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ON SYMMETRIES AND CONSTANTS OF MOTION IN HAMILTONIAN SYSTEMS WITH NONHOLONOMIC CONSTRAINTS

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To Włodzimierz M. Tulczyjew, with respect and admiration

1. Introduction. In classical Lagrangian or Hamiltonian systems, constants of motion are closely related to symmetries, as shown by Noether's theorem. For a Hamiltonian system on a symplectic manifold, the well known reduction theorem due to Marsden and Weinstein [38] uses both the symmetries and the constants of motion to produce a reduced Hamiltonian system on a lower dimensional symplectic manifold.

For mechanical systems with a nonholonomic constraint, constants of motion are still related to symmetries of the system, but in a subtler way. In section 2, we present the mathematical tools used in the description of mechanical systems with nonholonomic constraints, both in the Lagrangian and in the Hamiltonian formalisms. We will distinguish *ideal constraints* and *constraints of Chetaev type* (which, in our opinion, should not be considered as ideal), as well as constraints of a more general type. Then, in section 3, we will show that for these systems, two different types of Lie group actions should be distinguished, the first type allowing the reduction of the system, and the second giving rise to constants of motion. These results will be illustrated in section 4, where we consider what may be called a completely integrable nonholonomic system: a ball which rolls on the inner surface of a circular cylinder with vertical axis.

Bates, Graumann and MacDonnell [7] already observed that constants of motion in nonholonomic mechanical systems are related to symmetries, and discussed several examples (including that of the ball rolling on the inner surface of a cylinder). Our treatment rests on the same idea, and differs from theirs by the use of Poisson structures in the reduction procedure.

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2. Classical mechanical systems with a kinematic constraint

2.1. *General assumptions.* We consider a mechanical system whose configuration space (*i.e.* the set of possible positions of the various parts of the system at a given time) is a smooth n-dimensional manifold Q, which does not depend on time. A motion of the system is a parametrized curve $t \mapsto x(t)$ in Q, where the real parameter t (the time) runs over some open interval I. Assuming that the motion is smooth, for each $t \in I$, the vector $dx/dt \in T_{x(t)}Q$ is called the *kinematic state* of the system at time t, for the motion under consideration. The system is said to be *subject to a kinematic constraint* when not all vectors in TQ are possible kinematic states of the system. This occurs, for example, when two different parts of the system are rolling on each other without sliding, or when some part of the system rolls without sliding on an external object which is not a part of the system. We will assume that the kinematic constraint imposed upon the system is such that the set of possible kinematic states is a smooth submanifold C of TQ, which does not depend on time. The submanifold C will be called the *kinematic constraint submanifold*.

Remark 1. In many examples, the submanifold C is a vector subbundle of TQ; this happens in particular when the kinematic constraint is obtained by imposing to parts of the system to roll on each other without sliding, or to roll without sliding on an external object, that external object being at rest. In other examples, the submanifold C is an affine subbundle of TQ; this happens when some part of the system rolls without sliding on a moving external object, the motin of that external object being stationary. A ball which rolls without sliding on a horizontal disk rotating around a vertical axis at a constant velocity (as in a microwave oven) is an example of such a system.

Remark 2. Constraints obtained by means of servomechanisms may lead to kinematic submanifolds C much more general than vector or affine subbundles of TQ (see for example [34]). Even more generally, one may encounter systems in which the set of possible kinematic states depends on time, therefore can no longer be described by a fixed submanifold of TQ. A. Lewis [28] has proposed a general formalism for such systems.

2.2. *Dynamical properties of the system.* We will assume that a smooth Lagrangian $L : TQ \to \mathbf{R}$ accounts for the inertial properties of the system and for all forces acting upon it, other than the constraint force. As we will see in the next subsection, various assumptions can be made about that constraint force.

We denote by $J^2((\mathbf{R}, 0), Q)$ the space of jets of order 2, at the origin 0 of $\mathbf{R}$, of smooth parametrized curves $s \mapsto x(s)$ in Q (where the parameter s runs over some open interval which contains the origin 0). Let us recall that when a smooth Lagrangian $L : TQ \to \mathbf{R}$ is given, there is a map $\Delta(L)$, called the *Lagrange differential* of L, defined on $J^2((\mathbf{R}, 0), Q)$, with values in the cotangent bundle T^*Q, fibered over Q. The Lagrange differential is part of a complex, called the *Lagrange complex*, thoroughly studied by W. Tulczyjew [49]. Let us recall its expression in local coordinates. With a chart of Q, whose local coordinates are denoted by $(x^1, \dots, x^n)$, there are naturally associated charts of the tangent bundle TQ, the cotangent bundle T^*Q and the space of jets $J^2((\mathbf{R}, 0), Q)$, whose local coordinates are denoted by $(x^1, \dots, x^n, v^1, \dots, v^n)$, $(x^1, \dots, x^n, p_1, \dots p_n)$ and

$(x^1,\dots,x^n,\dot x^1,\dots,\dot x^n,\ddot x^1,\dots,\ddot x^n)$, respectively. Let a be an element in $J^2((\mathbf{R},0),Q)$, and $(a^1,\dots,a^n,\dot a^1,\dots,\dot a^n,\ddot a^1,\dots,\ddot a^n)$ its coordinates. Let $c: s\mapsto c(s)$ be a smooth parametrized curve in Q whose jet of order 2 at the origin is a, *i.e.* such that, for each i $(1\le i\le n)$,

$$x^i(c(s))|_{s=0}=a^i,\quad \frac{d}{ds}x^i(c(s))\Big|_{s=0}=\dot a^i,\quad \frac{d^2}{ds^2}x^i(c(s))\Big|_{s=0}=\ddot a^i.$$

We set

$$x^i(s)=x^i(c(s)),\qquad v^i(s)=\frac{d}{ds}x^i(c(s)).$$

The same letter L which denotes the Lagrangian will be used to denote the function of $2n$ real variables $(x^1,\dots,x^n,v^1,\dots,v^n)$ which is the expression of the Lagrangian in the chart of TQ under consideration.

With these notations, the coordinates $(x^1,\dots,x^n,p_1,\dots,p_n)$ of $\Delta(L)(a)$ are

$$\begin{cases} x^i=a^i,\\ p_i=\left(\dfrac{d}{ds}\left(\dfrac{\partial L}{\partial v^i}(x(s),v(s))\right)-\dfrac{\partial L}{\partial x^i}(x(s),v(s))\right)\Big|_{s=0}.\end{cases}$$

When no kinematic constraint is imposed on the system, every motion $c: t\mapsto c(t)$ must satisfy the well known Lagrange equation of motion

$$\Delta(L)(j^2c(t))=0,$$

where, for each time t in the interval of definition of c, $j^2c(t)$ denotes the jet of order 2 , at the origin $s=0$, of the map $s\mapsto c(t+s)$.

