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BASIC RELATIONS VALID FOR THE BERNSTEIN SPACES B_σ^2 AND THEIR EXTENSIONS TO LARGER FUNCTION SPACES VIA A UNIFIED DISTANCE CONCEPT

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Abstract. Some basic theorems and formulae (equations and inequalities) of several areas of mathematics that hold in Bernstein spaces B_σ^p are no longer valid in larger spaces. However, when a function f is in some sense close to a Bernstein space, then the corresponding relation holds with a remainder or error term. This paper presents a new, unified approach to these errors in terms of the distance of f from B_σ^p. The difficult situation of derivative-free error estimates is also covered.

This paper[1] is first concerned with several basic theorems which hold in Bernstein spaces B_σ^2, but are no longer valid in larger function spaces. These include the Whittaker–

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The paper is in final form and no version of it will be published elsewhere.

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Kotel'nikov–Shannon sampling theorem, the Valiron–Tschakaloff interpolation formula, Poisson's summation formula (special case), the general Parseval formula and the reproducing kernel formula.

Bernstein's inequality and Boas type formulae for higher order derivatives are treated as well.

Now if a function belongs to a space which is in some sense close to B^2_σ, such as Sobolev, Lipschitz, modulation or Hardy spaces, it is to be expected that the foregoing theorems are not violated drastically, but that their extensions to these spaces should be valid with a remainder which involves the distance of f in the wider space from B^2_σ.

The second part is devoted to the introduction of an appropriate metric for describing the distance between a function belonging to such a space from B^2_σ. A number of propositions and corollaries are presented involving estimates for this distance in case of the spaces mentioned. An innovation of this lecture are derivative free error estimates. Rates of convergence are also covered in detail.

In the third part this new unified theoretical approach is applied to estimate the remainders occurring in the extended versions of the theorems mentioned, including the approximate sampling theorem, the generalized Parseval decomposition formula, and the approximate reproducing kernel formula.

The four formulae for B^2_σ presented in Section 1 are all equivalent to each other in the sense that each is a corollary of the others. Likewise the six formulae of Section 2 for the space F^2, the largest space in which the Fourier transform, our basic tool, can be applied effectively, are also equivalent to each other. What is surprising is that all ten formulae are even equivalent. Proofs of the results not given here or in [7] will follow in later papers.

1. Motivation: Basic theorems for bandlimited functions. Let B^p_σ for $\sigma > 0$, $1 \le p \le \infty$, be the Bernstein space comprising all entire functions $f : \mathbb{C} \to \mathbb{C}$ that belong to $L^p(\mathbb{R})$ when restricted to the real axis and are as well of exponential type σ, so that they satisfy the inequality $f(z) = \mathcal{O}_f\big(\exp(\sigma\,|\mathfrak{Im}\,z|\big)$ as $|z| \to \infty$.

According to the Paley–Wiener theorem, the (distributional) Fourier transform of those functions has compact support contained in $[-\sigma, \sigma]$, where the Fourier transform is normalized by $\widehat{f}(v) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)e^{-ivu}\,du$.

There exist numerous relations (equations or inequalities) of the form

$$U(f) = V_\sigma(f) \quad \text{or} \quad U(f) \le V_\sigma(f) \qquad (f \in B^2_\sigma), \tag{1}$$

where U and V_σ are functionals; see [3]. Relations of this type are no longer valid outside B^2_σ. But if f belongs to a larger space, close to B^2_σ, then they should hold with an additional remainder $R_\sigma f$, such that

$$U(f) = V_\sigma(f) + R_\sigma f \quad \text{or} \quad U(f) \le V_\sigma(f) + R_\sigma f \tag{2}$$

with $R_\sigma f$ depending on the distance of f from B^2_σ.

An example of an inequality is Bernstein's inequality:

$$\|f^{(s)}\|_{L^2(\mathbb{R})} \le \sigma^s \|f\|_{L^2(\mathbb{R})} \qquad (f \in B^2_\sigma; s \in \mathbb{N}), \tag{3}$$

with $U(f) = \|f^{(s)}\|_{L^2(\mathbb{R})}$ and $V_\sigma(f) = \sigma^s \|f\|_{L^2(\mathbb{R})}$. If f belongs to the larger space, $f \in F^2$ (see below), and $v^s \widehat{f}(v) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then (3) holds only with an additional remainder $R_\sigma f = R^{\mathrm{Bern}}_{\sigma,s} f$, i.e., we have the approximate Bernstein inequality

$$\|f^{(s)}\|_{L^2(\mathbb{R})} \le \sigma^s \|f\|_{L^2(\mathbb{R})} + R^{\mathrm{Bern}}_{\sigma,s} f,$$

where the remainder can be estimated by

$$\left|R^{\mathrm{Bern}}_{\sigma,s} f\right| \le \left\{ \int_{|v|>\sigma} \left|v^s \widehat{f}(v)\right|^2 dv \right\}^{1/2}.$$

Now the integral on the right can be expressed in terms of the so-called distance functional $\mathrm{dist}_2(f^{(s)}, B^2_\sigma)$ (see (21)), measuring the distance of the function $f^{(s)} \in F^2$ from B^2_σ. Furthermore, this functional can be estimated in terms of the modulus of smoothness giving (see Proposition 4.2, Corollary 4.3),

$$\begin{aligned}\left|R^{\mathrm{Bern}}_{\sigma,s} f\right| \le \mathrm{dist}_2(f^{(s)}, B^2_\sigma) \le c\left\{ \int_\sigma^\infty v^{-1} \left[\omega_r(f^{(s)}, v^{-1}, L^2(\mathbb{R}))\right]^2 dv \right\}^{1/2} \\ = \mathcal{O}\left(\sigma^{-\alpha}\right) \qquad (\sigma \to \infty), \quad (4)\end{aligned}$$

the latter $\mathcal{O}$-estimate holding if additionally $f^{(s)} \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \le r$.

The distance functional itself as well as various estimates for it, one given e.g. in (4), are the new basic concepts of this paper; see Section 4.

Let us now consider some examples of equalities in (1). The classical sampling theorem of signal analysis, connected with the names of C. Shannon (1948/49), V. A. Kotel'nikov (1933), E. T. Whittaker (1915), and many others, states that a function $f \in B^2_\sigma$ has the following representation:

Classical Sampling Theorem (CST). For $f \in B^2_\sigma$ with some $\sigma > 0$ we have

$$f(z) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(z - \frac{k\pi}{\sigma}\Big) \quad (z \in \mathbb{C}), \tag{5}$$

the convergence being absolute and uniform on compact subsets of $\mathbb{C}$.

