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CARTAN CONNECTIONS AND MOMENTUM MAPS

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In honour of Professor W. Tulczyjew on the occasion of his 70th birthday

Introduction. The theory of Cartan connections was introduced and developed by C. Ehresmann in his famous paper on "Connexions infinitésimales" [E]. See also Kobayashi [Ko].

A Cartan connection gives rise to a special parallelism. We summarize a part of the results of a previous paper [L1] concerning this subject. We formulate in terms of jets and groupoids certain ideas of M. Lazard [La].

We explain links between parallelisms and momentum maps. We refer to [L.M] and [M2] for momentum maps. We recall some results concerning momentum maps relative to principal bundles, as developed in a recent paper [L4].

As was proved in [E], with a Cartan connection is associated a vector bundle $\mathcal{E} \to TM$ which is isomorphic to the tangent bundle TM. This isomorphism permits a generalization of the notion of Lagrangian differential introduced by W. Tulczyjew [W].

1. Definitions. All manifolds and maps are supposed to be C^∞. The projections $TN \to N$, $T^*N \to N$, $TT^*N \to TN$ will be denoted by p, q, Tq for any manifold N.

Let $\pi : E \to M$ be a locally trivial fibration, $VE = \ker T\pi$ be the vertical bundle. From the inclusion $i : VE \to TE$, we deduce the projection $j : T^*E \to V^*E$ whose kernel is the annihilator $(VE)^0$ of VE; this kernel may be identified with the subbundle $\pi^*T^*M = E \times_M T^*M$ of T^*E.

A section $\eta : E \to E \times_M T^*M$ is called a *semi-basic form* on E. This section induces a morphism $f = \mathrm{pr}_2 \circ \eta$ from E to T^*M (where pr_2 is the projection $E \times_M T^*M \to T^*M$). In particular for the fibration $q : T^*M \to M$, the natural Liouville form θ_M on T^*M corresponds to the identity mapping of T^*M. The form η on E is the pull-back $f^*\theta_M$. Conversely for any morphism $f : E \to T^*M$, the pull-back $f^*\theta_M$ is semi-basic.

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By means of adapted coordinates $(x^1, \dots, x^n, y^1, \dots, y^k)$ in $\pi^{-1}(U)$ and $(x^1, \dots, x^n, p_1, \dots, p_n)$ in $q^{-1}(U)$, the forms η and θ_M are written

$$\eta = \sum_{i=1}^{n} a_i(x^1, \dots, x^n, y^1, \dots, y^k)\, dx^i\,, \qquad \theta_M = \sum_{i=1}^{n} p_i\, dx^i\,,$$

with $n = \dim M$ and $n + k = \dim E$, and f is represented by $p_i = a_i$ $(i = 1, \dots, n)$.

A section $\mu : E \to V^*E$ is called a *vertical form*; it acts only on vertical vectors. This vertical form may be represented in $\pi^{-1}(U)$ by

$$\mu = \sum_{j=1}^{k} b_j(x^1, \dots, x^n, y^1, \dots, y^k)\, dy^j\,.$$

If λ is any form on E, λ is written

$$\lambda = \sum_{i=1}^{n} A_i\, dx^i + \sum_{j=1}^{k} B_j\, dy^j\,,$$

then

$$j\lambda = \sum_{j=1}^{k} B_j\, dy^j\,.$$

A foliation $\mathcal{F}$ on a symplectic manifold is said to be symplectically complete [L2] if for any pair of first integrals, the Poisson bracket is also a first integral. A foliation $\mathcal{F}$ is symplectically complete if and only if the orthogonal distribution $\mathcal{F}^\perp$ is completely integrable, defining on $\mathcal{F}^\perp$ a symplectically complete foliation.

2. Connections and parallelism [L1]**.** Let $\pi : E \to M$ be a locally trivial fibration, J_1E be the set of all 1-jets of local sections with the projection $\delta_1 : J_1E \to E$.

A *connection* C on E is a lifting

$$C : E \to J_1E\,.$$

It is a first order differential system. A solution of C is a section $\sigma : U \subset M \to E$ such that for any $x \in U$ the jet $j^1_x\sigma$ belongs to $C(E)$. The connection is said to be *integrable* if for any $y \in E$, there exists a solution σ such that $\sigma(\pi(y)) = y$. According to the Frobenius theorem, C is integrable if and only if the composed map $J_1C \circ C$ from E to J_1J_1E takes its values in J_2E (set of 2-jets of local sections). The obstruction to integrability is the *curvature*. It is a lifting $\rho : E \to L^2_{E,a}(\pi^*TM; VE)$, the set of alternate bilinear maps from $\pi^*TM \times \pi^*TM$ to VE.

