, B_1]_0 B_2 = \\ &B_1 A_1 [A_2, B_2]_0 + B_1 [A_1, B_2]_0 A_2 + A_1 [A_2, B_1]_0 B_2 + [A_1, B_1]_0 A_2 B_2. \end{aligned}$$

Doing the same calculations in reverse order we get

$$[A_1 A_2, B_1 B_2]_0 = A_1 B_1 [A_2, B_2]_0 + B_1 [A_1, B_2]_0 A_2 + A_1 [A_2, B_1]_0 B_2 + [A_1, B_1]_0 B_2 A_2.$$

Hence

$$A_1 B_1 [A_2, B_2]_0 + [A_1, B_1]_0 B_2 A_2 = B_1 A_1 [A_2, B_2]_0 + [A_1, B_1]_0 A_2 B_2$$

and the proposition follows. ■

THEOREM 1. *If \mathcal{A} is a unital associative ring which is strongly non-commutative, that is, the two-sided associative ideal of \mathcal{A} generated by the derived algebra $\mathcal{A}' = \text{span}\{[A, B] : A, B \in \mathcal{A}\}$ equals \mathcal{A} , then, under the assumptions of Proposition 1, we get*

$$(3) \quad [A, B]_0 = C[A, B]$$

for some central element $C \in \mathcal{A}$.

In particular, every bilinear operation $[\cdot, \cdot]_0$ on \mathcal{A} which satisfies the Leibniz rule is a Lie bracket, i.e. it is skew-symmetric and satisfies the Jacobi identity. If, moreover, \mathcal{A} is an algebra over a commutative ring \mathbf{k} and the center of \mathcal{A} is trivial, i.e. equals $\mathbf{k}\mathbf{1}$, then $[A, B]_0 = \lambda[A, B]$ for some $\lambda \in \mathbf{k}$.

Proof. By assumption, the unit $\mathbf{1} \in \mathcal{A}$ can be written in the form

$$(4) \quad \mathbf{1} = \sum_i C_i [A_i, B_i] + \sum_j [A'_j, B'_j] C'_j.$$

From (2) we get then

$$(5) \quad \left(\sum_i C_i [A_i, B_i]\right) [A, B]_0 = \left(\sum_i C_i [A_i, B_i]_0\right) [A, B].$$

Since both brackets satisfy the Leibniz rule, we have

$$(6) \quad [A'_j, B'_j] C'_j [A, B]_0 = [A'_j, B'_j C'_j] [A, B]_0 - B_j [A'_j, C'_j] [A, B]_0 = [A'_j, B'_j C'_j]_0 [A, B] - B_j [A'_j, C'_j]_0 [A, B] = [A'_j, B'_j]_0 C'_j [A, B],$$

so that

$$\begin{aligned} [A, B]_0 &= \left(\sum_i C_i [A_i, B_i] + \sum_j [A'_j, B'_j] C'_j\right) [A, B]_0 = \\ & \left(\sum_i C_i [A_i, B_i]_0 + \sum_j [A'_j, B'_j]_0 C'_j\right) [A, B] = C[A, B], \end{aligned}$$

where C is a fixed element of \mathcal{A} not depending on $A, B \in \mathcal{A}$.

To show that C is central, rewrite (6) in the form

$$(7) \quad [A', B'] X C [A, B] = C [A', B'] X [A, B],$$

so that

$$(8) \quad [[A', B'] X, C] \cdot [A, B] Y = 0,$$

where A, B, A', B', X, Y are arbitrary elements of \mathcal{A} . Since every one-sided associative ideal of \mathcal{A} containing the derived Lie algebra \mathcal{A}' is, due to the Leibniz rule, two-sided and thus equals \mathcal{A} , we get from (8) $[\mathcal{A}, C] \mathcal{A} = 0$, whence $[\mathcal{A}, C] = 0$ and the theorem follows. ■

REMARK 1. Obviously, if \mathcal{A} contains elements P, Q satisfying the canonical commutation rules $PQ - QP = \mathbf{1}$, then the assumption of the above theorem is satisfied automatically. This is exactly the Dirac case.