The sinc-function is given by $\operatorname{sinc} z := \sin(\pi z)/(\pi z)$ for $z \ne 0$, and $\operatorname{sinc} z := 1$ for $z = 0$. In the words of the motivation

$$U(f) = f(z), \quad V_\sigma(f) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(z - \frac{k\pi}{\sigma}\Big).$$

Let us first emphasize that this formula is equivalent to several other striking formulae of mathematical analysis (see [3]) such as:

Poisson's summation formula (PSF, particular case). For $f \in B^1_\sigma$

$$\int_{\mathbb{R}} f(t)\, dt = \frac{2\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\Big(\frac{2k\pi}{\sigma}\Big). \tag{6}$$

General Parseval formula (GPF). For $f, g \in B^2_\sigma$ with $\sigma > 0$

$$\int_{\mathbb{R}} f(t)\, \overline{g}(t)\, dt = \frac{\pi}{\sigma} \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \overline{g}\Big(\frac{k\pi}{\sigma}\Big). \tag{7}$$

Reproducing kernel formula (RKF). For $f \in B^2_\sigma$ with $\sigma > 0$

$$f(z) = \frac{\sigma}{\pi} \int_{\mathbb{R}} f(t)\, \operatorname{sinc}\Big(\frac{\sigma}{\pi}(z-t)\Big)\, dt \qquad (z \in \mathbb{C}). \tag{8}$$

This means that B^2_σ is a reproducing kernel Hilbert space, i.e., there exists a kernel function $k(\cdot, z)$ which belongs to B^2_σ for each $z \in \mathbb{C}$, such that

$$f(z) = \big\langle f(\cdot), k(\cdot, z)\big\rangle \qquad (z \in \mathbb{C}).$$

Valiron's or Tschakaloff's sampling/interpolation formula (VSF). For $f \in B^\infty_\sigma$ with $\sigma > 0$, we have for all $z \in \mathbb{C}$:

$$f(z) = \big(f'(0)z + f(0)\big) \operatorname{sinc}\Big(\frac{\sigma z}{\pi}\Big) + \sum_{k \in \mathbb{Z} \setminus \{0\}} f\Big(\frac{k\pi}{\sigma}\Big) \frac{\sigma z}{k\pi} \operatorname{sinc}\Big(\frac{\sigma z}{\pi} - k\Big), \tag{9}$$

the convergence being absolute and uniform on compact subsets of $\mathbb{C}$.

2. Extensions to non-bandlimited functions. We now weaken the assumption of $f \in B^2_\sigma$, i.e., the Fourier transform $\widehat{f}$ has support contained in $[-\sigma, \sigma]$, to $\widehat{f} \in L^1(\mathbb{R})$. In this respect we introduce the *Fourier inversion class* for $p \in [1, 2]$,

$$F^p := \big\{ f : \mathbb{R} \to \mathbb{C} \ :\ f \in L^p(\mathbb{R}) \cap C(\mathbb{R}),\ \widehat{f} \in L^1(\mathbb{R}) \big\},$$

as well as the ℓ^p *summability class* for *step size* $h > 0$

$$S^p_h := \big\{ f : \mathbb{R} \to \mathbb{C} \ :\ \big(f(hk)\big)_{k \in \mathbb{Z}} \in \ell^p(\mathbb{Z}) \big\}.$$

In the frame of these spaces all the formulae mentioned above hold only approximately in the sense that they have to be equipped with (additional) remainder terms. More precisely, the classical sampling theorem (5) is replaced by the

Approximate/extended sampling theorem (AST). For $f \in F^2 \cap S^1_{\pi/\sigma}$:

$$f(t) = \sum_{k \in \mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(t - \frac{k\pi}{\sigma}\Big) + (R^{\mathrm{WKS}}_\sigma f)(t) \qquad (t \in \mathbb{R}),$$

$$(R^{\mathrm{WKS}}_\sigma f)(t) := \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \Big(1 - e^{-i2k\sigma t}\Big) \int_{(2k-1)\sigma}^{(2k+1)\sigma} \widehat{f}(v) e^{ivt}\, dv, \tag{10}$$

the series converging absolutely and uniformly on $\mathbb{R}$. Moreover, the remainder $R^{\mathrm{WKS}}_\sigma f$ can be estimated by

$$\big|(R^{\mathrm{WKS}}_\sigma f)(t)\big| \le \sqrt{\frac{2}{\pi}} \int_{|v| \ge \sigma} |\widehat{f}(u)|\, du = o(1) \quad (\sigma \to \infty), \tag{11}$$

which yields

$$\lim_{\sigma \to \infty} \sum_{k \in \mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(t - \frac{k\pi}{\sigma}\Big) = f(t) \quad \text{(uniformly for } t \in \mathbb{R}).$$

This theorem is associated with the names of Weiss (1963), Brown (1967) and Butzer–Splettstößer (1977).

The particular case of Poisson's summation formula for $f \in B^1_\sigma$, thus (6), is generalized to the classical form:

Poisson's summation formula (PSF). For $f \in F^1$ with $\widehat{f} \in S^1_{\pi/\sigma}$

$$\sqrt{2\pi}\,\frac{\sigma}{\pi}\sum_{k\in\mathbb{Z}} f\Big(x+\frac{2k\sigma}{\pi}\Big) = \sum_{k\in\mathbb{Z}} \widehat{f}\Big(\frac{k\pi}{\sigma}\Big)e^{ik\pi x/\sigma} \quad (\text{a.e.}). \tag{12}$$

In case of the general Parseval formula (7) one has even to add two remainder terms, leading to

Generalized Parseval decomposition formula (GPDF). For $f \in F^2 \cap S^1_{\pi/\sigma}$, $\sigma > 0$, and $g \in F^2$, there holds $R^{\mathrm{WKS}}_\sigma f \in L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} f(u)\overline{g}(u)\,du = \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\overline{g}\Big(\frac{k\pi}{\sigma}\Big) + R_\sigma(f,g) \tag{13}$$

with

$$R_\sigma(f,g) := \int_{\mathbb{R}} (R^{\mathrm{WKS}}_\sigma f)(u)\overline{g}(u)\,du + \sqrt{\frac{\pi}{2}}\,\frac{1}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\int_{|v|\geq\sigma} \widehat{\overline{g}}(v)\,e^{ik\pi v/\sigma}\,dv,$$

where $R^{\mathrm{WKS}}_\sigma f$ is given by (2).