As any $z \in J_1E$ may be considered as an injective linear map from $T_{\pi(y)}M$ to T_yE (with $y = \delta_1(z)$), the connection induces a splitting

$$TE = VE \oplus \mathcal{H}$$

and conversely. According to a second version of the Frobenius theorem, the connection is integrable if and only if the distribution $\mathcal{H}$ is completely integrable.

A manifold M is said to be *parallelizable* if there exists a mapping

$$\omega : TM \to \mathbb{A}$$

(where $\mathbb{A}$ is a vector space) such that for any $x \in M$, the restriction ω_x of ω to T_xM is an isomorphism onto $\mathbb{A}$.

This yields a trivialization

$$\phi : TM \to M \times \mathbb{A}$$

defined for any $x \in M$ and any $v \in T_xM$ by

$$\phi(v) = (x, \omega_x v)\,.$$

With each mapping $f : U \subset M \to M$ we associate its *reduced differential*

$$D_{red}f : U \to L(\mathbb{A}, \mathbb{A})$$

defined by

$$D_{red}f(x) = \omega_{f(x)} \circ T_x f \circ \omega_x^{-1}\,.$$

The *reduced expression* of the vector field $X : M \to TM$ is the map $\omega \circ X : M \to \mathbb{A}$. The vector field X is said to be *invariant* if $\omega \circ X$ is constant.

For any pair $(x, x') \in M \times M$, the isomorphism $\omega_{x'}^{-1} \circ \omega_x$ is an isomorphism from T_xM onto $T_{x'}M$; it may be considered as the 1-jet $j_x^1 f$ of a diffeomorphism $f : U \ni x \to M$ such that $x' = f(x)$. So the parallelism ω induces a lifting

$$C : M \times M \to \Gamma^1(M)\,,$$

where $\Gamma^1(M)$ is the groupoid of all 1-jets of local diffeomorphisms on M.

As a diffeomorphism $f : U \subset M \to M$ induces a section $x \mapsto (x, f(x))$ of the fibration $\mathrm{pr}_1 : M \times M \to M$, the *lifting C is a connection.*

A solution of the differential system C is a diffeomorphism $f : U \subset M \to M$ such that for any $x \in U$, the jet $j_x^1 f$ is equal to $C(x, f(x))$. Such a solution is called a *translation*; identifying $j_x^1 f$ and $T_x f$, we check that $D_{red}f(x)$ is the identity map of $\mathbb{A}$.

The connection C lifts every vector field X tangent to M into a horizontal vector field on $M \times M$; it is the vector field $Y(x, x') = (X(x), \omega_{x'}^{-1} \circ \omega_x X(x))$. A vector field X on M is invariant if and only if its natural lift to $M \times M$ defined by $(x, x') \mapsto (X(x), X(x'))$ is horizontal.

We deduce that the *parallelism* (considered as a connection C) *is integrable if and only if the bracket of two invariant vector fields is an invariant vector field*. This yields a *Lie algebra* structure on $\mathbb{A}$.

Examples.

(i) A Lie group admits integrable parallelisms. The left and right translations as well as the left and right invariant vector fields are translations and invariant vector fields in our sense.

(ii) Let M be a simply connected manifold endowed with an integrable parallelism such that all translations are defined on the whole of M. If we fix a point on M, we get a Lie group structure on M.

(iii) The sphere S_7 admits a parallelism defined by means of the Cayley numbers. This parallelism is not integrable.

Remark. The usual point of view for introducing connections linked with a parallelism (not necessarily integrable) is the following. The choice of a basis of the tangent space $T_{x_0}M$ at a point x_0 determines n linearly independent invariant vector fields, hence a section of the frame bundle $H(M)$ (or $\{e\}$-structure); this yields a principal connection on $H(M)$ with null curvature. The trajectories of the invariant vector fields are the geodesics of this connection.

3. Parallelism and momentum maps. For any manifold M, the natural Liouville form θ_M is the form on T^*M which associates with any $v \in TT^*M$ the scalar

$$\big\langle p(v), Tq(v) \big\rangle\,.$$

It is known (see [L4]) that any vector field X on M is lifted on T^*M to an infinitesimal automorphism $\widetilde{X}$ of the form θ_M; it is the Hamiltonian vector field defined as follows. The vector field X induces a section $\overline{X}$ of the bundle $T^*M \times_M TM \to T^*M$; as θ_M is a section of $T^*M \times_M T^*M \to T^*M$, we may define $h = \langle \theta_M, X \rangle$ by $h = \langle \theta_M, \overline{X} \rangle$. Then $\widetilde{X}$ is the Hamiltonian vector field such that $i(\widetilde{X})d\theta_M = -dh$. It can be checked that $h = \langle \theta_M, \widetilde{X} \rangle$.