COROLLARY 1. Let \mathcal{A} be the algebra $gl(n, \mathbf{k})$ of $n \times n$ -matrices with coefficients in a commutative unital ring \mathbf{k} . Then every binary operation $[\cdot, \cdot]_0$ on \mathcal{A} , satisfying the Leibniz rule (1), is of the form

$$(9) \quad [A, B]_0 = \lambda(AB - BA)$$

for some $\lambda \in \mathbf{k}$.

Proof. It is easy to see that the derived Lie algebra \mathcal{A}' is the Lie algebra $sl(n, \mathbf{k})$ of trace-less matrices. Of course, $gl(n, \mathbf{k}) \neq sl(n, \mathbf{k})$, but $sl(n, \mathbf{k})$ contains invertible matrices, so that the associative ideal generated by \mathcal{A}' is the whole \mathcal{A} . Due to theorem 1, $[A, B]_0 = C(AB - BA)$ for some central element $C \in gl(n, \mathbf{k})$. Since, as is easily seen, the center of $gl(n, \mathbf{k})$ is trivial, we get (9). ■

3. Differential Loday brackets for commutative algebras. The important brackets of classical mechanics, like the Poisson bracket on a symplectic manifold or the Lagrange bracket on a contact manifold, were generalized by A. A. Kirillov [Ki] to *local Lie algebra* brackets on one-dimensional vector bundles over a manifold M , that is, to Lie brackets on their sections given by local operators. The fundamental fact discovered in [Ki] is that these operators have to be of the first order and then, locally, they reduce to the conformally symplectic Poisson or Lagrange brackets on the leaves of the corresponding generalized foliation of M . For the trivial bundle, i.e. for the algebra $C^\infty(M)$ of functions on M , the local brackets reduce to the so called *Jacobi brackets* associated with the corresponding *Jacobi structures* on the manifold M (cf. [Li]).

A purely algebraic version of Kirillov's result has been proved in [Gr], theorems 4.2 and 4.4, where Lie brackets on associative commutative algebras with no nilpotents, given by bidifferential operators, have been considered.

On the other hand, J.-L. Loday (cf. [Lo1]), while studying relations between Hochschild and cyclic homology in the search for obstructions to the periodicity of algebraic K-theory, discovered that one can skip the skew-symmetry assumption in the definition of a Lie algebra, still having a possibility to define appropriate (co)homology (see [Lo1, LP] and [Lo], Chapter 10.6). His Jacobi identity for such structures was formally the same as the classical Jacobi identity in the form

$$(10) \quad [x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

This time, however, this is no longer equivalent to

$$(11) \quad [[x, y], z] = [[x, z], y] + [x, [y, z]],$$

nor to

$$(12) \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

since we have no skew-symmetry. Loday called such structures *Leibniz algebras*, but since we have already associated the name of Leibniz with the Leibniz rule, we shall call them *Loday algebras*. This is in accordance with the terminology of [KS], where analogous structures in the graded case are defined. Of course, there is no particular reason not to define Loday algebras by means of (11) instead of (10) (and in fact, it was the original Loday definition), but both categories are equivalent via transposition of arguments. Similarly,

for associative algebras we can obtain associated algebras by transposing arguments, but in this case we still get associative algebras.

The natural question arises whether local Loday algebras, different from the local Lie algebras, do exist. In [GM] it has been proved that no new n -ary Loday-Poisson algebras and, in principle, no new Loday algebroids (in the sense of [GU]) are possible. Here we will prove that local Loday algebras on sections of one-dimensional vector bundles must be skew-symmetric, that is, they reduce to local Lie algebras of Kirillov. We will start with a general algebraic result on this subject.