This extended Parseval decomposition formula is due to Butzer–Gessinger [4].

Similarly, the reproducing kernel formula (8) is equipped with two additional terms:

Approximate reproducing kernel formula (ARKF). For $f \in F^2$:

$$f(t) = \frac{\sigma}{\pi}\int_{\mathbb{R}} f(u)\,\mathrm{sinc}\Big(\frac{\sigma}{\pi}(t-u)\Big)\,du + (R^{\mathrm{RKF}}_\sigma f)(t) \tag{14}$$

with

$$\begin{aligned}(R^{\mathrm{RKF}}_\sigma f)(t) &:= (R^{\mathrm{WKS}}_\sigma f)(t) - \frac{\sigma}{\pi}\int_{\mathbb{R}} (R^{\mathrm{WKS}}_\sigma f)(u)\,\mathrm{sinc}\Big(\frac{\sigma}{\pi}(t-u)\Big)du \\ &= \frac{1}{\sqrt{2\pi}}\int_{|v|\geq\sigma} \widehat{f}(v)e^{ivt}\,dv.\end{aligned}$$

Furthermore,

$$\big|(R^{\mathrm{RKF}}_\sigma f)(t)\big| \leq \frac{1}{\sqrt{2\pi}}\int_{|v|\geq\sigma} |\widehat{f}(u)|\,du\,.$$

The extension of the Valiron–Tschakaloff sampling formula (VSF) to non-bandlimited functions reads:

Approximate Valiron–Tschakaloff sampling formula (AVSF). Let $f \in F^2$ and let $v\widehat{f}(v)$ be absolutely integrable. Then for $\sigma > 0$ and $t \in \mathbb{R}$, we have

$$f(t) = \big(f'(0)t + f(0)\big)\,\mathrm{sinc}\Big(\frac{\sigma t}{\pi}\Big) + \sum_{k\in\mathbb{Z}\setminus\{0\}} f\Big(\frac{k\pi}{\sigma}\Big)\frac{\sigma t}{k\pi}\,\mathrm{sinc}\Big(\frac{\sigma t}{\pi}-k\Big) + (R^{\mathrm{VT}}_\sigma f)(t),$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{R}$ and

$$(R^{\mathrm{VT}}_\sigma f)(t) = (R^{\mathrm{WKS}}_\sigma f)(t) - i\sin(\sigma t)\sum_{k\in\mathbb{Z}} \frac{2k}{\sqrt{2\pi}}\int_{(2k-1)\sigma}^{(2k+1)\sigma} \widehat{f}(v)\,dv \tag{15}$$

with $R_\sigma^{\mathrm{WKS}} f$ again given by (2). Furthermore,

$$\big|(R_\sigma^{\mathrm{VT}} f)(t)\big| \le \frac{3}{\sqrt{2\pi}} \int_{|v|\ge\sigma} \big|\widehat{f}(v)\big|\,dv + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{|v|\ge\sigma} \big|v\widehat{f}(v)\big|\,dv = o(1) \quad (\sigma\to\infty).$$

Clearly, if the functions involved belong to the (particular) Bernstein space B_σ^2, then, according to the Paley–Wiener theorem, the remainder terms in (11), (13), (14) and (15) vanish, and one obtains the particular versions (5), (7), (8) and (9). Similarly, for $f\in B_\sigma^1$ and $x=0$, Poisson's summation formula (12) reduces to the particular case (6).

3. Boas-type formulae for higher derivatives. The following differentiation formula is due to Boas [1], who used it to give an elementary proof of Bernstein's inequality (3) for $f\in B_\sigma^\infty$.

Let $f\in B_{\pi/h}^\infty$, where $h>0$. Then, for $h=\pi/\sigma$, we have

$$f'(t) \;=\frac{1}{h}\sum_{k\in\mathbb{Z}} \frac{(-1)^{k+1}}{\pi(k-1/2)^2}\, f(t+h(k-1/2))\,. \tag{16}$$

As usual in numerical analysis, we have replaced the step size parameter σ by π/h. Formula (16) can be generalized to higher order derivatives (see [16, 6]):

THEOREM 3.1. *Let $f\in B_{\pi/h}^\infty$ for some $h>0$. Then for $s\in\mathbb{N}$,*

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k=-\infty}^{\infty} (-1)^{k+1} A_{s,k}\, f\Big(t+h\Big(k-\frac{1}{2}\Big)\Big) \qquad (t\in\mathbb{R}), \tag{17}$$

where

$$A_{s,k} := \frac{(2s-1)!}{\pi(k-1/2)^{2s}} \sum_{j=0}^{s-1} \frac{(-1)^j}{(2j)!} \Big[\pi\Big(k-\frac{1}{2}\Big)\Big]^{2j} \qquad (k\in\mathbb{Z}).$$

The extension to non-bandlimited functions reads [6]:

THEOREM 3.2. *Let $s\in\mathbb{N}$, $f\in F^2$ and let $v^{2s-1}f(v)$ be absolutely integrable. Then $f^{(2s-1)}$ exists and for $h>0$ formula (17) extends to*

$$f^{(2s-1)}(t) = \frac{1}{h^{2s-1}} \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{s,k} f\Big(t+h\Big(k-\frac{1}{2}\Big)\Big) + (R_{2s-1,h}^{\mathrm{Boas}} f)(t)\,, \tag{18}$$

where

$$(R_{2s-1,h}^{\mathrm{Boas}} f)(t) = \frac{i(-1)^{s-1}}{\sqrt{2\pi}\,h^{2s-1}} \int_{|v|\ge\pi/h} \big[(hv)^{2s-1} - \phi_{2s-1}(hv)\big]\,\widehat{f}(v)e^{ivt}\,dv$$

with ϕ_{2s-1} being the 4π-periodic function defined by

$$\phi_{2s-1}(v) = \begin{cases} v^{2s-1}, & -\pi\le v\le\pi,\\ (2\pi-v)^{2s-1}, & \pi<v\le 3\pi.\end{cases}$$

In particular,

$$\big|(R_{2s-1,h}^{\mathrm{Boas}} f)(t)\big| \le \sqrt{\frac{2}{\pi}} \int_{|v|\ge\pi/h} |v|^{2s-1}\big|\widehat{f}(v)\big|\,dv = o(1) \quad (h\to 0).$$

Similar results hold for even order derivatives.