Suppose now that M is parallelizable. From the map $\omega : TM \to \mathbb{A}$, we obtain by duality the mapping

$$\mu : T^*M \to \mathbb{A}^*$$

such that for any $x \in M$, the mapping $\mu_x : T^*M \to \mathbb{A}$ is the contragredient of ω_x.

This yields a trivialization

$$\psi : T^*M \to M \times \mathbb{A}^*$$

such that for any $x \in M$ and any $\eta \in T^*_xM$, we have

$$\psi(\eta) = (x, \mu_x \eta)\,.$$

Any morphism f from $E \to M$ to $T^*M \to M$ induces a map $E \to \mathbb{A}^*$ and conversely.

We have the notion of invariant 1-form; a form η on M is said to be *invariant* if $\mu \circ \eta$ is constant. This condition is equivalent to the following

$$(\omega_x^{-1})^*\eta(x) = (\omega_{x'}^{-1})^*\eta(x') \quad \text{for any pair} \quad (x, x') \in M \times M\,.$$

The Liouville form θ_M may be defined as the form associating with $v \in TT^*M$ the scalar

$$\big\langle \mu \circ p(v), \omega \circ Tq(v) \big\rangle\,.$$

Let X^a be the invariant vector field on M whose reduced expression is a. Then its lift $\widetilde{X}^a$ to T^*M is the vector field defined by the condition: for any $y \in T^*M$,

$$i(\widetilde{X}^a)d\theta_M|_y = -\big\langle \mu(y), a \big\rangle\,.$$

The bracket $[X, Y]$ of two vector fields X and Y on M is lifted to the bracket $[\widetilde{X}, \widetilde{Y}]$. So when the parallelism is integrable, the lifts of the invariant vector fields constitute a Lie algebra. We have the *Hamiltonian action of a Lie algebra on* T^*M in the sense of [L.M, chapter 4]. These considerations do not imply that the vector fields are complete.

The Hamiltonian action of a Lie group G on its cotangent bundle T^*G is studied in [L.M] and [M2].

Let L_g (resp. R_g) be the left (resp. the right) translation $s \mapsto gs$ (resp. $s \mapsto sg$). The left parallelism on G corresponds to the maps

$$\omega^L : TG \to \mathcal{G}\,, \qquad \mu^L : T^*G \to \mathcal{G}^*$$

such that for any $g \in G$

$$\omega^L_g = T_g L_{g^{-1}}\,, \qquad \mu^L_g = {}^t\!(\omega^L_g)^{-1} = {}^t\!(T_e L_g)\,.$$

We define ω^R and μ^R similarly.

We recall that a symplectic action of a Lie group G on a symplectic manifold M is said to be Hamiltonian if there exists a map $J : M \to \mathcal{G}^*$ (called *momentum map*) such that for any $c \in \mathcal{G}$ the associated fundamental vector field Y^c is Hamiltonian and admits as Hamiltonian the function f defined by $f(x) = \langle J(x), c \rangle$.

This is the case of the left action or the right action of G on T^*G, with momentum maps

$$J_L = \mu^R\,, \qquad J_R = \mu^L\,.$$

The exchange between R and L comes from the fact that the fundamental vector field Y^c corresponding to a left action of G is a right invariant vector field.

It is also proved that the left (resp. right) orbits of the left (resp. right) action of G are the level sets of J_R (resp. J_L). These orbits are orthogonal with respect to the symplectic form $d\theta_G$. The connected components of these orbits constitute symplectically complete foliations in the sense of section 1. When G is connected, it follows from the theory of symplectically complete foliations (see [L2]) that there exists on $\mathcal{G}^*$ a unique Poisson structure such that J_R is a Poisson map. The map J_L is a Poisson map for the opposite Poisson structure on $\mathcal{G}^*$. We recover the "Kirillov-Kostant-Souriau" Poisson structures on $\mathcal{G}^*$ (see [M2]).

These properties have led M. Condevaux, P. Dazord and P. Molino to introduce the notion of "generalized momentum map" [C.D.M]. The authors consider a symplectically complete foliation; an atlas of "local slidings" along the leaves of the orthogonal foliation constitutes the generalized momentum map.

