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BILINEAR OPERATORS AND LIMITING REAL METHODS

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Abstract. We investigate the behaviour of bilinear operators under limiting real methods. As an application, we show an interpolation formula for spaces of linear operators. Some results on norm estimates for bounded linear operators are also established.

1. Introduction. Interpolation of bilinear operators is a classical question already considered by Lions and Peetre [17] and Calderón [4] in their seminal papers on the real and the complex interpolation methods, respectively. Bilinear results have found a variety of interesting applications in analysis including boundedness of convolution operators, interpolation between a Banach space and its dual, stability of Banach algebras under interpolation or interpolation of spaces of bounded linear operators (see the articles by Peetre [19], Mastyło [18], Cobos and Fernández-Cabrera [6, 7] and the references given there).

In this paper we study the behaviour of bilinear operators under limiting real methods. These methods have been investigated by the present authors in [14] (see also the papers by Cobos, Fernández-Cabrera, Kühn and Ullrich [8], Cobos, Fernández-Cabrera and Mastyło [9] and Cobos, Fernández-Cabrera and Silvestre [10, 11]). The K-spaces $(A_0, A_1)_{q;K}$ are very close to the sum $A_0 + A_1$, while the J-spaces $(A_0, A_1)_{q;J}$ are near to the intersection $A_0 \cap A_1$. We recall their definitions in Section 2. Then, in Section 3, we show that the bilinear interpolation theorems $J \times J \to J$ and $J \times K \to K$ hold, and that there are no similar results of the type $K \times J \to J$ and $K \times K \to K$. As an application, we establish an interpolation formula for spaces of bounded linear operators. Finally,

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in Section 4, we compare norm estimates for bilinear operators with estimates for linear operators. We establish two results which complement those shown in [14].

2. Preliminaries. Let $\overline{A} = (A_0, A_1)$ be a *Banach couple*, that is to say, two Banach spaces A_0, A_1 which are continuously embedded in a common linear Hausdorff space.

Peetre's K- and J-functionals are defined for $t > 0$ by

$$K(t,a) = K(t,a;\overline{A}) = \inf\big\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\big\}, \quad a \in A_0 + A_1,$$

and

$$J(t,a) = J(t,a;\overline{A}) = \max\big\{\|a\|_{A_0}, t\|a\|_{A_1}\big\}, \qquad a \in A_0 \cap A_1.$$

Note that $K(1,\cdot)$ and $J(1,\cdot)$ are the usual norms on $A_0 + A_1$ and $A_0 \cap A_1$, respectively.

Let $0 < \theta < 1$ and $1 \le q \le \infty$. The *real interpolation space* $(A_0, A_1)_{\theta,q}$ is defined as the collection of all $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{\overline{A}_{\theta,q}} = \Big(\int_0^\infty (t^{-\theta} K(t,a))^q \, \frac{dt}{t}\Big)^{1/q}$$

(when $q = \infty$ the integral should be replaced by the supremum). We refer to [2, 20, 3, 1] for full details on this construction.

The limiting space $\overline{A}_{q;K} = (A_0, A_1)_{q;K}$, corresponding to the value $\theta = 1$, is formed by all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\overline{A}_{q;K}} = \Big(\int_0^1 K(t,a)^q \, \frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty (t^{-1}K(t,a))^q \, \frac{dt}{t}\Big)^{1/q}.$$

Limiting spaces for $\theta = 0$ are defined by means of the J-functional: the space $\overline{A}_{q;J} = (A_0, A_1)_{q;J}$ consists of all those $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t) \, \frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1\text{)}$$

and

$$\Big(\int_0^1 (t^{-1}J(t,u(t)))^q \, \frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty J(t,u(t))^q \, \frac{dt}{t}\Big)^{1/q} < \infty.$$

The norm on $\overline{A}_{q;J}$ is

$$\|a\|_{\overline{A}_{q;J}} = \inf\Big\{ \Big(\int_0^1 (t^{-1}J(t,u(t)))^q \, \frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty J(t,u(t))^q \, \frac{dt}{t}\Big)^{1/q} \Big\}.$$

See [14] (and also [8, 9, 10, 11] for properties of limiting spaces). We just recall that

$$\overline{A}_{q;K} = (A_0, A_1, A_1, A_0)_{(1/2,1/2),q;K}, \tag{1}$$

where $(\cdot,\cdot,\cdot,\cdot)_{(\alpha,\beta),q;K}$ is the K-method of interpolation associated to the unit square (see [15, 13]). A similar result is valid for the limiting J-space.