4. Foundations for a unified approach to extensions: A hierarchy of wider spaces and estimates for the distance of f from B^2_σ. We have seen that there exist formulae for functions in B^p_σ that hold for $f \in F^p$ (or a subspace of it) with a remainder $R_\sigma f$ tending to zero as $\sigma \to \infty$. Now we aim at a unified approach to such extensions with error estimates presented in terms of the distance of f from B^p_σ.

A hierarchy of spaces. In our approach, the Fourier inversion class F^p for $p \in [1,2]$ is the largest space beyond B^p_σ for which the remainder $R_\sigma f$ in the representation (2) can be evaluated and estimated effectively. However, if we want $R_\sigma f$ to converge rapidly to zero as $\sigma \to \infty$, we should rather consider a subspace of F^p. It is therefore desirable to know a hierarchy of spaces lying between B^p_σ and F^p. Our considerations include:

- The *modulation space* $M^{2,1}$ comprising all functions $f \in L^2(\mathbb{R})$ such that

$$\|f\|_{M^{2,1}} := \sum_{n\in\mathbb{Z}} \Big\{ \int_n^{n+1} \big|\widehat{f}(v)\big|^2\, dv \Big\}^{1/2} < \infty. \tag{19}$$

- The subspace $M^{2,1}_*$ comprising all functions $f \in M^{2,1}$ such that the series

$$\sum_{n\in\mathbb{Z}} \frac{1}{h} \Big\{ \int_n^{n+1} \Big|\widehat{f}\Big(\frac{v}{h}\Big)\Big|^2\, dv \Big\}^{1/2}$$

  converges uniformly with respect to h on bounded subintervals of $(0,\infty)$.

- The *Lipschitz spaces* for $r \in \mathbb{N}$ and $0 < \alpha \le r$,

$$\mathrm{Lip}_r(\alpha; L^2(\mathbb{R})) := \big\{ f \in L^2(\mathbb{R}) : \omega_r(f;\delta;L^2(\mathbb{R})) = \mathcal{O}\big(\delta^\alpha\big),\ \delta \to 0+ \big\},$$

  where

$$\omega_r(f;\delta;L^2(\mathbb{R})) := \sup_{|h|\le\delta} \|\Delta_h^r f\|_{L^2(\mathbb{R})}$$

  is the modulus of smoothness of order r with respect to the $L^2(\mathbb{R})$ norm, and $(\Delta_h^r f)(u)$ is the forward difference of order r, defined by

$$(\Delta_h^r f)(u) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(u + jh).$$

- The *Sobolev spaces*

$$W^{r,p}(\mathbb{R}) := \big\{ f \in L^p(\mathbb{R}) : v^r\, \widehat{f}(v) = \widehat{g}(v),\ g \in L^p(\mathbb{R}) \big\},$$

$$\|f\|_{W^{r,p}(\mathbb{R})} := \Big\{ \sum_{k=0}^{r} \|f^{(k)}\|^p_{L^p(\mathbb{R})} \Big\}^{1/p}.$$

- The *Hardy spaces* of functions f analytic in the strip

$$\mathcal{S}_d := \big\{ z \in \mathbb{C} : |\Im z| < d \big\}$$

  such that

$$\|f\|_{H^p(\mathcal{S}_d)} := \Big[\sup_{0<y<d} \int_{\mathbb{R}} \frac{|f(t-iy)|^p + |f(t+iy)|^p}{2}\, dt \Big]^{1/p} < \infty.$$

Let us observe that modulation spaces are connected with the well-known Wiener amalgam spaces $W(L^p, \ell^q)$ which comprise all measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{p,q} := \Big\{\sum_{n\in\mathbb{Z}} \Big\{\int_n^{n+1} |f(t)|^p\,dt\Big\}^{q/p}\Big\}^{1/q} = \Big\|\|f\|_{L^p(n,n+1)}\Big\|_{\ell^q} < \infty$$

with the usual convention applying when p or q is infinite. Note that $L^p(\mathbb{R}) = W(L^p, \ell^p)$. In fact, the elements f of $M^{2,1}$ are exactly the Fourier transforms of the elements g in the amalgam space $W(L^2, \ell^1)$; this gives the norm in (19). Whereas amalgam spaces were introduced by N. Wiener [17], and first studied systematically by F. Holland [14] (based on earlier work by J. L. B. Cooper [10] on positive definite functions), modulation spaces were first presented by H. G. Feichtinger at Oberwolfach [11, 12].

For the spaces above, we have the following inclusions

$$B^p_\sigma|_{\mathbb{R}} \subsetneqq H^p(\mathcal{S}_d)|_{\mathbb{R}} \subsetneqq W^{r,p}(\mathbb{R}) \cap C(\mathbb{R}) \subsetneqq F^p \cap S^p_h \subsetneqq F^p \subsetneqq L^p(\mathbb{R}). \tag{20}$$

Here $|_{\mathbb{R}}$ means that the functions of the corresponding space are restricted to $\mathbb{R}$. For $p = 2$ these inclusions can be further refined by invoking the Lipschitz and the modulation spaces. We have

$$W^{r,2}(\mathbb{R}) \cap C(\mathbb{R}) \subsetneqq M^{2,1}_* \subsetneqq M^{2,1} \subsetneqq F^2 \cap S^2_h$$

and

$$M^{2,1}_* \subsetneqq \mathrm{Lip}_r(\tfrac{1}{2}, L^2(\mathbb{R})) \cap F^2.$$

The strict inclusion relation $M^{2,1} \subsetneqq F^2 \cap S^2_h$ will be proven in Section 6.

Norms and distances. In order to measure the distance of a function f belonging to F^2 (or to one of its subspaces) from B^2_σ, we need to introduce a metric in F^2. For $q \in [1, 2]$ and $f \in F^2$, we define

$$|\!|\!| f |\!|\!|_q := \Big\{\int_{\mathbb{R}} |\widehat{f}(v)|^q\,dv\Big\}^{1/q} \equiv \|\widehat{f}\|_{L^q(\mathbb{R})},$$

which endows F^2 with a norm. It induces a metric

$$|\!|\!| f - g |\!|\!|_q =: \mathrm{dist}_q(f, g) \qquad (f, g \in F^2),$$

which allows us to define the distance of $f \in F^2$ from B^2_σ as

$$\mathrm{dist}_q(f, B^2_\sigma) := \inf_{g\in B^2_\sigma} |\!|\!| f - g |\!|\!|_q \equiv \inf_{g\in B^2_\sigma} \|\widehat{f} - \widehat{g}\|_{L^q(\mathbb{R})}. \tag{21}$$

For $q = 2$, $\mathrm{dist}_2(f, g) = \|f - g\|_{L^2(\mathbb{R})}$, by the isometry of the $L^2(\mathbb{R})$-Fourier transform.