With the kinematic constraint, the equations of motion become

$$\begin{cases} \Delta(L)(j^2c(t))=f_c(t),\\ \dfrac{dc(t)}{dt}\in C,\end{cases}$$

where the map $f_c: t\mapsto f_c(t)$ is the constraint force. That map, which is in general unknown, must be such that for each time t, $f_c(t)\in T^*_{c(t)}Q$ and that $dc(t)/dt\in C$.

That system of equations is in general underdetermined. In order to get a well posed system of equations, additional assumptions about the constraint force are made, which should express the physical properties of the constraint. Among these assumptions, one of the most frequently made is that the constraint is *ideal*. That assumption is discussed in the next subsection.

2.3. *Ideal constraints and d'Alembert's principle.* In classical treatises of mechanics such as [11, 32, 45, 54], a constraint is said to be *ideal* if, for each time t, the virtual work of the constraint force vanishes, for any infinitesimal virtual displacement compatible with the constraint *frozen at time t*. In what follows, we will use that definition, and we will try to translate it in more precise, geometrical terms, without altering its physical meaning.

Several authors, following Arnol'd [3, chapter IV, § 21, p. 91], say that a constraint satisfies *d'Alembert's principle* when that constraint is ideal. But according to the celebrated treatise of Lagrange [27, Seconde partie, Section première, page 182], to [45, p. 88] or to [32], d'Alembert's principle has a much more general meaning: it states that the sum of all the forces acting on a system are, at any time, exactly balanced by the inertial forces of that system.

Some authors use the terms *ideal constraint* with a different meaning; for example, in [20], it is said that a constraint is ideal when the virtual work of the constraint force vanishes for any infinitesimal virtual displacement compatible with the constraint, without imposing to that constraint to be *frozen* at time t. Such a definition may of course be used as long as it is not self-contradictory, but we think that it is not in agreement with the physical meaning of the word *ideal*, nor with the meaning given to the terms *ideal constraint* in classical treatises.

In order to translate the meaning of the terms *ideal constraint* into precise geometrical properties, we must say what is an infinitesimal virtual displacement at time t, when such an infinitesimal virtual displacement is said to be compatible with the constraint frozen at time t, and what is the virtual work of the constraint force in such an infinitesimal virtual displacement.

Let us assume that the system has a motion $t \mapsto c(t)$.

For us, an infinitesimal virtual displacement at time t is simply a vector $w \in T_{c(t)}Q$. We must stress the fact that an infinitesimal virtual displacement at time t *is not* a velocity: the principle of virtual work has its origin in *statics* (see the beautiful book of Tulczyjew [51]), in which the time plays no part.

When should we say that a virtual displacement w is compatible with the constraint frozen at time t? Let us first assume that C is an affine subbundle of TQ or, more generally, an affine subbundle of TQ_1, where Q_1 is a submanifold of Q. In that case, the answer to that question is clear. As seen in Section 2.1, the constraint submanifold is an affine subbundle of TQ when the constraint is obtained by imposing that a component of the system rolls without sliding on a moving external object, the motion of that external object being stationary (the reader may think of a ball rolling on a rotating horizontal disk, the angular velocity of the disk being constant). To freeze the constraint at time t means clearly to consider that the motion of the external moving part is stopped (or, rather, that the time is stopped, the parameter of which the infinitesimal virtual displacement depends being an abstract parameter other than the time). Therefore, an infinitesimal virtual displacement $w \in T_{c(t)}Q$ will be said to be *compatible with the constraint frozen at time t* if w lies in the *vector* subbundle $\overrightarrow{C}$ associated to the affine subbundle C. Observe that w *is not*, in general, an element of the constraint submanifold C! Observe also that the set of infinitesimal virtual displacements compatible with the constraint frozen at time t is a vector subspace $\overrightarrow{C}_{c(t)}$ of $T_{c(t)}Q$ which does not depend on the motion c under consideration, nor on the kinematic state $dc(t)/dt$ of the system at time t; it depends only on the configuration $c(t)$ of the system at time t. When the motion under consideration c runs over all possible motions, and when the time t takes all possible values, we obtain, as set of all possible infinitesimal virtual displacements compatible with the constraint frozen at a fixed time the vector subbundle $\overrightarrow{C}$ of TQ (or of TQ_1, if C is an affine subbundle not of TQ, but of the tangent bundle to some submanifold Q_1 of Q).

When C is an arbitrary submanifold of TQ, what one should call an infinitesimal virtual displacement compatible with the constraint frozen at time t is not so clear. Appell [1, 2] and Delassus [19] have used (more or less implicitly) the following definition, made precise more recently by Chetaev [16] and widely used in many recent works [9, 18, 20, 39, 40, 53]. That definition amounts to "linearizing" C around the point $dc(t)/dt$. Let $C_{c(t)} = T_{c(t)}Q \cap C$. It is a subset of the vector space $T_{c(t)}Q$, and it contains the point $dc(t)/dt$. Let us assume that it is a submanifold of $T_{c(t)}Q$. Since $T_{c(t)}Q$ is a vector space, the tangent space to that submanifold at $dc(t)/dt$, denoted by $T_{dc(t)/dt}C_{c(t)}$, is a vector subspace of $T_{c(t)}$Q. For Chetaev, the virtual displacement w is said to be *compatible with the constraint frozen at time t* if $w \in T_{dc(t)/dt}C_{c(t)}$. We will say that these virtual infinitesimal displacements are *compatible with the constraint in the sense of Chetaev*. Observe that now the set of virtual infinitesimal displacements compatible with the constraint in the sense of Chetaev at time t is a vector subspace of $T_{c(t)}Q$ which depends not only on the configuration $c(t)$ of the system, but also on the kinematic state $dc(t)/dt$ of the system at time t. Therefore, the set of all possible virtual infinitesimal displacements compatible with the constraint in the sense of Chetaev, for all possible motions and all values of the time, is no more a vector subbundle of TQ, nor of TQ_1 where Q_1 is a submanifold of Q. Rather, it is a vector bundle over the base C, called by some authors [5, 28] the *Chetaev bundle*.

Now what is the virtual work of the constraint force $f_c(t)$ for a virtual infinitesimal displacement w at time t? The answer is clear, since $f_c(t) \in T^*_{c(t)}Q$ and $w \in T_{c(t)}Q$: it is simply the pairing $\langle f_c(t), w \rangle$. Clearly, that virtual work vanishes for any w in some vector subspace of $T_{c(t)}Q$ if and only if $f_c(t)$ lies in the annihilator of that vector subspace.

Finally, we may state the following definition.

DEFINITION 1. We consider a mechanical system whose configuration space is a smooth n-dimensional manifold Q, with a kinematic constraint for which the set of admissible kinematic states is a smooth submanifold C of TQ.

1. The constraint is said to be *linear* (resp. *affine*) in the velocities if C is a vector subbundle (resp. an affine subbundle) of a tangent bundle TQ_1, where Q_1 is a submanifold of Q (which may be equal to Q).