Remark. C. Albert and P. Dazord [A.D] have exhibited a symplectic groupoid structure on T^*G as follows. The set of unities is $T^*_e G = \mathcal{G}^*$ with projections μ^L and μ^R from T^*G onto $\mathcal{G}^*$ (in our notations). The product $u \circ v$ of $u \in T^*_g G$ and $v \in T^*_s G$ is defined if and only if

$$\mu^L_g(u) = \mu^R_s(v) = b\,.$$

Then

$$u \circ v = (\mu^L_g)^{-1}(\mu^R_s)^{-1} b = (\mu^L_g)^{-1} v\,.$$

By means of the left trivialization $\psi^L : T^*G \to G \times \mathcal{G}^*$ (with $\psi^L(\eta) = (g, \mu^L_g \eta)$ for any $g \in G$ and $\eta \in T^*G$), u and v may be written respectively $u = (g,\xi)$, $v = (s,\lambda)$. Then $(g,\xi) \circ (s,\lambda) = (gs,\lambda)$ if $\xi = \mathrm{Ad}^*_s \lambda$.

4. Principal connections and momentum maps. In this section, as we shall have to consider the left parallelism on a Lie group and a parallelism on a principal bundle P, we shall denote by α the left invariant form $\omega^L : TG \to \mathcal{G}$. This form α (called the

Maurer-Cartan form of G) satisfies the relations

$$L_g^*\alpha = \alpha\,, \quad R_g^*\alpha = \mathrm{Ad}(g^{-1})\alpha\,, \quad d\alpha + \frac{1}{2}\,[\alpha,\alpha] = 0\,.$$

Let $\pi : P \to M$ be a principal G-bundle (where M is not assumed to be parallelizable). The action of G on P by right translations $z \mapsto zg$ being free and regular, any $z \in P$ determines a diffeomorphism from $P_{\pi(z)}$ onto G which maps z on e. Hence we get an isomorphism ϖ_z from $T_zP_{\pi(z)}$ onto $\mathcal{G}$. We deduce a map

$$\varpi : VP \to \mathcal{G}$$

called the *vertical parallelism* in the sense of [L1]. We have

$$\varpi_{zg} = \mathrm{Ad}(g^{-1})\varpi_z \quad \text{for} \quad g \in G\,.$$

Considering the contragredient ${}^t\,\varpi_z^{-1}$ of ϖ_z, we obtain a map

$$\kappa : V^*P \to \mathcal{G}^*\,.$$

The right action of G on P (whose orbits are the fibers) lifts to a Hamiltonian action on T^*P. It is proved in [L4] that the corresponding momentum map is

$$J = \kappa \circ j : T^*P \to \mathcal{G}^*$$

where j is the natural projection $T^*P \to V^*P$.

So $J^{-1}(0)$, kernel of J, is the kernel of the projection j; according to the first section, $J^{-1}(0)$ is the set $\widetilde{P} = P \times_M T^*M$ of semi-basic forms on P.

Let $\beta : TP \to \mathcal{G}$ be a connection form on P inducing a principal connection. We recall that $\beta_{zg} = \mathrm{Ad}(g^{-1})\beta_z$ for $z \in P$, $g \in G$. The restriction of β to VP is the form ϖ defined above.

In the splitting (*cf.* section 2)

$$TP = VP \oplus \mathcal{H}\,,$$

the horizontal bundle $\mathcal{H} = \ker\beta$ is G-invariant.

By duality we get

$$T^*P = V^*P \oplus \mathcal{H}^* = \mathcal{H}^0 \oplus \widetilde{P}\,,$$

identifying $\mathcal{H}^*$ with the annihilator $(VP)^0$ of VP in TP *i.e.* with $\widetilde{P}$ and identifying V^*P with $\mathcal{H}^0$.

We have proved in [L4] that any $\varphi \in T_z^*P$ may be written $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 = \varphi - \beta_z^* J(\varphi)$ belonging to $\widetilde{P}$ and $\varphi_2 = \beta_z^* J(\varphi)$ vanishing on $\ker\beta$.

5. Cartan connections. Let $\widehat{G}$ be a Lie group (with Lie algebra $\widehat{\mathcal{G}}$) such that G is a closed subgroup of $\widehat{G}$ and P be a G-principal bundle.

A *Cartan connection* on P is a form

$$\omega : TP \to \widehat{\mathcal{G}}$$

satisfying the conditions

1) The restriction of ω to VP is the form $\varpi : VP \to \mathcal{G}$ defined above.
2) $\omega_{zg} = \mathrm{Ad}(g^{-1})\omega_z$ for $g \in G$.
3) ω defines a parallelism on P.

It follows that $\dim \widehat{G} = \dim G + \dim M$. A Cartan connection is *not* a connection in the usual sense because ω takes its values in $\widehat{\mathcal{G}}$. But it induces a principal connection on a principal $\widehat{G}$-bundle $\widehat{P}$ obtained by enlarging the structure group of G to $\widehat{G}$. This bundle is

$$\widehat{P} = P \times_G \widehat{G}\,,$$

quotient of $P \times \widehat{G}$ by the equivalence relation $(z, s) \sim (zg, g^{-1}s)$ for any $z \in P$, $s \in \widehat{G}$, $g \in G$. Let Λ be the projection $P \times \widehat{G} \to \widehat{P}$.