Classical Banach space norms are generally too strong for measuring distances in our problems. Indeed, an example in the instance of Sobolev spaces with $p = 2$ is given in

Proposition 4.1. *There exists $f \in B^2_\sigma$ and a sequence $f_n \in W^{r,2}(\mathbb{R})$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \mathrm{dist}_2(f_n, f) = 0$, but $\lim_{n\to\infty} \|f_n - f\|_{W^{r,2}(\mathbb{R})} = \infty$.*

In fact for the sequence

$$f_n(t) := \sqrt{\frac{2}{\pi}}\,\frac{e^{i\pi n^2 t}}{n}\,\mathrm{sinc}\Big(\frac{t}{n\pi}\Big) \qquad (n \in \mathbb{N}),$$

$$\widehat{f}_n(v) = \operatorname{rect}(n\pi v - n^2\pi) = \begin{cases} 1 & \text{if } |v-n| < \frac{1}{n}\,, \\ \frac{1}{2} & \text{if } |v-n| = \frac{1}{n}\,, \\ 0 & \text{if } |v-n| > \frac{1}{n}\,. \end{cases}$$

Using the isometry of the Fourier transform, we see that

$$\|f_n\|^2_{L^2(\mathbb{R})} = \frac{2}{n}$$

and, for $k \in \mathbb{N}$,

$$\|f_n^{(k)}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |v^k \widehat{f}_n(v)|^2\, dv = \int_{n-1/n}^{n+1/n} v^{2k}\, dv > 2n^{2k-1}.$$

Thus, for $r \in \mathbb{N}$, we have

$$\|f_n\|_{W^{r,2}(\mathbb{R})} \geq \sqrt{2n}\,.$$

This yields the desired result.

The following fundamental results hold, the first certainly being a derivative free error estimate. It is always assumed that $1 \leq q \leq 2$.

Proposition 4.2.
a) *If $f \in F^2$, then*

$$\operatorname{dist}_q(f, B^2_\sigma) = \Big\{\int_{|v|>\sigma} \big|\widehat{f}(v)\big|^q dv\Big\}^{1/q} \leq c\Big\{\int_\sigma^\infty v^{-q/2}\big[\omega_r(f, v^{-1}, L^2(\mathbb{R}))\big]^q\, dv\Big\}^{1/q}.$$

b) *If $f \in W^{s,2}(\mathbb{R}) \cap C(\mathbb{R})$ with $v^s\widehat{f}(v) \in L^1(\mathbb{R})$ for some $s \in \mathbb{N}$, then $f^{(s)}$ exists and*

$$\begin{aligned}\operatorname{dist}_q(f^{(s)}, B^2_\sigma) &= \Big\{\int_{|v|>\sigma} \big|v^s\widehat{f}(v)\big|^q\, dv\Big\}^{1/q} \\ &\leq c\Big\{\int_\sigma^\infty v^{-q/2}\big[\omega_r(f^{(s)}, v^{-1}, L^2(\mathbb{R}))\big]^q\, dv\Big\}^{1/q}.\end{aligned}$$

These estimates enable us to determine the rates of convergence as $\sigma \to \infty$ for the subspaces of F^2 listed above.

Corollary 4.3.
a) *If $f \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R})) \cap F^2$ and $1/q - 1/2 < \alpha \leq r$, then*

$$\operatorname{dist}_q(f, B^2_\sigma) = \mathcal{O}\big(\sigma^{-\alpha-1/2+1/q}\big) \quad (\sigma \to \infty).$$

b) *If $f \in F^2$ and $f^{(s)} \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R}))$, $s \in \mathbb{N}$, $0 < \alpha \leq r$, then*

$$\operatorname{dist}_q(f, B^2_\sigma) = \mathcal{O}\big(\sigma^{-\alpha-s-1/2+1/q}\big) \quad (\sigma \to \infty).$$

Corollary 4.4. *If $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ and $f^{(r)} \in M^{2,1}_\star$ for some $r \in \mathbb{N}$, then*

$$\operatorname{dist}_q(f, B^2_\sigma) = \mathcal{O}\big(\sigma^{-r-1+1/q}\big) \quad (\sigma \to \infty),$$

and for $s \leq r$,

$$\operatorname{dist}_q(f^{(s)}, B^2_\sigma) = \mathcal{O}\big(\sigma^{-r-1+s+1/q}\big) \quad (\sigma \to \infty).$$

In Sobolev spaces the distances also converge to zero like a power of $1/\sigma$ and in Hardy spaces they converge to zero exponentially. Indeed, we have:

Corollary 4.5.
a) *Let* $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$, $r \in \mathbb{N}$. *Then for* $s \in \mathbb{N}$,

$$\operatorname{dist}_q(f, B^2_\sigma) = \mathcal{O}\big(\sigma^{-r-1/2+1/q}\big) \quad (\sigma \to \infty),$$
$$\operatorname{dist}_q(f^{(s)}, B^2_\sigma) = \mathcal{O}\big(\sigma^{-r-1/2+s+1/q}\big) \quad (r > s + 1/q - 1/2;\ \sigma \to \infty).$$

b) *Let* $f \in H^2(\mathcal{S}_d)$, $s \in \mathbb{N}$, *and* $q \in [1, 2]$. *Then*

$$\operatorname{dist}_q(f, B^2_\sigma) = \mathcal{O}\big(e^{-d\sigma}\big) \quad (\sigma \to \infty),$$
$$\operatorname{dist}_q(f^{(s)}, B^2_\sigma) = \mathcal{O}\big(\sigma^s e^{-d\sigma}\big) \quad (\sigma \to \infty).$$

Since dist_2 is the euclidean distance, the characterization of Lip-functions due to Junggeburth–Scherer–Trebels [15] gives