As usual, if A and B are Banach spaces, $\mathcal{L}(A,B)$ stands for the space of all bounded linear operators from A into B.

Given two Banach couples $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$, by $T \in \mathcal{L}(\overline{A}, \overline{B})$ we denote a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction to each A_j defines a bounded operator from A_j into B_j $(j = 0, 1)$. It is not hard to check that if $T \in \mathcal{L}(\overline{A}, \overline{B})$ then the restrictions

$$T : \overline{A}_{q;K} \to \overline{B}_{q;K} \qquad \text{and} \qquad T : \overline{A}_{q;J} \to \overline{B}_{q;J}$$

are bounded too.

By $\mathbb{K}$ we denote the scalar field, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $|\cdot|$ is its usual norm. Given two positive functions f, g, we write $f \sim g$ if the quotient f/g is bounded from below and from above by positive constants.

3. Interpolation of bilinear operators. In this section we study the behaviour of bilinear operators under limiting real methods. It will be useful to work with the following discrete norm

$$\|a\|_{q;K} = \Big(\sum_{m=-\infty}^{\infty} \big(\min(1, 2^{-m}) K(2^m, a)\big)^q \Big)^{1/q},$$

which is equivalent to $\|\cdot\|_{\overline{A}_{q;K}}$. A first consequence of this discrete representation of $\overline{A}_{q;K}$ is that

$$\overline{A}_{1;K} \hookrightarrow \overline{A}_{q;K}\,, \quad 1 \le q \le \infty. \tag{2}$$

For the J-space, we work with

$$\|a\|_{q;J} = \inf\Big\{ \Big(\sum_{m=-\infty}^{\infty} \big(\max(1, 2^{-m}) J(2^m, u_m)\big)^q \Big)^{1/q} \Big\},$$

where the infimum is taken over all possible representations $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) with $(u_m) \subset A_0 \cap A_1$ satisfying

$$\Big(\sum_{m=-\infty}^{\infty} \big(\max(1, 2^{-m}) J(2^m, u_m)\big)^q \Big)^{1/q} < \infty. \tag{3}$$

REMARK 3.1. Note that if $(u_m) \subset A_0 \cap A_1$ satisfies (3), then the series is absolutely convergent in $A_0 + A_1$ because

$$\begin{aligned} \sum_{m=-\infty}^{\infty} K(1, u_m) &\le \sum_{m=-\infty}^{\infty} \min(1, 2^{-m}) J(2^m, u_m) \\ &\le \Big(\sum_{m=-\infty}^{\infty} (\max(1, 2^{-m}) J(2^m, u_m))^q \Big)^{1/q} \Big(\sum_{m=-\infty}^{\infty} \Big(\frac{\min(1, 2^{-m})}{\max(1, 2^{-m})} \Big)^{q'} \Big)^{1/q'} < \infty. \end{aligned}$$

Here $1/q + 1/q' = 1$.

The following two theorems are a consequence of the results of [5] and the connection (1) between limiting methods and interpolation methods associated to the unit square (see [10, 11]). However, we give here more simple direct proofs.

THEOREM 3.2. *Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$, $\overline{C} = (C_0, C_1)$ be Banach couples and let $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1 + 1/r$. Suppose that*

$$R : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$$

is a bounded bilinear operator whose restrictions to $A_j \times B_j$ *define bounded operators*

$$R : A_j \times B_j \to C_j$$

with norms M_j $(j = 0, 1)$*. Then the restriction*

$$R : (A_0, A_1)_{p;J} \times (B_0, B_1)_{q;J} \to (C_0, C_1)_{r;J}$$

is also bounded, with norm $M \le \max(M_0, M_1)$*.*

Proof. Take any $a \in (A_0, A_1)_{p;J}$ and $b \in (B_0, B_1)_{q;J}$, and consider any J-representations $a = \sum_{m=-\infty}^{\infty} a_m$, $b = \sum_{m=-\infty}^{\infty} b_m$. For each $k \in \mathbb{Z}$, put