Let us consider the form $\omega - \alpha$ on $P \times \widehat{G}$ (where α is the Maurer-Cartan form on $\widehat{G}$). As

$$\omega_{zg} = \mathrm{Ad}(g^{-1})\omega_z\,, \qquad \alpha_{sg} = \mathrm{Ad}(g^{-1})\alpha_s$$

we have

$$\omega_{zg} - \alpha_{sg} = \mathrm{Ad}(g^{-1})(\omega_z - \alpha_s)$$

and the form $\omega - \alpha$ is the pullback $\Lambda^*\widehat{\beta}$ of a *connection form $\widehat{\beta}$ on $\widehat{P}$*. The restriction to P of the form $\widehat{\beta}$ is the form ω.

It can be checked (see [E], [L1]) that if X and Y are vector fields on P and $\widehat{G}$ whose reduced expression is the same (*i.e.* there exists $a \in \widehat{\mathcal{G}}$ such that $\omega(X_z) = \alpha(Y_s) = a$ for any pair $(z, s) \in P \times \widehat{G}$), then the pair (X, Y) is projectable by Λ onto a vector field Z tangent to the horizontal distribution $\widehat{\mathcal{H}} = \ker \widehat{\beta}$. We shall say that the *Cartan connection ω is integrable* if the corresponding parallelism is *integrable* in the sense of section 2. As the parallelism on $\widehat{G}$ is integrable, *the Cartan connection ω is integrable if and only if the connection $\widehat{\beta}$* on $\widehat{P}$ is *integrable*.

The obstruction to the integrability is the curvature Ω defined by

$$\Omega = d\omega + \frac{1}{2}\,[\omega, \omega]\,.$$

Let $T_P\widehat{P}$, $V_P\widehat{P}$, $T_P^*\widehat{P}$, $V_P^*\widehat{P}$ be the restriction of $T\widehat{P}$, $V\widehat{P}$, $T^*\widehat{P}$, $V^*\widehat{P}$ to P The vertical parallelism $\widehat{\varpi} : V\widehat{P} \to \widehat{\mathcal{G}}$ and the Cartan connection $\omega : TP \to \widehat{\mathcal{G}}$ induce an isomorphism φ from $V_P\widehat{P}$ onto TP; for $z \in P$, the restriction φ_z of φ is the isomorphism $\omega_z^{-1} \circ \widehat{\varpi}_z$ from $V_z\widehat{P}$ onto T_zP.

Let us define $\widehat{\kappa} : V^*\widehat{P} \to \widehat{\mathcal{G}}^*$ and $\widehat{j} : T^*\widehat{P} \to V^*\widehat{P}$ as were defined κ and j for T^*P and V^*P. The momentum map $\widehat{J}$ of the symplectic action of $\widehat{G}$ on $T^*\widehat{P}$ is

$$\widehat{J} = \widehat{\kappa} \circ \widehat{j} : T^*\widehat{P} \to \widehat{\mathcal{G}}^*\,.$$

Then the restriction $\widehat{J}_P$ of $\widehat{J}$ to $T_P^*\widehat{P}$ may be written

$$\widehat{J}_P = \mu \circ \mathcal{P}\,,$$

where $\mathcal{P}$ is the projection $T_P^*\widehat{P} \to T^*P$ and

$$\mu : T^*P \to \widehat{\mathcal{G}}^*$$

is defined by the trivialization $T^*P \approx P \times \widehat{\mathcal{G}}^*$ associated with the parallelism ω (see section 3).

The momentum map $J : T^*P \to \mathcal{G}^*$ may be written

$$J = \gamma \circ \mu$$

where γ is the projection $\widehat{\mathcal{G}}^* \to \mathcal{G}^*$.

While μ depends on the choice of ω, the map J, by its very definition, is independent of ω.

REMARKS.

(i) The restriction $\widehat{\mathcal{H}}_P$ to P of the horizontal distribution $\widehat{\mathcal{H}} = \ker \widehat{\beta}$ is transversal to both vector bundles $V_P\widetilde{P}$ and TP. So if the Cartan connection (and hence $\widehat{\mathcal{H}}$) is *integrable*, then any $z \in P$ admits a neighborhood U in $\widehat{P}$ such that the trace on U of every horizontal leaf (if not empty) intersects P and the fiber $\widehat{P}_{\widehat{\pi}(z)}$ in one point. This yields a local diffeomorphism of P onto $\widehat{P}_{\widehat{\pi}(z)}$ (hence on $\widehat{G}$).