Proposition 4.6. *Let* $f \in F^2$, $r \in \mathbb{N}$, *and* $0 < \alpha \le r$. *Then*

$$f \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R})) \iff \operatorname{dist}_2(f, B^2_\sigma) = \mathcal{O}\big(\sigma^{-\alpha}\big) \quad (\sigma \to \infty).$$

5. Applications of the distance approach to the remainders of the formulae under discussion. After these preparations we turn to the errors involved under the extensions to larger spaces. The new results include:

Theorem 5.1 (AST, [8, 9]). *For* $f \in F^2 \cap S^2_{\pi/\sigma}$:

$$\Big| f(t) - \sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big) \operatorname{sinc} \frac{\sigma}{\pi}\Big(t - \frac{k\pi}{\sigma}\Big)\Big| = \big|(R^{\mathrm{WKS}}_\sigma f)(t)\big|$$
$$\le \sqrt{\frac{2}{\pi}}\, \operatorname{dist}_1(f, B^2_\sigma) \le c \int_\sigma^\infty v^{-1/2}\omega_r(f, v^{-1}, L^2(\mathbb{R}))\, dv \quad (t \in \mathbb{R};\ \sigma > 0).$$

Observe that the integral may be infinite, but under the assumptions of Corollaries 4.3, 4.5 it is finite, and we also have the following rates of convergence for $\sigma \to \infty$:

Corollary 5.2.
a) (*Lipschitz space*) *If* $f \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R})) \cap F^2 \cap S^2_{\pi/\sigma}$, $r \in \mathbb{N}$, $1/2 < \alpha \le r$, *then*

$$(R^{\mathrm{WKS}}_\sigma f)(t) = \mathcal{O}\big(\sigma^{-\alpha+1/2}\big) \quad (\sigma \to \infty).$$

b) (*Sobolev space*) *If* $f \in W^{r,2} \cap C(\mathbb{R})$ *for some* $r \in \mathbb{N}$, *then*

$$(R^{\mathrm{WKS}}_\sigma f)(t) = \mathcal{O}\big(\sigma^{-r+1/2}\big) \quad (\sigma \to \infty).$$

c) (*Hardy space*) *If* $f \in H^2(\mathcal{S}_d)$, *then*

$$(R^{\mathrm{WKS}}_\sigma f)(t) = \mathcal{O}\big(e^{-d\sigma}\big) \quad (\sigma \to \infty).$$

d) (*Modulation space*) *If* $f \in W^{r,2} \cap C(\mathbb{R})$ *and* $f^{(r)} \in M^{2,1}_\star$ *for some* $r \in \mathbb{N}$, *then*

$$(R^{\mathrm{WKS}}_\sigma f)(t) = \mathcal{O}\big(\sigma^{-r}\big) \quad (\sigma \to \infty).$$

THEOREM 5.3 (GPDF, [4, 2]). *If $f \in F^2 \cap S^1_{\pi/\sigma}$, $\sigma > 0$, and $g \in F^1$, then $R^{\mathrm{WKS}}_\sigma f \in L^2(\mathbb{R})$ and*

$$\Big| \int_{\mathbb{R}} f(u)\overline{g}(u)\,du - \frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}} f\Big(\frac{k\pi}{\sigma}\Big)\overline{g}\Big(\frac{k\pi}{\sigma}\Big)\Big| = \big|R_\sigma(f,g)\big|$$
$$\le \sqrt{\frac{2}{\pi}}\,\|g\|_{L^1(\mathbb{R})}\operatorname{dist}_1(f, B^2_\sigma) + \sigma\sqrt{\frac{\pi}{2}}\operatorname{dist}_1(g, B^2_\sigma)\sum_{k\in\mathbb{Z}}\Big|f\Big(\frac{k\pi}{\sigma}\Big)\Big|.$$

If, in addition, $f, g \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$, then

$$R_\sigma(f,g) \le \frac{C}{\sigma}\Big\{\operatorname{dist}_2(f', B^2_\sigma) + \operatorname{dist}_2(g', B^2_\sigma) + \frac{\pi}{\sigma}\operatorname{dist}_2(f', B^2_\sigma)\operatorname{dist}_2(g', B^2_\sigma)\Big\}.$$

This theorem leads to the following corollary.

COROLLARY 5.4.
a) *Let $f, g \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$ such that $v\widehat{f}(v)$ and $v\,\widehat{g}(v)$ are absolutely integrable. If $f', g' \in \operatorname{Lip}_r(\alpha, L^2(\mathbb{R}))$, $0 < \alpha \le r$, then*

$$R_\sigma(f,g)(t) = \mathcal{O}\big(\sigma^{-\alpha-1}\big) \quad (\sigma \to \infty).$$

b) *Let $f, g \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$, $r \in \mathbb{N}$, such that $v\widehat{f}(v)$ and $v\,\widehat{g}(v)$ are absolutely integrable. Then*

$$R_\sigma(f,g)(t) = \mathcal{O}\big(\sigma^{-r}\big) \quad (\sigma \to \infty).$$

c) *If $f, g \in H^2(\mathcal{S}_d)$, then*

$$R_\sigma(f,g)(t) = \mathcal{O}\big(e^{-d\sigma}\big) \quad (\sigma \to \infty).$$

d) *Let $f, g \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$, $r \in \mathbb{N}$. If $f^{(r)}$ and $g^{(r)}$ both belong to $M^{2,1}_\star$, then*

$$R_\sigma(f,g)(t) = \mathcal{O}\big(\sigma^{-r-1/2}\big) \quad (\sigma \to \infty).$$

THEOREM 5.5 (ARKF, [7]). *For $f \in F^2$*

$$\Big| f(t) - \frac{\sigma}{\pi}\int_{\mathbb{R}} f(u)\operatorname{sinc}\Big(\frac{\sigma}{\pi}(t-u)\Big)\,du\Big| = \big|(R^{\mathrm{RKF}}_\sigma f)(t)\big| \le \frac{1}{\sqrt{2\pi}}\operatorname{dist}_1(f, B^2_\sigma).$$

Comparing Theorems 5.5 and 5.1 shows that Corollary 5.2 remains valid, if $R^{\mathrm{WKS}}_\sigma f$ is replaced by $R^{\mathrm{RKF}}_\sigma f$.

Next we turn to the approximate Valiron sampling formula.