2. The constraint is said to be *ideal* if it is affine in the velocities and such that the constraint force takes its values in the annihilator $\overrightarrow{C}^0$ (in $T^*_{Q_1}Q$) of the vector subbundle $\overrightarrow{C}$ of TQ_1 associated with the affine subbundle C.

3. The constraint is said to be *of Chetaev type* if for each $x \in TQ$, $C_x = T_xQ \cap C$ is a submanifold of T_xQ and if for each $v \in C_x$, the constraint force when the system's kinematic state is v takes its value in the annihilator $(T_vC_x)^0$ (in T^*_xQ) of the vector subspace T_vC_x of T_xQ.

REMARK 3. A particular type of kinematic constraint linear in the velocities is when $C = TQ_1$, where Q_1 is a submanifold of Q. If in addition the constraint is ideal, the constraint force takes its values in the annihilator of TQ_1 (in the cotangent bundle $T^*_{Q_1}Q$ restricted to the submanifold Q_1). By replacing by Q_1 the original configuration space Q, one obtains an ordinary Lagrangian system without kinematic constraint, whose Lagrangian is the restriction of L to TQ_1 (considered as a subset of TQ).

REMARK 4. We have avoided calling *ideal* a constraint of Chetaev type, because one may wonder whether constraints of that type are encountered in real mechanical systems, even with some idealization. P. Appell [2] has described a mechanical system with a kinematic constraint not linear, nor affine in the velocities, obtained by a combination of a wheel rolling without sliding on a horizontal plane, of a thread wound around that wheel with a heavy material point tied at one of its ends, and several other components. It is known that if one assumes that this constraint is of Chetaev type, one does not obtain the correct equations of motion for the system [19, 46]. As for kinematic constraints obtained by means of servomechanisms, they are generally not of Chetaev type [34].

2.4. *The equations of motion in the Lagrangian formalism.* We will use in what follows a description of the kinematic constraint which contains, as particular cases, both ideal constraints and constraints of Chetaev type. Our description may also be suitable for more general constraints, such as some of those obtained by means of servomechanisms. As shown by P. Dazord [18] and the author [34], one should use, for the description of such constraints, two separate ingredients:

(i) the set C of admissible kinematic states (which may be assumed to be a smooth submanifold of TQ, at least when the constraint does not depend on time),

(ii) for each $v \in C$, whose projection on Q is denoted by x, a subset U_v of T_x^*Q, which will be the set of possible values of the constraint force when the kinematic state of the system is v. In what follows, that subset will be assumed to be a vector subspace of T_x^*Q. We will denote by $A_v \subset T_xQ$ its kernel, *i.e.* the set of all $w \in T_xQ$ such that $\langle \xi, v \rangle = 0$ for all $\xi \in U_v$. In other words, U_v is the annihilator A_v^0 of A_v.

Let $v \in C$ be an admissible kinematic state, and $x = \tau_Q(v) \in Q$ the corresponding configuration of the system (we have denoted by $\tau_Q : TQ \to Q$ the canonical projection). For an ideal constraint, $A_v = \overrightarrow{C}_x$, the vector subspace of T_xQ, associated to the affine subspace C_x, and $U_v = \overrightarrow{C}_x^0$ is its annihilator. For a constraint of Chetaev type, $A_v = T_vC_x$, considered as a vector subspace of T_xQ, where $C_x = T_xQ \cap C$ is assumed to be a submanifold of the vector space T_xQ, and U_v is its annihilator $(T_vC_x)^0$. We see that in both these cases, U_v is determined by C; therefore all the properties of the constraint are known when C is known.

For more general types of constraints encountered in real mechanical systems, the set C of admissible kinematic states and the set U_v of possible values of the constraint force when the system's kinematic state is v, are given independently of each other.

With such a description, the equations of motion of the system are:

$$
\begin{cases} \Delta(L)(j^2c(t)) = f_c(t), \\ \dfrac{dc(t)}{dt} \in C, \\ f_c(t) \in U_{dc(t)/dt}. \end{cases} \tag{1}
$$

The reader will easily see that for an ideal constraint, these equations become the well known Lagrange equations with multipliers [15].

2.5. *Another interpretation of the equations of motion.* The space $J^2((\mathbf{R},0),Q)$ of jets of order 2, at the origin 0 of $\mathbf{R}$, of smooth parametrized curves in Q can be canonically identified with the subset of the second tangent bundle $T(TQ)$ formed by elements $\xi \in T(TQ)$ which satisfy

$$\tau_{TQ}(\xi) = T\tau_Q(\xi),$$

where we have denoted by $\tau_{TQ} : T(TQ) \to TQ$ and $\tau_Q : TQ \to Q$ the canonical tangent bundle projections, and by $T\tau_Q : T(TQ) \to TQ$ the canonical lift to vectors of the map τ_Q.

On the other hand, for any $v \in TQ$, with $\tau_Q(v) = x \in Q$, the cotangent space T^*_xQ can be canonically identified with the subspace of $T^*_v(TQ)$ made by semi-basic covectors, *i.e.* covectors $\eta \in T^*_v(TQ)$ which vanish on the kernel of the map $T_v\tau_Q : T_v(TQ) \to T_xQ$.

By using these two canonical identifications, we can consider the Lagrange differential as a map, defined on the affine subbundle of $T(TQ)$:

$$\left\{\, \xi \in T(TQ) \;\middle|\; \tau_{TQ}(\xi) = T\tau_Q(\xi) \,\right\},$$

with values to the set of semi-basic elements in $T^*(TQ)$, and fibered over TQ, *i.e.* which maps each $\xi \in T(TQ)$ satisfying $\tau_{TQ}(\xi) = T\tau_Q(\xi) = v$, in the set of semi-basic covectors in $T^*_v(TQ)$.

When the Lagrange differential is considered in that way, the constraint force $f_c(t)$ at time t, which appears in the equations of motion (1) must be considered as a semi-basic covector in $T^*_{dc(t)/dt}(TQ)$.

In what follows, we will assume that for each $v \in C$, U_v is the fibre at v of a vector subbundle U of $T^*_C(TQ)$, contained in the subbundle of semi-basic covectors.

Moreover, we observe that the dual of the subbundle of semi-basic covectors of $T^*(TQ)$ is the quotient bundle $T(TQ)/\ker T\tau_Q$. Therefore, the vector bundle U of possible values of the constraint force is the annihilator of a vector subbundle A of the restriction to C of the quotient bundle $T(TQ)/\ker T\tau_Q$.