$$c_k = \sum_{m=-\infty}^{\infty} R(a_m, b_{k-m}).$$

Then $c_k \in C_0 \cap C_1$ because

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} J(2^k, R(a_m, b_{k-m})) \\ &\quad \le \sum_{m=-\infty}^{\infty} \max\big(M_0 \|a_m\|_{A_0} \|b_{k-m}\|_{B_0}, M_1 2^m \|a_m\|_{A_1} 2^{k-m} \|b_{k-m}\|_{B_1}\big) \\ &\quad \le \max(M_0, M_1) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J(2^{k-m}, b_{k-m}) \end{aligned}$$

and the last sum is finite as we will show in the course of the next paragraph. Hence, $(c_k)_{k=-\infty}^{\infty} \subset C_0 \cap C_1$ with

$$J(2^k, c_k) \le \max(M_0, M_1) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J(2^{k-m}, b_{k-m}).$$

Next we show that the series $\sum_{k=-\infty}^{\infty} c_k$ is absolutely convergent in $C_0 + C_1$. According to Remark 3.1, this holds if (c_k) satisfies (3). We check this last fact by using Young's inequality. We have

$$\begin{aligned} &\Big(\sum_{k=-\infty}^{\infty} (\max(1, 2^{-k}) J(2^k, c_k))^r \Big)^{1/r} \\ &\quad \le \max(M_0, M_1) \Big(\sum_{k=-\infty}^{\infty} \Big(\sum_{m=-\infty}^{\infty} \max(1, 2^{-m}) J(2^m, a_m) \\ &\qquad\qquad \times \max(1, 2^{-(k-m)}) J(2^{k-m}, b_{k-m}) \Big)^r \Big)^{1/r} \qquad (4) \\ &\quad \le \max(M_0, M_1) \Big(\sum_{m=-\infty}^{\infty} \big(\max(1, 2^{-m}) J(2^m, a_m)\big)^p \Big)^{1/p} \\ &\qquad\qquad \times \Big(\sum_{k=-\infty}^{\infty} \big(\max(1, 2^{-k}) J(2^k, b_k)\big)^q \Big)^{1/q} < \infty\,. \end{aligned}$$

These arguments allow us also to show that

$$\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K(1, R(a_m, b_{k-m})) < \infty.$$

Indeed, since

$$\begin{aligned} K(1, R(a_m, b_{k-m})) &\\ &\le \min\big(M_0 \|a_m\|_{A_0} \|b_{k-m}\|_{B_0}, 2^{-k} M_1 2^m \|a_m\|_{A_1} 2^{k-m} \|b_{k-m}\|_{B_1}\big) \\ &\le \max(M_0, M_1) \min(1, 2^{-k}) J(2^m, a_m) J(2^{k-m}, b_{k-m}), \end{aligned}$$

proceeding as in Remark 3.1, we obtain with

$$L = \Big(\sum_{k=-\infty}^{\infty} \Big(\frac{\min(1, 2^{-k})}{\max(1, 2^{-k})} \Big)^{r'} \Big)^{1/r'}$$

the estimates

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} & K(1, R(a_m, b_{k-m})) \\ &\le \max(M_0, M_1) \sum_{k=-\infty}^{\infty} \min(1, 2^{-k}) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J(2^{k-m}, b_{k-m}) \\ &\le L \max(M_0, M_1) \Big(\sum_{k=-\infty}^{\infty} \Big(\max(1, 2^{-k}) \sum_{m=-\infty}^{\infty} J(2^m, a_m) J(2^{k-m}, b_{k-m}) \Big)^r \Big)^{1/r} \\ &\le L \max(M_0, M_1) \Big(\sum_{k=-\infty}^{\infty} \Big(\sum_{m=-\infty}^{\infty} \max(1, 2^{-m}) J(2^m, a_m) \\ &\qquad\qquad \times \max(1, 2^{-(k-m)}) J(2^{k-m}, b_{k-m}) \Big)^r \Big)^{1/r}. \end{aligned}$$

Using now Young's inequality, we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} & K(1, R(a_m, b_{k-m})) \\ &\le L \max(M_0, M_1) \Big(\sum_{m=-\infty}^{\infty} \big(\max(1, 2^{-m}) J(2^m, a_m) \big)^p \Big)^{1/p} \\ &\qquad \times \Big(\sum_{k=-\infty}^{\infty} \big(\max(1, 2^{-k}) J(2^k, b_k) \big)^q \Big)^{1/q} < \infty. \end{aligned}$$

A change in the order of summation in the double series yields that

$$R(a, b) = \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R(a_m, b_{k-m}) = \sum_{k=-\infty}^{\infty} c_k \,.$$