THEOREM 5.6 (AVSF, [7]). *Let $f \in F^2$ with $v\widehat{f}(v) \in L^1(\mathbb{R})$. Then for $\sigma > 0$,*

$$\Big| f(t) - \Big\{\big(f'(0)t + f(0)\big)\operatorname{sinc}\Big(\frac{\sigma t}{\pi}\Big) + \sum_{k\in\mathbb{Z}\setminus\{0\}} f\Big(\frac{k\pi}{\sigma}\Big)\frac{\sigma t}{k\pi}\operatorname{sinc}\Big(\frac{\sigma t}{\pi} - k\Big)\Big\}\Big|$$
$$= \big|(R^{\mathrm{VT}}_\sigma f)(t)\big| \le \frac{3}{\sqrt{2\pi}}\operatorname{dist}_1(f, B^2_\sigma) + \frac{1}{\sqrt{2\pi}\,\sigma}\operatorname{dist}_1(f', B^2_\sigma) \le \frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\operatorname{dist}_1(f', B^2_\sigma).$$

The following corollaries result.

Corollary 5.7.
a) *Let* $f \in W^{1,2}(\mathbb{R}) \cap C(\mathbb{R})$ *be such that* $v\widehat{f}(v) \in L^1(\mathbb{R})$*. If* $1/2 < \alpha \leq r$ *and* $f' \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$*, then*

$$(R_\sigma^{\mathrm{VT}} f)(t) = \mathcal{O}\big(\sigma^{-\alpha-1/2}\big) \quad (\sigma \to \infty).$$

b) *Let* $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$*,* $r \geq 2$*. Then*

$$(R_\sigma^{\mathrm{VT}} f)(t) = \mathcal{O}\big(\sigma^{-r+1/2}\big) \quad (\sigma \to \infty).$$

c) *If* $f \in H^2(\mathcal{S}_d)$*, then*

$$(R_\sigma^{\mathrm{VT}} f)(t) = \mathcal{O}\big(e^{-d\sigma}\big) \quad (\sigma \to \infty).$$

d) *If* $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$ *and* $f^{(r)} \in M_\star^{2,1}$*, then*

$$(R_\sigma^{\mathrm{VT}} f)(t) = \mathcal{O}\big(\sigma^{-r}\big) \quad (\sigma \to \infty).$$

Next to the extended version of Boas' formula for higher order derivatives (18):

Theorem 5.8 ((16), [16, 6, 7]). *Let* $s \in \mathbb{N}$*,* $f \in F^2$*, and let* $v^{2s-1}f(v)$ *be absolutely integrable. Then*

$$\left| f^{(2s-1)}(t) - \frac{1}{h^{2s-1}} \sum_{k\in\mathbb{Z}} (-1)^{k+1} A_{s,k} f\Big(t + h\Big(k - \frac{1}{2}\Big)\Big) \right|$$
$$= \big|(R_{2s-1,h}^{\mathrm{Boas}} f)(t)\big| \leq \sqrt{\frac{2}{\pi}}\, \mathrm{dist}_1(f^{(2s-1)}, B_{\pi/h}^2) \qquad (h > 0).$$

Corollary 5.9.
a) *Let* $s \in \mathbb{N}$*,* $f \in W^{2s-1,2}(\mathbb{R}) \cap C(\mathbb{R})$*, and let* $v^{2s-1}f(v)$ *be absolutely integrable. If* $f^{(2s-1)} \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$*,* $1/2 < \alpha \leq r$*, then*

$$(R_{2s-1,h}^{\mathrm{Boas}} f)(t) = \mathcal{O}\big(h^{\alpha-1/2}\big) \quad (h \to 0).$$

b) *Let* $r, s \in \mathbb{N}$*,* $r \geq 2s$*, and* $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$*. Then*

$$(R_{2s-1,h}^{\mathrm{Boas}} f)(t) = \mathcal{O}\big(h^{r+1/2-2s}\big) \quad (h \to 0).$$

c) *If* $f \in H^2(\mathcal{S}_d)$*, then*

$$(R_{2s-1,h}^{\mathrm{Boas}} f)(t) = \mathcal{O}\big(h^{-2s+1}e^{-d\pi/h}\big) \quad (h \to 0).$$

d) *Let* $r, s \in \mathbb{N}$*,* $r \geq 2s$*, and* $f \in W^{r,2}(\mathbb{R}) \cap C(\mathbb{R})$*. If* $f^{(r)}$ *belongs to* $M_\star^{2,1}$*, then*

$$(R_{2s-1,h}^{\mathrm{Boas}} f)(t) = \mathcal{O}\big(h^{r+1-2s}\big) \quad (h \to 0).$$

We finally turn to Bernstein's inequality, already discussed in the "motivation" of Section 1. Returning to the estimate (4) for the remainder, we have the following corollary concerning the rates of convergence.

Corollary 5.10.
a) *Let* $f \in W^{s,2}(\mathbb{R})$ *with* $v^s\widehat{f}(v) \in L^1(\mathbb{R})$ *for some* $s \in \mathbb{N}$*. If* $f^{(s)} \in \mathrm{Lip}_r(\alpha, L^2(\mathbb{R}))$*,* $0 < \alpha \leq r$*, then*

$$R_{\sigma,s}^{\mathrm{Bern}} f = \mathcal{O}\big(\sigma^{-\alpha}\big) \quad (\sigma \to \infty).$$

b) *Let* $r, s \in \mathbb{N}$*,* $r > s$*, and* $f \in W^{r,2}(\mathbb{R})$*. Then*

$$R_{\sigma,s}^{\mathrm{Bern}} f = \mathcal{O}\big(\sigma^{-r+s}\big) \quad (\sigma \to \infty).$$

c) *If* $f \in H^2(\mathcal{S}_d)$, *then*

$$R^{\mathrm{Bern}}_{\sigma,s} f = \mathcal{O}\big(\sigma^s e^{-d\sigma}\big) \quad (\sigma \to \infty).$$

d) *Let* $r, s \in \mathbb{N}$, $r \geq s$, *and* $f \in W^{r,2}(\mathbb{R})$. *If* $f^{(r)}$ *belongs to* $M^{2,1}_\star$, *then*

$$R^{\mathrm{Bern}}_{\sigma,s} f = \mathcal{O}\big(\sigma^{-r-1/2+s}\big) \quad (\sigma \to \infty).$$

6. The modulation space and F^2. We here return to the inclusion chain (20) of Section 4.

PROPOSITION 6.1. *The following strict inclusion relation is true*:

$$M^{2,1} \subsetneqq F^2 \cap S^2_1.$$

Proof. If $f \in M^{2,1}$, then $f = \widehat{g}$ with $g \in W(L^2, \ell^1)$. In view of two standard inclusion relations for Wiener amalgam spaces [13, (2.3), (2.4)],

$$W(L^2,\ell^1) \subset W(L^2,\ell^2) = L^2(\mathbb{R}), \quad W(L^2,\ell^1) \subset W(L^1,\ell^1) = L^1(\mathbb{R}),$$

one has that $f = \widehat{g} \in L^2(\mathbb{R}) \cap C(\mathbb{R})$, noting [5, Prop. 5.1.2, Prop. 5.2.1]. Furthermore $\widehat{f}(v) = \widehat{\widehat{g}}(v) = g(-v) \in L^1(\mathbb{R})$. Altogether we see that f belongs to F^2.