2.6. *The Legendre transformation.* The Lagrangian $L : TQ \to \mathbf{R}$ determines a smooth map $\mathcal{L} : TQ \to T^*Q$ called the *Legendre transformation*. Let us recall that in a chart of Q with local coordinates $(x^1,\dots,x^n)$ and the associated charts of TQ and T^*Q, whose coordinates are denoted by $(x^1,\dots,x^n,v^1,\dots,v^n)$ and $(x^1,\dots,x^n,p_1,\dots,p_n)$, respectively, the Legendre transformation is given by

$$\mathcal{L} : (x^1,\dots,x^n,v^1,\dots,v^n) \mapsto \left(x^1,\dots,x^n,\, p_1 = \frac{\partial L(x,v)}{\partial v^1},\dots,p_n = \frac{\partial L(x,v)}{\partial v^n}\right).$$

The Lagrangian is said to be *regular* (or *hyper-regular* in the terminology of C. Godbillon [21]) when the Legendre transformation is a smooth diffeomorphism of TQ onto T^*Q (or, more generally, onto an open subset of T^*Q).

W. Tulczyjew and his co-workers [50, 37] have made a thorough study of the Legendre transformation for nonregular Lagrangians, and shown that in the Hamiltonian formalism, the motion of the mechanical system is then described by an *implicit* differential equation. Inspired by his work, Barone, Grassini and Mendella [4, 5] have proposed a very general setting for mechanical systems with a nonregular Lagrangian and a kinematic constraint of very general type.

When the Lagrangian L is regular, there exists a smooth function $H : T^*Q \to \mathbf{R}$, called the *Hamiltonian* of the system, given, in the above defined local coordinates, by

$$H(x,p) = \sum_{i=1}^{n} p_i v^i - L(x,v),$$

where $(x,p) = (x^1,\dots,x^n,p_1,\dots,p_n)$ and $(x,v) = (x^1,\dots,x^n,v^1,\dots,v^n) = \mathcal{L}^{-1}(x,p)$.

Let us recall that on the cotangent bundle T^*Q, there is a canonical 1-form θ, called the *Liouville* 1-form, whose expression in local coordinates is

$$\theta = \sum_{i=1}^{n} p_i \, dx^i.$$

Its differential,

$$d\theta = \sum_{i=1}^{n} dp_i \wedge dx^i$$

is the canonical symplectic form of the cotangent bundle. We will denote by Λ_{T^*Q} the corresponding Poisson tensor, given in local coordinates by

$$\Lambda_{T^*Q} = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i}.$$

The transition from the Lagrangian formalism to the Hamiltonian formalism rests on the following well known properties.

Proposition 1. *Let $L : TQ \to \mathbf{R}$ be a smooth Lagrangian, $\mathcal{L} : TQ \to T^*Q$ be the corresponding Legendre transformation, and $E : TQ \to \mathbf{R}$ be the function*

$$E(v) = \bigl\langle \mathcal{L}(v), v \bigr\rangle - L(v).$$

We denote by θ the Liouville 1-form on T^*Q, by $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ the canonical projections of the tangent and cotangent bundles, respectively. Let $x \in Q$, $v \in T_xQ$, $p = \mathcal{L}(v) \in T^*_xQ$. Then:

(i). *For each $\xi \in T_v(TQ)$ which satisfies $T\tau_Q(\xi) = \tau_{TQ}(\xi) = v$, we have*

$$\Delta(L)(\xi) = (i(\xi)(\mathcal{L}^* \, d\theta) + dE)(v).$$

(ii). *Let $f \in T^*_v(TQ)$ be a semi-basic covector, also considered as an element in T^*_xQ. We denote by $\lambda(p,f)$ the vector tangent at p to the fibre T^*_xQ, and equal to f (since T^*_xQ is a vector space, a vector tangent to that space at any point can be considered as an element of that space). Then*

$$\mathcal{L}^*(i(\lambda(p,f))d\theta) = f.$$

(iii). *The tangent linear map $T_v\mathcal{L}$ maps the set of elements $\xi \in T_v(TQ)$ which satisfy $T\tau_Q(\xi) = v$ into the set of elements $\eta \in T_p(T^*Q)$ which satisfy $T\pi_Q(\eta) = v$. If in addition L is regular, that map is a bijection.*

(iv). *We assume now that L is regular. Let $H : T^*Q \to \mathbf{R}$ be the corresponding Hamiltonian (one has then $E = H \circ \mathcal{L}$). Let $v \in TQ$, $x = \tau_Q(v) \in Q$ and $p = \mathcal{L}(v) \in T^*Q$. The differential at p of the Hamiltonian H restricted to the fibre $\pi_Q^{-1}(x)$ is v.*

2.7. *The constraint submanifold and the projection bundle.* Since in the present paper we are interested mostly in symmetries, we will, for simplicity, assume in what follows that the Lagrangian L is regular, *i.e.* that the Legendre transformation $\mathcal{L}$ is a diffeomorphism. The image $D = \mathcal{L}(C)$ is then a submanifold of T^*Q, called the *constraint submanifold* (in the Lagrangian formalism).

We define the *projection bundle* W as the vector subbundle of $T_D(T^*Q)$ whose fibre at each point $p \in D$ is

$$W_p = \left\{ \lambda(p, f) \;\middle|\; f \in U_{\mathcal{L}^{-1}(p)} \right\},$$

where $\lambda : T^*Q \times_Q T^*Q \to T(T^*Q)$ is the map defined in part (ii) of Proposition 1, section 2.6.

We see that the projection bundle is contained in the vertical subbundle of $T_D(T^*Q)$, *i.e.* in the kernel of $T_D\pi_Q$.

The projection bundle can be defined in another, equivalent way, given in the following proposition.

PROPOSITION 2. *Let $p \in D$, $x = \pi_Q(p) \in Q$, $v = \mathcal{L}^{-1}(p) \in T_xQ$. Let*

$$A_v = \left\{ w \in T_xQ \;\middle|\; \langle f, w\rangle = 0 \text{ for all } f \in U_v \right\}.$$

In other words, A_v is the vector subspace of T_xQ whose annihilator is U_v defined in section 2.4. Then the fibre W_p of the projection bundle at p is

$$W_p = \operatorname{orth}((T_p\pi_Q)^{-1}(A_v)),$$

where orth *denotes the orthogonal complement with respect to the symplectic form $d\theta$.*

When the kinematic constraint is ideal, $A_v = \overrightarrow{C}_x$, and

$$W_p = \operatorname{orth}((T_p\pi_Q)^{-1}(\overrightarrow{C}_x)).$$

Proof. Let $\zeta \in T_p(T^*Q)$. Any $\eta \in W_p$ can be written as $\eta = \lambda(p, f)$, with $f \in U_v$. Using property (ii) of Proposition 1, section 2.6, we have

$$d\theta(\eta, \zeta) = d\theta(\lambda(p, f), \zeta) = \bigl\langle f, T\mathcal{L}^{-1}(\zeta)\bigr\rangle.$$

On the right hand side, f is considered as a semi-basic covector at v. By looking at f as a covector at x, we may also write

$$d\theta(\eta, \zeta) = \bigl\langle f, T\tau_Q \circ T\mathcal{L}^{-1}(\zeta)\bigr\rangle = \bigl\langle f, T\pi_Q(\zeta)\bigr\rangle,$$

since $\tau_Q \circ \mathcal{L}^{-1} = \pi_Q$, therefore $T\tau_Q \circ T\mathcal{L}^{-1} = T\pi_Q$. The result follows. ∎