Let now \mathcal{A} be an associative commutative algebra with unit $\mathbf{1}$ over a field \mathbf{k} of characteristic 0. Recall that a (linear) *differential operator of order $\leq n$* on \mathcal{A} is a \mathbf{k} -linear operator $D : \mathcal{A} \rightarrow \mathcal{A}$ such that $\delta(x)^{n+1}D = 0$ for all $x \in \mathcal{A}$, where $\delta(x)$ is the commutator with the multiplication by x :

$$(13) \quad (\delta(x)D)(y) = D(xy) - xD(y).$$

This is the same as to say that $\delta(x_1) \cdots \delta(x_{n+1})D = 0$ for any $x_1, \dots, x_{n+1} \in \mathcal{A}$. Note that a zero-order differential operator D is just the multiplication by $D(\mathbf{1})$ and that $\delta(x)$ acts as a derivation of the associative algebra $End_{\mathbf{k}}(\mathcal{A})$ of linear operators on \mathcal{A} :

$$(14) \quad \delta(x)(D_1D_2) = (\delta(x)D_1)D_2 + D_1(\delta(x)D_2).$$

For multilinear operators we define analogously

$$(15) \quad \delta_{y_i}(x)D(y_1, \dots, y_p) = D(y_1, \dots, xy_i, \dots, y_p) - xD(y_1, \dots, y_p)$$

the corresponding derivations with respect to the i th variable and call the multilinear operator D to be *of order $\leq n$* if $\delta_{y_i}(x)^{n+1}D = 0$ for all $x \in \mathcal{A}$ and all i , i.e. D is of order $\leq n$ with respect to each variable separately. This means that fixing $p - 1$ arguments we get a differential operator of order $\leq n$. One can also consider multilinear operators such that, fixing $p - 1$ arguments, we get a differential operator of some order depending on what we have fixed (and possibly unbounded). In this paper, however, by a multilinear differential operator we mean an operator of order n for some n . Note also that the differentials $\delta_{y_i}(x)$ and $\delta_{y_j}(u)$ commute.

DEFINITION 1. A *differential Loday bracket* on \mathcal{A} is a Loday algebra bracket on \mathcal{A} , i.e. a bracket satisfying (10), given by a bidifferential operator of a certain order n .

PROPOSITION 2. *If \mathcal{A} has no nontrivial nilpotent elements, then every differential Loday bracket on \mathcal{A} is of order ≤ 1 .*

Proof. Let $D : \mathcal{A} \times \mathcal{A} \ni (x, y) \rightarrow [x, y] \in \mathcal{A}$ be a differential Loday bracket. Denote $D_x = [x, \cdot]$, $D'_x = [\cdot, x]$, and let n be the order of D with respect to the second argument, i.e. the maximum of the orders of D_x for all $x \in \mathcal{A}$. Then, $\delta(w)^n D_x$ is the multiplication by $\delta(w)^n D_x(\mathbf{1})$ for all $w, x \in \mathcal{A}$ and $\delta(w)^n D_x(\mathbf{1})$ is different from zero for some $w, x \in \mathcal{A}$. Let now k be the maximum of the orders of $\delta(w)^n D_x(\mathbf{1})$ with respect to x . Obviously, k is not greater than the order m of D with respect to the first variable. In other words,

$$(16) \quad \delta_x(u)^p (\delta(w)^q D_x) = \delta_0(u)^p \delta_1(w)^q D(x, \cdot) = 0$$

(where δ_0, δ_1 are the corresponding differentials with respect to the first and the second

variable, respectively), if $q > n$, or $q = n$ and $p > k$, and

$$(17) \quad \delta_x(u)^k (\delta(w)^n D_x) = \delta_0(u)^k \delta_1(w)^n D(x, \cdot)$$

is the multiplication by

$$(18) \quad \delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1})$$

which is different from zero for some $u = u_0$ and $w = w_0$. Note that since we do not assume that the bracket is skew-symmetric, we do not have $D_x + D_x^r = 0$ and that n is the order of D with respect to the first argument. We claim first that $n \leq 1$.