Consequently, by (4), we derive

$$\begin{aligned}\|R(a,b)\|_{r;J} &\leq \Big(\sum_{k=-\infty}^{\infty}(\max(1,2^{-k})J(2^k,c_k))^r\Big)^{1/r}\\ &\leq \max(M_0,M_1)\Big(\sum_{m=-\infty}^{\infty}\big(\max(1,2^{-m})J(2^m,a_m)\big)^p\Big)^{1/p}\\ &\qquad\times\Big(\sum_{k=-\infty}^{\infty}\big(\max(1,2^{-k})J(2^k,b_k)\big)^q\Big)^{1/q}.\end{aligned}$$

Now the result follows by taking the infimum over all possible J-representations of a and b. ■

Theorem 3.3. *Let $\overline{A}=(A_0,A_1)$, $\overline{B}=(B_0,B_1)$, $\overline{C}=(C_0,C_1)$ be Banach couples and let $1\leq p,q,r\leq\infty$ with $1/p+1/q=1+1/r$. Assume that*

$$R:(A_0+A_1)\times(B_0+B_1)\to C_0+C_1$$

is a bounded bilinear operator whose restrictions to $A_j\times B_j$ define bounded operators

$$R:A_j\times B_j\to C_j$$

with norms M_j $(j=0,1)$. Then the restriction

$$R:(A_0,A_1)_{p;J}\times(B_0,B_1)_{q;K}\to(C_0,C_1)_{r;K}$$

is also bounded, with norm $M\leq\max(M_0,M_1)$.

Proof. Take any $a\in(A_0,A_1)_{p;J}$ and $b\in(B_0,B_1)_{q;K}$. Let $(\lambda_m)_{m=-\infty}^{\infty}$ be a sequence of positive numbers such that

$$\sum_{m=-\infty}^{\infty}\min(1,2^{-m})^q\lambda_m^q=1,$$

and let $\varepsilon>0$. For each $m\in\mathbb{Z}$ choose a representation $b=b_0^{(m)}+b_1^{(m)}$ of b in B_0+B_1 such that

$$\big\|b_0^{(m)}\big\|_{B_0}+2^m\big\|b_1^{(m)}\big\|_{B_1}\leq K(2^m,b)+\varepsilon\lambda_m.$$

Pick any J-representation $a=\sum_{m=-\infty}^{\infty}a_m$ of a. Then, for each $k\in\mathbb{Z}$, we have

$$\begin{aligned}K(2^k,R(a,b)) &\leq \sum_{m=-\infty}^{\infty}K(2^k,R(a_m,b))\\ &\leq \sum_{m=-\infty}^{\infty}\big(K(2^k,R(a_m,b_0^{(k-m)}))+K(2^k,R(a_m,b_1^{(k-m)}))\big)\\ &\leq \sum_{m=-\infty}^{\infty}\big(M_0\|a_m\|_{A_0}\big\|b_0^{(k-m)}\big\|_{B_0}+2^kM_1\|a_m\|_{A_1}\big\|b_1^{(k-m)}\big\|_{B_1}\big)\end{aligned}$$

$$\le \max(M_0,M_1)\sum_{m=-\infty}^{\infty} J(2^m,a_m)\big(\|b_0^{(k-m)}\|_{B_0}+2^{k-m}\|b_1^{(k-m)}\|_{B_1}\big)$$
$$\le \max(M_0,M_1)\sum_{m=-\infty}^{\infty} J(2^m,a_m)\big(K(2^{k-m},b)+\varepsilon\lambda_{k-m}\big).$$

Therefore, by Young's inequality, we derive

$$\begin{aligned}\|R(a,b)\|_{r;K} &= \Big(\sum_{k=-\infty}^{\infty}\big(\min(1,2^{-k})K(2^k,R(a,b))\big)^r\Big)^{1/r}\\
&\le \max(M_0,M_1)\Big[\sum_{k=-\infty}^{\infty}\Big(\sum_{m=-\infty}^{\infty}\max(1,2^{-m})J(2^m,a_m)\\
&\qquad\times\min(1,2^{-(k-m)})\big(K(2^{k-m},b)+\varepsilon\lambda_{k-m}\big)\Big)^r\Big]^{1/r}\\
&\le \max(M_0,M_1)\Big[\sum_{m=-\infty}^{\infty}\big(\max(1,2^{-m})J(2^m,a_m)\big)^p\Big]^{1/p}\\
&\qquad\times\Big[\sum_{m=-\infty}^{\infty}\big(\min(1,2^{-m})(K(2^m,b)+\varepsilon\lambda_m)\big)^q\Big]^{1/q}\\
&\le \max(M_0,M_1)\Big(\sum_{m=-\infty}^{\infty}\big(\max(1,2^{-m})J(2^m,a_m)\big)^p\Big)^{1/p}\big(\|b\|_{q;K}+\varepsilon\big).\end{aligned}$$