(ii) According to section 2, the parallelism on P induces a lifting $C : P \times P \to \Gamma^1(P)$ (groupoid of the 1-jets of all local diffeomorphisms on P). In the case of a Cartan connection the result may be improved. We shall sum up the investigations of [L1].

The lifting C takes its values in the groupoid $\Theta^1 \subset \Gamma^1(P)$, the set of the 1-jets of all local automorphisms of P; such a local automorphism φ is defined in an open set $\pi^{-1}(U)$ (where U is an open subset of M) and satisfies the relation $\varphi(zg) = \varphi(z)g$ for any $g \in G$.

Let us consider the gauge groupoid Φ of P, *i.e.* the quotient of $P \times P$ by the diagonal right action of G. Let π_1 and π_2 be the projections of Φ onto M. An invertible π_1-section $\sigma_1 : U \to \Phi$ (with $\pi_1 \circ \sigma_1 = \mathrm{id}_U$) satisfies the condition: the map $\pi_2 \circ \sigma_1$ is a diffeomorphism f inducing the π_2-section $\sigma_2 = \sigma_1 \circ f^{-1}$. There exists a (1-1) correspondence between the set of all local automorphisms of P and the set of all invertible π_1-sections of Φ. The set $\Phi^{(1)}$ of 1-jets of all invertible π_1-sections is a groupoid and the groupoid Θ^1 may be identified with the fibered product $P \times_M \Phi^{(1)}$; then the lifting C induces a lifting

$$C' : \Phi \to \Phi^{(1)} ,$$

which is a groupoid morphism. Conversely such a lifting C' induces a Cartan connection on P.

(iii) Let $\pi : P \to M$ be a principal G-bundle which is not necessarily endowed with a Cartan connection. The groupoid Θ^1, with base P, defined in remark (ii) acts on the manifold $\mathcal{T}P$ of *transverse frames* defined in the following way. An element u of $\mathcal{T}P$ is a 1-jet $j^1_0 f$ (identified with the linear map $T_0 f$), where f is a map from $\mathbb{R}^n$ to P whose source contains $0 \in \mathbb{R}^n$, such that $T_z\pi \circ T_0 f$ is an isomorphism from $T_0\mathbb{R}^n$ to $T_x M$, *i.e.* an element of the frame bundle $H(M)$. We have set $z = f(0)$ and $x = \pi(z)$. The image of u is transverse to the vertical bundle.

In [L1] we have proved that the projection $\mathcal{T}P \to M$ is a principal bundle map whose gauge groupoid is the groupoid $\Phi^{(1)}$ defined in remark (ii).

Suppose now that P is endowed with a Cartan connection ω. Then if we specify an element u of $\mathcal{T}P$, the lifting $C' : \Phi \to \Phi^{(1)}$ induces a lifting $s : P \to \mathcal{T}P$. As we have a natural projection $\mathcal{T}P \to H(M)$, we obtain a mapping $P \to H(M)$ which is a principal bundle morphism, as proved in [E].

The manifold $\mathcal{T}P$ is a submanifold of the frame bundle $H(P)$. Indeed with any $u \in \mathcal{T}P$ (with target $z \in P$), we associate the isomorphism (ϖ_z^{-1}, u), from $\mathcal{G} \times T_0\mathbb{R}^n$ to $T_z P$ which belongs to $H(P)$, the mapping ϖ defining the vertical parallelism as in section 4.

We have here a special case of the situation examined in the remark of section 2.

6. Cartan connections on homogeneous spaces. Let $\widehat{G}$ be a Lie group and G a closed subgroup. The right action of G on $\widehat{G}$ defines a principal G-bundle structure on $\widehat{G}$, whose base M is the homogeneous space $F = \widehat{G}/G$. We shall set $P = \widehat{G}$ when considering $\widehat{G}$ as a principal G-bundle.

The Maurer-Cartan form α on $\widehat{G}$ is an integrable Cartan connection form. It induces a principal connection on the $\widehat{G}$-bundle $\widehat{P} = P \times_G \widehat{G}$, as we have seen in section 5. This bundle $\widehat{P}$ can be identified with the trivial principal bundle $\widehat{G} \times \widehat{G}/G$, with horizontal leaves $\{s\} \times \widehat{G}/G$. The manifold $\widehat{G} \times \widehat{G}/G$ is also diffeomorphic to the gauge groupoid of P, quotient of $P \times \widehat{G}$ by the equivalence relation $(s_1, s_2) \sim (s_1 g, s_2 g)$ with $s_1, s_2 \in \widehat{G}$, $g \in G$.