But $M^{2,1}$ is also a subset of S^2_1. Indeed, by a basic result on Fourier transforms in Wiener amalgam spaces [14, Theorem 2], [13, Theorem 2.8], it follows that $g \in W(L^2, \ell^1)$ implies $\widehat{g} \in W(L^\infty, \ell^2)$. Thus $M^{2,1}$ is a subset of $W(L^\infty, \ell^2)$, which is obviously a subset of S^2_1.

It remains to show that $M^{2,1} \neq F^2 \cap S^2_1$, thus that there exists a function $f \in (F^2 \cap S^2_1) \setminus M^{2,1}$. We here reproduce such a fitting example, sketched and provided by Przemysław Wojtaszczyk.[2]

Let $\varphi \in C^5(\mathbb{R})$ such that $\varphi(t) > 0$ for $t \in (-\frac{1}{2}, \frac{1}{2})$ and zero outside. Define

$$g(t) := \sum_{n=3}^{\infty} \frac{1}{n^{1/2} \ln n}\, \varphi\big(n(t - n - \tfrac{1}{2})\big) \qquad (t \in \mathbb{R}), \tag{22}$$

and $f := \widehat{g}$, which is the example in question. Then

$$\sum_{n\in\mathbb{Z}} \Big\{ \int_n^{n+1} |g(t)|^2\, dt \Big\}^{1/2} = \|\varphi\|_{L^2[-1/2,1/2]} \sum_{n=3}^{\infty} \frac{1}{n \ln n} = \infty$$

and so $g \notin W(L^2, \ell^1)$, or equivalently, $f \notin M^{2,1}$.

On the other hand,

$$\|g\|^2_{L^2(\mathbb{R})} = \|\varphi\|^2_{L^2[-1/2,1/2]} \sum_{n=3}^{\infty} \frac{1}{n^2 \ln^2 n} < \infty,$$

$$\|g\|_{L^1(\mathbb{R})} = \|\varphi\|_{L^1[-1/2,1/2]} \sum_{n=3}^{\infty} \frac{1}{n^{3/2} \ln n} < \infty,$$

which implies that $f \in F^2$.

[2] In private conversation at the workshop *From Abstract to Computational Harmonic Analysis* held at Strobl, June 2011, Hans Feichtinger mentioned properties of amalgam and modulation spaces for supporting the opinion that $M^{2,1}$ and $F^2 \cap S^2_1$ cannot be equal while Prof. Wojtaszczyk (Warsaw) kindly came up with the example reproduced here.

Using (22), one finds that for $t \in \mathbb{R}$,

$$|f(t)| = \left| \sum_{n=3}^{\infty} \frac{1}{n^{3/2} \ln n} \,\widehat{\varphi}\Big(\frac{t}{n}\Big) e^{-it(n+1/2)} \right| \le \sum_{n=3}^{\infty} \frac{1}{n^{3/2} \ln n} \Big|\widehat{\varphi}\Big(\frac{t}{n}\Big)\Big|. \tag{23}$$

Since $\varphi \in C^5(\mathbb{R})$, one has for some constant c_0 depending only on φ,

$$|\widehat{\varphi}(t)| \le c_0 \min\Big\{1, \frac{1}{|t|^5}\Big\}. \tag{24}$$

Now let k be an integer outside the interval $[-3, 3]$. Splitting the summation in (23) into the ranges $3 \le n \le |k|$ and $|k| + 1 \le n < \infty$, and using (24), one may estimate

$$|f(k)| \le \frac{c_0}{|k|^5} \,\Sigma_k + c_0 \Sigma'_k, \tag{25}$$

where

$$\Sigma_k := \sum_{n=3}^{|k|} \frac{n^{7/2}}{\ln n} \qquad \text{and} \qquad \Sigma'_k := \sum_{n=|k|+1}^{\infty} \frac{1}{n^{3/2} \ln n}\,.$$

Clearly,

$$\Sigma'_k < \frac{1}{\ln |k|} \sum_{n=|k|+1}^{\infty} \frac{1}{n^{3/2}} < \frac{1}{\ln |k|} \int_{|k|}^{\infty} \frac{dx}{x^{3/2}} = \frac{2}{|k|^{1/2} \ln |k|}\,. \tag{26}$$

As regards Σ_k, one notes that $n^{7/2}/\ln n$ is increasing for $n \ge 3$ and so, by integral comparison and subsequent integration by parts one finds that

$$\begin{aligned} \Sigma_k < \int_e^{|k|+1} \frac{x^{7/2}}{\ln x}\, dx &= \frac{2}{9} \cdot \frac{x^{9/2}}{\ln x}\Big|_e^{|k|+1} + \frac{2}{9} \int_e^{|k|+1} \frac{x^{7/2}}{\ln^2 x}\, dx \\ &< \frac{2}{9} \cdot \frac{x^{9/2}}{\ln x}\Big|_e^{|k|+1} + \frac{2}{9} \int_e^{|k|+1} \frac{x^{7/2}}{\ln x}\, dx\,. \end{aligned}$$

From this, one deduces that

$$\Sigma_k < \frac{2}{7} \cdot \frac{x^{9/2}}{\ln x}\Big|_e^{|k|+1} \le c_1 \frac{|k|^{9/2}}{\ln |k|} \tag{27}$$

with a constant c_1. Combining (25)–(27), one arrives at

$$|f(k)| < \frac{c_0(2 + c_1)}{|k|^{1/2} \ln |k|} \qquad (k \in \mathbb{Z} \setminus [-3, 3]).$$

This shows that $f \in S_1^2$. This completes the proof of the proposition. ∎

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