2.8. *The equations of motion in the Hamiltonian formalism.* Let $t \mapsto c(t)$ be a smooth curve in Q. Let

$$t \mapsto \widehat{c}(t) = \mathcal{L} \circ \frac{dc(t)}{dt}$$

be the corresponding parametrized curve in T^*Q. Using Proposition 1 of Section 2.6, we see that c satisfies the equations of motion in the Lagrangian formalism (1) if and only if $\widehat{c}$ satisfies the equations

$$(2)\qquad \begin{cases} i\left(\dfrac{d\widehat{c}(t)}{dt}\right) d\theta + dH(\widehat{c}(t)) = i(f_D(t))d\theta, \\ \widehat{c}(t) \in D, \\ f_D(t) \in W_{\widehat{c}(t)}. \end{cases}$$

In these equations, $f_D(t)$ is an element in $T_{\widehat{c}(t)}(T^*Q)$, called the *constraint force in the Hamiltonian formalism*. It is related to the constraint force $f_c(t)$ in the Lagrangian formalism by

$$f_D(t) = \lambda(\widehat{c}(t), f_c(t)).$$

Conversely, if $t \mapsto \widehat{c}(t)$ is a smooth parametrized curve in T^*Q which satisfies the equations of motion in the Hamiltonian formalism, there exists a unique smooth parametrized curve $t \mapsto c(t)$ in Q such that $\widehat{c}(t) = \mathcal{L} \circ dc(t)/dt$; this results easily from property (iii) of Proposition 1, section 2.6.

Let X_H be the Hamiltonian vector field on T^*Q associated with H, defined by

$$i(X_H)d\theta = -dH, \quad \text{or} \quad X_H = \Lambda^\sharp_{T^*Q}(dH).$$

We have denoted by $\Lambda^\sharp_{T^*Q} : T^*Q \to TQ$ the vector bundle morphism related to the Poisson tensor Λ_{T^*Q} by $\Lambda_{T^*Q}(\alpha, \beta) = \langle \beta, \Lambda^\sharp_{T^*Q}(\alpha) \rangle$. Equations (2) are clearly equivalent to the equations

$$(3)\qquad \begin{cases} \dfrac{d\widehat{c}(t)}{dt} = X_H(\widehat{c}(t)) + f_D(t), \\ \widehat{c}(t) \in D, \\ f_D(t) \in W_{\widehat{c}(t)}. \end{cases}$$

Equations (2) or (3) are the equations of motion in the Hamiltonian formalism.

Definition 2. The mechanical system is said to be *regular* if it satisfies the two conditions:

(i) The intersection of the vector subbundles W and TD of the bundle $T_D(T^*Q)$ is the zero bundle,

(ii) The Hamiltonian vector field X_H, restricted to the submanifold D, is a section of the direct sum $TD \oplus W$.

When the system is regular, the equations of motion (2) or (3) are well behaved: the Hamiltonian vector field X_H, restricted to the submanifold D, splits, in a unique way, into a sum

$$X_H\big|_D = X_D + X_W,$$

where X_D is a vector field on D, and X_W a section of the projection bundle W. The equations of motion (3) become

$$(4)\qquad \begin{cases} \dfrac{d\widehat{c}(t)}{dt} = X_D(\widehat{c}(t)), \\ f_D(t) = -X_W(\widehat{c}(t)). \end{cases}$$

In that system, the first equation is a smooth, autonomous differential equation on the manifold D, while the second equation gives, at each time t, the value of the constraint force f_D in the Hamiltonian formalism.

Under very mild assumptions about the Lagrangian, mechanical systems with an ideal kinematic constraint, or with a constraint of Chetaev type, are regular (see for example [34]).

Remark 5. For a regular Hamiltonian system with constraint, the Poisson tensor Λ_{T^*Q} can be projected onto D, and yields a pseudo-Poisson tensor Λ_D on that submanifold. Van der Schaft and Maschke [52] have shown that for an ideal constraint linear in the velocities, Λ_D is a Poisson tensor if and only if the constraint is holonomic. That result is closely related to a property of distributions on a symplectic manifold due to P. Libermann [29]. We have discussed the properties of Λ_D in [35, 36]. In [35], we have written, incorrectly, that the evolution vector field X_D is simply equal to $\Lambda_D^\sharp(dH|_D)$. This is true when the Hamiltonian H is a constant of motion, but not in general, as Robert McLachlan [41] pointed out to us. The additional terms which appear in X_D are discussed by Cantrijn and his coworkers in [12, 13].

3. Symmetries and constants of motion

3.1. *Conservation of energy.* When there is no kinematic constraint, the Hamiltonian H is a constant of motion (*i.e.* for each motion $t \mapsto c(t)$ of the system, $t \mapsto H(\widehat{c}(t))$ is a constant). This is no longer true when there is a kinematic constraint. The following proposition indicates, for a system with a kinematic constraint, conditions under which the Hamiltonian H is still a constant of motion.

Proposition 3. *Under the assumptions of Section 2.7, the following two properties are equivalent:*

(i) *for each $p \in D$, $W_p \subset \ker dH(p)$,*
(ii) *each $v \in C$ is an element of the vector subspace A_v of T_xQ defined in Proposition 2, where $x = \tau_Q(v)$.*

When these two equivalent properties are satisfied, the Hamiltonian H is a constant of motion.

Proof. As seen in 2.7, for each $p \in D$, W_p is contained in $\ker T_p\pi_Q$. But property (iv) of Proposition 1 shows that $dH(p)$, restricted to $\ker T_p\pi_Q$, is equal to v. The equivalence of properties (i) and (ii) follows immediately.

Using Equations (3) we obtain

$$\frac{d}{dt}H(\widehat{c}(t)) = \big\langle dH(\widehat{c}(t)), f_D(t)\big\rangle,$$

since $\big\langle dH(\widehat{c}(t)), X_H(\widehat{c}(t))\big\rangle = 0$. Since $f_D(t) \in W_{\widehat{c}(t)}$, we see that when the equivalent properties (i) and (ii) are satisfied, H is a constant of motion. ∎

Remark 6. When the kinematic constraint is affine in the velocities and ideal, the equivalent conditions (i) and (ii) of Proposition 2 are satisfied if and only if the constraint is *linear* in the velocities.

REMARK 7. When the kinematic constraint is of Chetaev type, these two equivalent conditions are satisfied if and only if the constraint submanifold C is, at each of its points, tangent to the Liouville vector field of TQ. We recall that the Liouville vector field of TQ is the vector field whose value, at each $v \in TQ$, is the vector tangent at v to the fibre T_xQ (with $x = \tau_Q(v)$), and equal to v.