Remark. Let $\mathcal{J}_L$ be the momentum map of the left action of $\widehat{G}$ on $T^*\widehat{G}$. If $b \in \widehat{\mathcal{G}}^*$ is invariant for the coadjoint action of G on $\widehat{\mathcal{G}}^*$, then the submanifold $\mathcal{J}_L^{-1}(b)$ is invariant under the left action of G on T^*G as it is shown in [M2] and the quotient manifold $\mathcal{J}_L^{-1}(b)/G$ (*i.e.* the set of orbits of the left action of G on $\mathcal{J}_L^{-1}(b)$) has a reduced symplectic structure.

We are in a special case of the situation studied in [Ku] as follows: $\pi : P \to M$ is a principal G-bundle and the Lie group G is a subgroup of a Lie group $\widehat{G}$ acting on P.

7. Connections on frame bundles. Among the homogeneous spaces let us consider $G = L_n = GL(n, \mathbb{R})$ and $\widehat{G} = L_n \times \mathbb{R}^n$ (semi-direct product); then $\widehat{G}/G = \mathbb{R}^n$; the group $\widehat{G}$ is the affine group $\mathcal{A}(n, \mathbb{R})$.

If we restrain L_n to the orthogonal group $SO(n)$, then $\widehat{G}$ is the group of euclidian displacements and $\widehat{G}/G$ is endowed with a euclidian structure. For $n = 3$, C. Marle [M2] has considered this situation when studying the motion of the rigid body.

For a n-dimensional manifold M, the frame bundle $\pi : H(M) \to M$ is a principal L_n-bundle. An L_n-connection on $H(M)$ is called a linear connection. It is known that there exists on $H(M)$ an equivariant form η with values in $\mathbb{R}^n$. It is the form defined by

$$\eta(v) = h^{-1}(\pi(v)) \quad \text{for} \quad v \in T_h H(M)\,.$$

As $\ker \eta = VH$, with any L_n-connection form β on $H(M)$ is associated a form $\omega = \beta + \eta$ which is a Cartan connection. So *every frame bundle is parallelizable*. The form ω is the restriction of a $\mathcal{A}(n)$-connection form on the bundle of affine frames. The parallelism associated with ω is integrable if the corresponding $\mathcal{A}(n)$-connection has a null curvature. This is equivalent to the nullity of the curvature and torsion of the linear connection β.

Every subbundle of the frame bundle is also parallelizable. This is in particular true for Riemannian structures.

Instead of L_n, we could consider the group L_n^r (whose elements are r-jets of local diffeomorphisms of $\mathbb{R}^n$ with source and target 0). Then $H^r(M)$ is the set of frames of order r. In particular the projective and conformal structures are structures of order 2. See [E] and [Ko].

In [A.G.M] are considered principal bundles for which there exists a form η with values in $\mathbb{R}^n$ such that η is equivariant.

Other examples of parallelizable principal bundles are the SU_2-bundle $S_7 \to S_4$ and the S_1-bundle $S_7 \to \mathbb{P}_3(\mathbb{C})$.

8. Cartan connections and generalized Lagrangian differentials. We keep the notations of section 5. Let $\widehat{\pi} : \widehat{P} \to M$, $\pi : P \to M$ the projections. We consider a Cartan connection ω on a principal G-bundle, with values in the Lie algebra $\widehat{\mathcal{G}}$ of $\widehat{G}$. We shall sketch some results due to C. Ehresmann [E].

As $\widehat{G}$ acts on the left on the homogeneous $F = \widehat{G}/G$, we may define the bundle $\mathcal{F}$ associated with $\widehat{P}$, with standard fiber F. It is the manifold $\widehat{P} \times_{\widehat{G}} F$, quotient of $\widehat{P} \times F$ by the equivalence relation $(z, y) \sim (zg, g^{-1}y)$ for any $g \in \widehat{G}$.

The bundle $\mathcal{F} \to M$ admits a natural section $\sigma : M \to \mathcal{F}$ defined as follows. The subgroup G of $\widehat{G}$ is the set of all $g \in \widehat{G}$ leaving invariant $a_0 \in F$, where a_0 is the image of e by the projection $\Phi : \widehat{G} \to \widehat{G}/G$. Any $z \in \widehat{P}_x$ can be considered as a diffeomorphism from F to the fiber $\mathcal{F}_x$. If z belongs to P_x, this fiber P_x may be written zG. So the image a_x of a_0 by an element of P_x is independent of the choice of that element. We set $\sigma(x) = a_x$. We shall identify the base M with its image by σ.