Suppose the contrary. Then $2n - 1 > n$ and

$$(19) \quad \delta_y(v)^k \delta_x(u)^{k+1} \delta_z(w)^{2n-1} (D(D(x, y), z) + D(y, D(x, z)) - D(x, D(y, z))) = (k + 1) \binom{2n - 1}{n} (\delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1}) \cdot \delta_0^k(v) \delta_1(u) \delta_1(w)^{n-1} D(\mathbf{1}, \mathbf{1})),$$

where we identify the right-hand-side term with the corresponding zero-order differential operator. Indeed, $\delta(w)^{2n-1}$ vanishes on the first summand $D_{[x,y]}$. For the rest we get

$$(20) \quad \delta(w)^{2n-1} [D_y, D_x]_c(z) = \binom{2n - 1}{n} ([\delta(w)^n D_y, \delta(w)^{n-1} D_x]_c + [\delta(w)^{n-1} D_y, \delta(w)^n D_x]_c)(z),$$

where the brackets are the commutator brackets. Since $\delta(w)^n D_y$ is multiplication by $\delta(w)^n D_y(\mathbf{1})$, the first summand is just $\delta(\delta(w)^n D_y(\mathbf{1})) \delta(w)^{n-1} D_x(z)$, so it vanishes after applying $\delta_x(u)^{k+1}$ in view of (16). From the second summand, after applying $\delta_x(u)^{k+1}$, we get clearly

$$(21) \quad \delta_x(u)^{k+1} (\delta(w)^{n-1} D_y \circ \delta(w)^n D_x)(z) = (k + 1) (\delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1}) \cdot \delta_1(u) \delta_1(w)^{n-1} D_y(z)).$$

Applying now $\delta_y(v)^k$ we get (19). Since

$$(22) \quad D(D(x, y), z) + D(y, D(x, z)) - D(x, D(y, z)) = 0$$

due to the Jacobi identity, we get

$$(23) \quad \delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1}) \cdot \delta_0^k(v) \delta_1(u) \delta_1(w)^{n-1} D(\mathbf{1}, \mathbf{1}) = 0.$$

Putting now $u := u + tw$ in (23) with t varying through \mathbf{k} and passing to the first derivative with respect to t (the coefficient by t) we get

$$(24) \quad 0 = k \delta_0(u)^{k-1} \delta_0(w) \delta_1(w)^n D(\mathbf{1}, \mathbf{1}) \cdot \delta_0^k(v) \delta_1(u) \delta_1(w)^{n-1} D(\mathbf{1}, \mathbf{1}) + \delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1}) \cdot \delta_0^k(v) \delta_1(w)^n D(\mathbf{1}, \mathbf{1}).$$

Multiplying both sides of (24) by $\delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1})$ and using (23) we get, after putting $u = v$,

$$(25) \quad (\delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1}))^3 = 0,$$

thus

$$(26) \quad \delta_0(u)^k \delta_1(w)^n D(\mathbf{1}, \mathbf{1}) = 0,$$

since there are no nontrivial nilpotents in \mathcal{A} . But the latter is different from zero for $u = u_0, w = w_0$; a contradiction.

Thus $n \leq 1$ and we will finish by showing that m , the order of D with respect to the first argument, is also ≤ 1 . As above, suppose the contrary and let s be the maximal order of $\delta(w)^m D_y^r(\mathbf{1})$ with respect to y , so that there are $w_0, u_0 \in \mathcal{A}$ such that

$$(27) \quad \delta_0(w_0)^m \delta_1(u_0)^s D(\mathbf{1}, \mathbf{1}) \neq 0.$$

We get from (22)

$$(28) \quad 0 = \delta_z(u)^s \delta_y(u)^s \delta_x(w)^{2m} (D(D(x, y), z) + D(y, D(x, z)) - D(x, D(y, z))) = \binom{2m}{m} (\delta_1(u)^s \delta_0(w)^m D(\mathbf{1}, \mathbf{1}))^2$$

which contradicts (27). Indeed, since $2m > m > 1$ and

$$(29) \quad D(D(x, y), z) + D(y, D(x, z)) - D(x, D(y, z)) = (D_z^r D_y^r + D_y D_z^r - D_{[y,z]})(x),$$