3.2. *Symmetries and reduction.* We assume that a Lie group G acts on the left, by a Hamiltonian action Φ, on the symplectic phase space $(T^*Q, d\theta)$. We denote by $\mathcal{G}$ the Lie algebra of G, by $\mathcal{G}^*$ its dual and by $J : T^*Q \to \mathcal{G}^*$ the momentum map of that action.

When the Hamiltonian H is invariant under the action Φ, and when there is no kinematic constraint, the well known reduction theorem due to Marsden and Weinstein [38], together with Noether's theorem [43], allows us to reduce the system. Noether's theorem tells us first that the momentum map J is a constant of motion. Then, if $\mu \in \mathcal{G}^*$ is a regular value (or, more generally, a weakly regular value) of J, $J^{-1}(\mu)$ is a submanifold of T^*Q, invariant under the restriction of the action Φ to a subgroup G_μ of G. The subgroup G_μ is the stabilizer of μ, for the affine action of G on $\mathcal{G}^*$ for which the momentum map J is equivariant. Under some regularity assumptions, the set of orbits $P_\mu = J^{-1}(\mu)/G_\mu$ is endowed with a reduced symplectic 2-form Ω_μ. The Hamiltonian H, restricted to $J^{-1}(\mu)$, induces a reduced Hamiltonian H_μ on P_μ. A first step in the study of solutions of the Hamiltonian system $(T^*Q, d\theta, H)$ contained in $J^{-1}(\mu)$ is the study of their projections on the reduced phase space P_μ, which are simply the solutions of the reduced Hamiltonian system $(P_\mu, \Omega_\mu, H_\mu)$.

For systems with a kinematic constraint, the momentum map is no more a constant of motion. Therefore it is no more possible to restrict to a level set of the momentum map, as in the reduction theorem of Marsden and Weinstein. Many authors have used direct Poisson reduction of the phase space and derived evolution equations for the momentum map, to obtain generalizations of Noether's theorem [10, 12, 25, 26, 44, 47]. Since Poisson reduction may produce singularities, several authors have used singular reduction [8, 17], or used a different kind of reduction founded on differential forms defined on subbundles of the tangent bundle [6, 7, 48]. For properties of Poisson manifolds, we refer to [31].

We would like to present here an idea which seems to us new (although a very similar idea is present in the work of Bates, Grauman and MacDonnell [7]; however these authors do not use Poisson structures in their reduction procedure, which differs slightly from ours). That idea is that when there is a kinematic constraint, two different types of group actions should be considered, because the conditions under which the momentum map is a constant of motion, and those under which the phase space can be reduced by using the symmetries, are not the same. Moreover, an action of the first type may sometimes be deformed into an action of the second type, maybe after restriction to a subgroup.

We must first extend slightly the framework of Hamiltonian systems with a constraint, so that it will allow reduction by symmetries [33].

DEFINITION 3. We will say that (P, Λ, H, D, W) is a *Hamiltonian system with constraint* when (P, Λ) is a Poisson manifold called the *phase space* of the system, $H : P \to \mathbf{R}$ is a smooth function called the *Hamiltonian*, D is a submanifold of P called the *con-*

straint submanifold and W a vector subbundle of $T_D P$. We will say that (P, Λ, H, D, W) is *regular* when the two subbundles TD and W of $T_D P$ are such that

(i) $TD \cap W = \{0\}$,
(ii) the Hamiltonian vector field $\Lambda^\sharp(dH)$, with the notations of section 2.8, restricted to the submanifold D, is a section of $TD \oplus W$.

When these conditions are satisfied, the vector field $X_H|_D$ splits into a sum

$$X_H|_D = X_D + X_W,$$

where X_D is a vector field on D called the *evolution vector field* of the system, and X_W is a section of W; its opposite, $-X_W$, is called the *constraint force*.

PROPOSITION 4. *Let (P, Λ, H, D, W) be a regular Hamiltonian system with constraint, and $\Phi : G \times P \to P$ an action of a Lie group G on the manifold P, which satisfies the following conditions:*

(i) *the action Φ is a Poisson action, i.e. it preserves the Poisson tensor Λ,*

(ii) *the action Φ leaves the Hamiltonian H invariant,*

(iii) *the constraint submanifold D and the vector subbundle W remain invariant under the action Φ, i.e. for each $p \in D$ and $g \in G$, $\Phi_g(p) \in D$ and $T_p\Phi_g(W_p) = W_{\Phi_g(p)}$.*

Moreover, we assume that the set $\widehat{P} = P/G$ of orbits of the action Φ is a smooth manifold and that the projection $\pi : P \to \widehat{P}$ is a submersion. Then there exists on $\widehat{P}$ a unique Poisson tensor Λ such that $\pi : P \to \widehat{P}$ is a Poisson map. Let $\widehat{H} : \widehat{P} \to \mathbf{R}$ be such that $H = \widehat{H} \circ \pi$. Let $\widehat{D} = \pi(D)$ and, for each $p \in D$, let $\widehat{p} = \pi(p)$, $\widehat{W_{\widehat{p}}} = T_p\pi(W_p)$. Then $\widehat{D}$ is a submanifold (maybe with multiple points) of $\widehat{P}$, and $(\widehat{P}, \widehat{\Lambda}, \widehat{H}, \widehat{D}, \widehat{W})$ is a regular Hamiltonian system with constraint, called the reduced system. The projection π maps each integral curve $t \mapsto p(t)$ of the evolution vector field X_D of the initial system, onto an integral curve $t \mapsto \widehat{p}(t) = \pi(p(t))$ of the evolution vector field $X_{\widehat{D}}$ of the reduced system.

The proof is similar to that given in [33] for systems with a constraint of Chetaev type.

The above proposition allows the reduction by symmetries of a Hamiltonian system with constraint. But contrary to what occurs in the reduction theorem of Marsden and Weinstein, the momentum map J cannot be used to reduce further to submanifolds on which J keeps a constant value, since it is not a constant of motion. The next proposition indicates conditions in which a group action produces an integral of motion.

PROPOSITION 5. *Let (P, Λ, H, D, W) be a regular Hamiltonian system with constraint, and $\Phi : G \times P \to P$ an action of a Lie group G on the manifold P, which satisfies the following conditions:*

(i) *the action Φ is a Hamiltonian action, i.e. it admits a momentum map $J : P \to \mathcal{G}^*$ such that, for each $X \in \mathcal{G}$, the fundamental vector field on P associated with X is the Hamiltonian vector field $\Lambda^\sharp(d\langle J, X\rangle)$,*

(ii) *the action Φ leaves the Hamiltonian H invariant,*

(iii) *for each $p \in D$, the vector subspace W_p of T_pP is contained in the kernel of T_pJ.*

Then the momentum map J, restricted to the submanifold D, is constant on each integral curve of the evolution vector field X_D.