Let $\mathcal{E} = \bigcup_{x \in M} T_x\mathcal{F}_x$. We thus define a vector bundle associated with P whose standard fiber is $\widehat{\mathcal{G}}/\mathcal{G}$. Indeed each $z \in P_x$ induces an isomorphism from $T_{a_0}F$ to $T_x\mathcal{F}_x$. Consider the map $T_e\Phi : T_e\widehat{G} \to T_{a_0}\widehat{G}/G$; its kernel is T_eG. So $T_{a_0}F$ can be identified with the vector space $\widehat{\mathcal{G}}/\mathcal{G}$. For any $z \in P$, the map $T_z\pi \circ \omega_z^{-1}$ from $\widehat{\mathcal{G}}$ to T_xM (with $x = \pi(z)$) vanishes on $\mathcal{G}$, inducing an isomorphism $f_x : \mathcal{E}_x \to T_xM$. This gives a vector bundle isomorphism $f : \mathcal{E} \to TM$. With the terminology of [E], we say that the Cartan parallelism induces a *soldering* of the fiber bundles $\mathcal{F}$ and $\mathcal{E}$ to their base M.

We recall the notion of *Lagrangian differential* introduced by W. Tulczyjew [T] as explained in [L3]. See also [M1].

The set T^2M of 2-velocities on M is the subbundle of TTM defined by $T^2M = \{\, v^2 \in TTM\,;\, Tp(v^2) = p(v^2)\,\}$. An element of TTM belonging to T^2M is said to be *holonomic*. The difference $v^2 - w^2$ of holonomic tangent vectors with the same origin in TM is a vertical vector. Let L be a hyperregular function on TM defining a Legendre transformation $\mathcal{L} : TM \to T^*M$; then TM is endowed with a symplectic structure and the fibers of the projection $TM \to M$ constitute a Lagrangian foliation. Let $A = i(Z)dL - L$ (where Z is the Liouville vector field on TM). The hamiltonian vector field X_A on TM is holonomic. For any other holonomic vector field X, the vector field $X - X_A$ is vertical; its image by the symplectic duality is a semi-basic form, hence defines a morphism from TM to T^*M. So we obtain a morphism $\Delta(L) : T^2M \to T^*M$ which is the Lagrangian differential.

Consider now the vector bundle $\mathcal{E} \to M$. The isomorphism $Tf^{-1} : TTM \to T\mathcal{E}$ maps T^2M onto a subbundle $\overline{T}\mathcal{E}$ which may be called the set of pseudoholomorphic tangent vectors to $\mathcal{E}$; this subbundle $\overline{T}\mathcal{E}$ is $\{w \in T\mathcal{E}\,;\, Tp(w) = f \circ p(w)\}$. Utilizing the isomorphism f and its contragredient ${}^t f^* : \mathcal{E}^* \to T^*M$, we may show as above that a hyperregular function S on $\mathcal{E}$ generates a symplectic structure on $\mathcal{E}$ and a Hamiltonian vector field $X_{S-i(Z')dS}$. Hence we deduce a morphism $\overline{T}\mathcal{E} \to T^*M$ which can be called the generalized Lagrangian differential of S.

Remark. In particular cases of Cartan connections, we may define a morphism $\overline{T}\mathcal{E} \to T^*M$ without using a Lagrangian function.

According to [K.N] the homogeneous space $F = \widehat{G}/G$ is said to be *reductive* if there exists a splitting $\widehat{\mathcal{G}} = \mathcal{G} \oplus \mathcal{M}$ where $\mathcal{M}$ is an $\mathrm{Ad}(G)$-invariant subspace of $\widehat{\mathcal{G}}$. Let P be endowed with a Cartan connection ω. It is proved in [L1] that the subspace $\mathcal{M}$ defines a principal connection on P whose horizontal distribution is invariant under the parallelism induced by ω. The horizontal subspaces are the images of $\mathcal{M}$ by the isomorphisms ω_z^{-1} when z generates P.

It is proved also that the morphism $P \to H(M)$ introduced in remark (iii) of section 5 induces a principal connection on $H(M)$, hence a spray $TM \to T^2M$. Using the morphism $f : \mathcal{E} \to TM$, we obtain a lifting $\mathcal{E} \to \overline{T}\mathcal{E}$; hence with any element of $\overline{T}\mathcal{E}$ we associate a vertical tangent vector to $\mathcal{E}$. Given a Liouville form θ on $\mathcal{E}$ (*i.e.* according to [A.G.M] a semi-basic form such that $d\theta$ is symplectic), the symplectic duality on $\mathcal{E}$ maps each vertical vector field to a semi-basic form. Hence we obtain a morphism $\overline{T}\mathcal{E} \to T^*M$. The examples investigated in section 7 correspond to the reductive case.

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