$\delta_x(w)^{2m}$ vanishes on $D_{[y,z]}$ ($2m > m$) and on $D_y D_z^r$ (we already know that D_y is of order ≤ 1), so for the middle term of (28) we get

$$(30) \quad \begin{aligned} &\delta_z(u)^s \delta_y(u)^s \delta_x(w)^{2m} (D_z^r \cdot D_y^r)(x) = \\ &x \binom{2m}{m} \delta_z(u)^s \delta_y(u)^s (\delta(w)^m (D_z^r) \cdot \delta(w)^m (D_y^r)) = \\ &xyz \binom{2m}{m} (\delta_1(u)^s \delta_0(w)^m D(\mathbf{1}, \mathbf{1}))^2. \blacksquare \end{aligned}$$

REMARK 2. The bi-orders (k, n) and (m, s) correspond to bidifferential parts of D of maximal orders with respect to the anti-lexicographical and the lexicographical orderings, respectively. We could say that they are the corresponding bi-symbols of this bidifferential operator. Note, however, that it is not true in the general algebraic case that differential operators of higher orders are polynomials in derivatives, like for differential operators on $C^\infty(M)$ (cf. [Gr], remark 3.2). It is only for first-order differential operators on \mathcal{A} that we have the decomposition $Der(\mathcal{A}) \oplus \mathcal{A}$ into the direct sum of derivations and zero-order operators (multiplications by elements of \mathcal{A}).

REMARK 3. The assumption concerning the nilpotent elements is essential. For instance, if \mathcal{A} is freely generated by elements x, y with the constraint $x^2 = 0$, then

$$(31) \quad D(u, v) = xu \frac{\partial^n}{\partial y^n}(v)$$

is a differential Loday bracket of order n .

THEOREM 2. *If \mathcal{A} has no nontrivial nilpotent elements, then every differential Loday bracket on \mathcal{A} is a standard skew-symmetric Jacobi bracket given by*

$$(32) \quad [x, y] = \Lambda(x, y) + x\Gamma(y) - y\Gamma(x),$$

where Λ and Γ are, respectively, a biderivation and a derivation on \mathcal{A} which satisfy the compatibility conditions

$$(33) \quad \begin{aligned} [\Gamma, \Lambda]_{NR} &= 0; \\ [\Lambda, \Lambda]_{NR} &= -2\Lambda \wedge \Gamma, \end{aligned}$$

for $[\cdot, \cdot]_{NR}$ being the Nijenhuis-Richardson bracket of skew-multilinear maps.

REMARK 4. The above theorem generalizes theorem 4.4 in [Gr]. The Nijenhuis-Richardson graded Lie algebra bracket [NR] (see also [Gr]) is in this algebraic case the analog of the Schouten-Nijenhuis bracket of multivector fields.

Proof. In view of proposition 2, we can assume that the bracket $D = [\cdot, \cdot]$ is of first order and, according to [Gr], theorem 4.4, it is sufficient to prove that the bracket is skew-symmetric. The bracket being of the first order satisfies the generalized Leibniz rule:

$$(34) \quad \begin{aligned} [x, yz] &= y[x, z] + [x, y]z - yz[x, 1]; \\ [yz, x] &= y[z, x] + [y, x]z - yz[1, x]. \end{aligned}$$

Indeed, D_x is of the first order, so $\delta(z)D_x$ is the multiplication by $\delta(z)D_x(\mathbf{1})$. But $\delta(z)D_x(y) = [x, yz] - z[x, y]$, so

$$(35) \quad [x, yz] - z[x, y] = (\delta(z)D_x(\mathbf{1}))y = ([x, z] - z[x, 1])y.$$