Proof. By Noether's theorem, the value $X_H(p)$ of the Hamiltonian vector field X_H, at each point $p \in D$, is in the kernel of T_pJ. Since $X_H(p) = X_D(p) + X_W(p)$, and since $X_W(p) \in W_p$, we see that $X_D(p)$ is in the kernel of T_xJ. The result follows immediately. ∎

REMARK 8. In Proposition 5, we do not need to assume that the submanifold D and the projection bundle W are invariant under the action Φ.

REMARK 9. The assumptions being those of Proposition 5, we assume in addition that the Poisson structure of the phase space P is associated to a symplectic structure (this happens, for example, when P is a cotangent bundle T^*Q equipped with its canonical symplectic 2-form $d\theta$). Then condition (iii) of Proposition 5 is equivalent to the following condition:

(iii bis) *For each $p \in D$, the tangent space at p to the G-orbit $\Phi(G,p)$ is contained in* $\operatorname{orth} W_p$.

That follows easily from the fact that the kernel of T_pJ is the symplectic orthogonal of the tangent space at p to the G-orbit $\Phi(G,p)$ (see for example [30], proposition 5.5 page 215).

Let us assume in addition that P is a cotangent bundle T^*Q equipped with its canonical symplectic 2-form $d\theta$, that the Hamiltonian H on T^*Q comes from a regular Lagrangian L on TQ, that $D = \mathcal{L}(C)$ (as in sections 1 and 2) and that the action Φ is the canonical lift to the cotangent bundle of an action Φ_Q of the Lie group G on the configuration space Q. Proposition 2 shows that

$$W_p = \operatorname{orth}((T_p\pi_Q)^{-1}(A_v)),$$

where $v = \mathcal{L}^{-1}(p) \in TQ$. Since the projection on Q of the tangent space at p to the G-orbit $\Phi(G,p)$ is the tangent space at $x = \pi_Q(p)$ to the G-orbit $\Phi_Q(G,x)$, we see that condition (iii) of Proposition 5 is equivalent to the following condition:

(iii ter) *For each $v \in C$, the tangent space at $x = \tau_Q(v)$ to the G-orbit $\Phi_Q(G,x)$ is contained in the vector subspace A_v of T_xQ defined in Proposition 2.*

Under that form, condition (iii) has a clear meaning: any possible values of the constraint force must vanish when coupled with any vector tangent to a G-orbit $\Phi_Q(G,x)$.

As observed by Bates, Grauman and MacDonnell [7], in several examples, a group action satisfying the assumptions of Proposition 4 can be deformed to produce, maybe after reduction to a subgroup, another group action which satisfies the assumptions of Proposition 5. This fact will be illustrated in the next section.

4. Example

4.1. *The configuration space, the phase space and the Hamiltonian.* We consider a spherical ball of radius r which rolls without sliding on the inner surface of a circular cylinder of radius R. The symmetry axis of that cylinder is vertical. We denote by m the mass of the ball, by I its moment of inertia with respect to an axis through its centre.

The configuration space of the system is $\mathbf{R}\times S^1\times G$, with $G = SO(E)$, where E is the Euclidean three-dimensional space. A point in the configuration space is a triple (z,θ,g), where z is the height of the centre of the ball over some reference horizontal plane, θ is the angle made by a fixed horizontal oriented line chosen as reference with the horizontal oriented line which joins the centre of the ball to its point of contact with the cylinder, and g is the element in the rotation group $SO(E)$ which maps a given orientation of the ball, chosen as reference, onto its actual orientation.

In the Hamiltonian formalism, the phase space is $\mathbf{R}\times\mathbf{R}^*\times S^1\times\mathbf{R}^*\times T^*G$, where $\mathbf{R}^*$ denotes the dual of the real line $\mathbf{R}$ (which may be identified with $\mathbf{R}$). A point in phase space will be a multiplet $(z,p_z,\theta,p_\theta,(g,p_g))$, where $g\in G = SO(E)$ and $p_g\in T^*_gG$.

The Hamiltonian of the system is

$$H = \frac{1}{2m}\left(\frac{1}{(R-r)^2}\,p_\theta^2 + p_z^2\right) + \frac{1}{2I}\,|p_g|^2 + m\gamma z,$$

where γ is the gravity acceleration. We have denoted by $|p_g|$ the length of the covector $p_g\in T^*_gG$ for the canonical Euclidean structure of that vector space.

4.2. *The constraint submanifold.* The constraint submanifold in the Hamiltonian formalism is the submanifold D of the phase space made by elements $(z,p_z,\theta,p_\theta,g,p_g)$ which satisfy the two equations

$$\begin{cases} \dfrac{1}{m(R-r)}\,p_\theta + \dfrac{r}{I}\,M_z = 0,\\[2mm] \qquad\qquad \dfrac{1}{m}\,p_z - \dfrac{r}{I}M_\theta = 0. \end{cases}$$

In these equations, we have set $M = \widehat{R}_{g^{-1}}p_g$, where $\widehat{R}_{g^{-1}} : T^*G\to T^*G$ denotes the canonical lift to the cotangent bundle of the right translation $R_{g^{-1}} : G\to G$ which maps $h\in G$ onto hg^{-1}. Observe that M is an element of the cotangent space T^*_eG at the unit element e of G; that space will be identified with the dual of the Lie algebra $\mathcal{G}$ of G.

Let us now explain the meaning of M_θ and M_z which appear in these equations. Once an orientation and a fixed orthonormal basis of positive orientation (e_1,e_2,e_3) of E are chosen, an element in $\mathcal{G}$ can be considered as a 3×3 skew-symmetric matrix, and may be identified with a vector in E, according to

$$\begin{pmatrix} 0 & -M_3 & M_2\\ M_3 & 0 & -M_1\\ -M_2 & M_1 & 0\end{pmatrix}\quad\text{identified with}\quad M_1e_1 + M_2e_2 + M_3e_3.$$

That identification does not depend on the choice of the orthonormal basis (e_1,e_2,e_3), as long as that basis is of positive orientation. Therefore, instead of the fixed orthonormal basis (e_1,e_2,e_3) of E, we may use a moving orthonormal basis (e_n,e_θ,e_z), where e_n and e_θ are horizontal, e_n being normal to the cylinder at the contact point of the ball and directed outwards, and e_θ being tangent to the cylinder and directed towards the positive trigonometric sense of rotation around the cylinder. The unit vector e_z is vertical and directed upwards. We will denote by M_n, M_θ and M_z the components of M in the basis (e_n,e_θ,e_z).

4.3. *The projection bundle.* The projection bundle W is the image, by the map λ defined in Proposition 1, of pairs of covectors $(z, p_z, \theta, p_\theta, g, p_g)$ and $(z, p'_z, \theta, p'_\theta, g, p'_g)$, attached to the same point (z, θ, g) of the configuration space, which satisfy the following conditions:

(i) the first covector $(z, p_z, \theta, p_\theta, g, p_g)$ is in D,
(ii) the second covector $(z, p'_z, \theta, p'_\theta, g, p'_g)$ satisfies

$$M'_\theta + r p'_z = 0 \quad \text{and} \quad M'_z - \frac{r}{R-r}\, p'_\theta = 0,$$

where M'_n, M'_θ and M'_z are the components of $M' = \widehat{R}_{g^{-1}} p'_g$ in the basis (e_n, e_θ, e_z), with the same identifications as those made for M.