One proves analogously the second equation. The equations (34) mean that the operators R_x and L_y defined by

$$(36) \quad \begin{aligned} L_x(y) &= [x, y] - [x, \mathbf{1}]y, \\ R_y(x) &= [x, y] - [\mathbf{1}, y]x, \end{aligned}$$

are derivations of \mathcal{A} . They are just obtained from the decompositions of the corresponding first-order differential operators mentioned in remark 4. Moreover,

$$(37) \quad \begin{aligned} (\delta(y)D_x)(z) &= L_x(y)z, \\ (\delta(y)D_x^r)(z) &= R_x(y)z. \end{aligned}$$

From the Jacobi identity (10) we get immediately that

$$(38) \quad [[x, x], z] = 0$$

and, using the Jacobi identity once more, that

$$(39) \quad [[y, [x, x]], z] = 0$$

for all $x, y, z \in \mathcal{A}$. The linearized version of (38) reads

$$(40) \quad [[x, y] + [y, x], z] = 0.$$

From (40) we get easily

$$(41) \quad \begin{aligned} 0 &= \delta_x(u)^2([[x, y] + [y, x], z]) = x\delta(u)^2(D_z^r(D_y + D_y^r))(1) = \\ &= 2xR_z(u)(L_y(u) + R_y(u)), \end{aligned}$$

so

$$(42) \quad R_z(u)(L_y(u) + R_y(u)) = 0.$$

In view of (38), $D_{[x, x]} = 0$, so that $L_{[x, x]} = 0$. But putting $y = z = [x, x]$ in (42) we get $(R_{[x, x]}(u))^2 = 0$, so that $R_{[x, x]} = 0$ and hence

$$(43) \quad [z, [x, x]] = z[\mathbf{1}, [x, x]].$$

Putting now $y = [1, [x, x]]$ in (43) we get, in view of (39),

$$(44) \quad ([1, [x, x]])^2 = [[1, [x, x]], [x, x]] = 0,$$

so that $[1, [x, x]] = 0$ and finally

$$(45) \quad [z, [x, x]] = 0.$$

Now, using (45) instead of (38) we get, similarly to (42),

$$(46) \quad L_z(u)(L_y(u) + R_y(u)) = 0$$

which, together with (42), yields

$$(47) \quad (L_z(u) + R_z(u))(L_y(u) + R_y(u)) = 0.$$

The latter, after putting $z = y$, implies clearly $L_y + R_y = 0$. But $(L_y + R_y)(x) = 0$ means

$$(48) \quad [x, y] + [y, x] = x([y, \mathbf{1}] + [\mathbf{1}, y]).$$

Since the right-hand side of (48) is symmetric with respect to x and y ,

$$(49) \quad x([y, \mathbf{1}] + [\mathbf{1}, y]) = y([x, \mathbf{1}] + [\mathbf{1}, x])$$

which implies

$$(50) \quad [y, \mathbf{1}] + [\mathbf{1}, y] = 2y[\mathbf{1}, \mathbf{1}].$$

Putting in (50) $y = [\mathbf{1}, \mathbf{1}]$ we get, due to (38), $[\mathbf{1}, \mathbf{1}]^3 = 0$, so that

$$(51) \quad [y, \mathbf{1}] + [\mathbf{1}, y] = 0$$

and the skew-symmetry $[x, y] + [y, x] = 0$ follows from (48). ■

COROLLARY 2. *Every local binary operation on sections of a one-dimensional bundle over a manifold M which satisfies the Jacobi identity (10) is skew-symmetric, i.e. it is a local Lie algebra bracket.*

In particular, every such operation on the algebra $C^\infty(M)$ is a Jacobi bracket associated with a Jacobi structure on M .

Proof. Since the operation is local, we can reduce locally to the algebra of smooth functions and, due to locality of the operation, to a bidifferential operator. In the algebra of smooth functions there are no nontrivial nilpotents, so, according to theorem 2, the operation is of first order and skew-symmetric. ■

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