4.4. *Two Lie group actions of the first type.* There are two Lie group actions which leave invariant the symplectic 2-form of the phase space, the Hamiltonian, the constraint submanifold D and the projection bundle W. These two actions commute and may be used, in any order, to reduce the phase space of the system. The first of these actions is an action of the group $G = SO(E)$, given by

$$(h, (z, p_z, \theta, p_\theta, g, p_g)) \mapsto (z, p_z, \theta, p_\theta, gh, \widehat{R}_h p_g).$$

The second is an action of the group S^1, given by

$$(\beta, (z, p_z, \theta, p_\theta, g, p_g)) \mapsto (z, p_z, \theta + \beta, p_\theta, g_\beta g, \widehat{L}_{g_\beta} p_g),$$

where $g_\beta \in G = SO(E)$ is the rotation of angle β around the vertical axis, and $\widehat{L}_{g_\beta}$ is the canonical lift to the cotangent bundle of the left translation $L_{g_\beta} : G \to G$ which maps $h \in G$ onto $g_\beta h$.

These two actions are Hamiltonian, but they do not satisfy condition (iii) of Proposition 5. Therefore, we cannot say whether their momentum maps are constants of motion.

After reduction by these two actions, the reduced phase space is $\mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^* \times \mathcal{G}^*$, with the coordinates (z, p_z, p_θ, M_0). The canonical projection of the inital phase space onto the reduced phase space is the map

$$(z, p_z, \theta, p_\theta, g, p_g) \mapsto (z, p_z, p_\theta, M_0 = \mathrm{Ad}^*_{g_\theta^{-1}} \circ \widehat{R}_{g^{-1}} p_g).$$

The reduced constraint submanifold $\widehat{D}$ is the set of elements (z, p_z, p_θ, M_0) in the reduced phase space which satisfy the two equations

$$\frac{1}{m(R-r)}\, p_\theta + \frac{r}{I}\, M_{0z} = 0, \quad \frac{1}{m}\, p_z - \frac{r}{I}\, M_{0y} = 0,$$

where M_{0x}, M_{0y} and M_{0z} are the components of M_0 in a fixed, orthonormal, positively oriented basis (e_x, e_y, e_z) of E, with e_x and e_y horizontal and e_z vertical directed upwards.

The reduced projection bundle $\widehat{W}$ is generated by the two vector fields along $\widehat{D}$:

$$\widehat{W}_1 = \frac{r}{R-r}\,\frac{\partial}{\partial M_{0z}} + \frac{\partial}{\partial p_\theta}, \quad \widehat{W}_2 = -r\,\frac{\partial}{\partial M_{0y}} + \frac{\partial}{\partial p_z}.$$

4.5. *A group action of the second type.* Let us consider the action of $\mathbf{R}$ on the phase space, given by

$$(\delta, (z, p_z, \theta, p_\theta, g, p_g)) \mapsto \left(z, p_z, \theta - \frac{r}{R-r}\,\delta, p_\theta, g_\delta g, \widehat{L}_{g_\delta} p_g \right).$$

We have denoted by $g_\delta \in G = SO(E)$ the rotation around the vertical axis of angle δ. Observe that $\delta \in \mathbf{R}$ is considered as an angle by taking its value modulo 2π.

That action is Hamiltonian, and has as a momentum map

$$J : (z, p_z, \theta, p_\theta, g, p_g) \mapsto -\frac{r}{R-r}\, p_\theta + M_{0z},$$

where M_{0z} is the component on e_z of $M_0 = \mathrm{Ad}^*_{g_\theta^{-1}} \circ \widehat{R}_{g^{-1}} p_g$.

This $\mathbf{R}$-action does not preserve the constraint submanifold D, nor the projection bundle W, but it satisfies the conditions for application of Proposition 5. Therefore its momentum map J is a constant of motion.

That $\mathbf{R}$-action has a clear physical meaning, in agreement with Remark 9: under that action, the ball rolls on the cylinder without sliding, around a horizontal circle of that cylinder.

So $-\dfrac{r}{R-r}\, p_\theta + M_{0z}$ is a constant of motion. But on $\widehat{D}$, we have $\dfrac{1}{m(R-r)}\, p_\theta + \dfrac{r}{I}\, M_{0z} = 0$. Therefore, we see that p_θ and M_{0z} are both constants of motion.

4.6. *The equations of motion in the reduced phase space.* Keeping in mind that p_θ and M_{0z} are constants of motion, the equations of motion for the other coordinates z, p_z M_{0x} and M_{0y} are

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{1}{m} p_z, \\ \dfrac{dp_z}{dt} = \dfrac{mr}{I}\, \dfrac{dM_{0y}}{dt}, \\ \dfrac{dM_{0x}}{dt} = \dfrac{1}{m(R-r)^2}\, \dfrac{Ip_\theta}{mr}\, p_z, \\ \dfrac{dM_{0y}}{dt} = -\dfrac{I}{m(R-r)^2(I+mr^2)}\, p_\theta M_{0x} - \dfrac{rI}{I+mr^2}\, m\gamma. \end{cases}$$

That system reduces to

$$\begin{cases} \dfrac{dp_z}{dt} = A + BM_{0x}, \\ \dfrac{dM_{0x}}{dt} = Cp_z, \end{cases}$$

where A, B and C are the constants

$$A = -\frac{m^2r^2\gamma}{I+mr^2}, \quad B = -\frac{rp_\theta}{(I+mr^2)(R-r)^2}, \quad C = \frac{Ip_\theta}{m^2r(R-r)^2}.$$

We observe that $BC < 0$. We obtain finally

$$\begin{cases} z = z_0 + \dfrac{p_{z0}}{m\sqrt{-BC}}\, \sin(\sqrt{-BC}(t-t_0)), \\ \theta = \theta_0 + \dfrac{p_\theta}{m(R-r)^2}\, (t-t_0). \end{cases}$$

Once the constant p_θ is given and z ant θ are known, p_z, M_{0x}, M_{0y} and the constant M_{0z} are given by

$$p_z = m\,\frac{dz}{dt}, \quad M_{0x} = \frac{1}{B}\left(\frac{dp_z}{dt} - A\right), \quad M_{0y} = \frac{I}{mr}\,p_z, \quad M_{0z} = -\frac{Ip_\theta}{mr(R-r)}.$$

We obtain the well known result [42]: all the motions are quasi-periodic.

A similar treatment may be made for the problem, solved by J. Hermans [22, 23], of a ball which rolls on a surface of revolution with a vertical axis; however in that problem the constants of motion cannot be obtained explicitly (they are solutions of differential equations).

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