Taking the infimum over all J-representations of a and letting ε go to 0, we get

$$\|R(a,b)\|_{r;K}\le \max(M_0,M_1)\|a\|_{p;J}\|b\|_{q;K},$$

as desired. ∎

Remark 3.4. In applications, sometimes one is only given a bounded bilinear operator $R:(A_0+A_1)\times(B_0\cap B_1)\to C_0+C_1$ whose restrictions $R:A_j\times\big(B_0\cap B_1,\|\cdot\|_{B_j}\big)\to C_j$ are bounded for $j=0,1$, and where the couple $\overline{B}$ is such that $B_0\cap B_1$ is dense in B_j for $j=0,1$. The question is to show that R has a bounded extension to the interpolation spaces. This means, for the case of Theorem 3.3, an extension from $\overline{A}_{p;J}\times\overline{B}_{q;K}$ into $\overline{C}_{r;K}$.

This problem has a positive answer provided that $q<\infty$. Namely, if $b\in B_0\cap B_1$, the argument in the proof of Theorem 3.3 gives

$$\|R(a,b)\|_{r;K}\le \max(M_0,M_1)\|a\|_{p;J}\|b\|_{q;K}.$$

Since $B_0\cap B_1$ is dense in $\overline{B}_{q;K}$ when $q<\infty$ (see [14, Corollary 5.5]), the bounded extension is possible.

Next we show an application of this remark to interpolation of operator spaces.

Theorem 3.5. *Let $\overline{A}=(A_0,A_1)$, $\overline{B}=(B_0,B_1)$ be Banach couples with $A_0\cap A_1$ dense in A_j for $j=0,1$. Assume that $1\le p,q,r\le\infty$ with $q<\infty$ and $1/p+1/q=1+1/r$. Then*

$$\big(\mathcal{L}(A_0,B_0),\mathcal{L}(A_1,B_1)\big)_{p;J}\subset \mathcal{L}(\overline{A}_{q;K},\overline{B}_{r;K}).$$

Proof. Let $R : \mathcal{L}(A_0 \cap A_1, B_0 + B_1) \times (A_0 \cap A_1) \to B_0 + B_1$ be the bounded bilinear operator defined by $R(T, a) = Ta$. It is clear that $R : \mathcal{L}(A_j, B_j) \times \big(A_0 \cap A_1, \|\cdot\|_{A_j}\big) \to B_j$ is also bounded for $j = 0, 1$. Whence, by Remark 3.4, R has a bounded extension $R : \big(\mathcal{L}(A_0, B_0), \mathcal{L}(A_1, B_1)\big)_{p;J} \times (A_0, A_1)_{q;K} \to (B_0, B_1)_{r;K}$. Therefore, the wanted inclusion follows. ∎

If we exchange the role of J- and K-methods in Theorem 3.3, then the corresponding statement does not hold as the next example shows.

COUNTEREXAMPLE 3.6. Let (A_0, A_1) be a Banach couple such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$. Put $R : (A_0 + A_1) \times (\mathbb{K} + \mathbb{K}) \to A_0 + A_1$ for the bounded bilinear operator defined by $R(a, \lambda) = \lambda a$. It is clear that restrictions $R : A_j \times \mathbb{K} \to A_j$ are bounded for $j = 0, 1$. If the bilinear theorem $K \times J \to J$ were true, then for any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1 + 1/r$ we would deduce that $R : (A_0, A_1)_{p;K} \times (\mathbb{K}, \mathbb{K})_{q;J} \to (A_0, A_1)_{r;J}$ is bounded. This yields that $(A_0, A_1)_{p;K} \hookrightarrow (A_0, A_1)_{r;J}$. Take any $0 < \theta < 1$ and $1 \le s \le \infty$. By [14, Lemmata 3.2 and 4.2], we know that $(A_0, A_1)_{r;J} \hookrightarrow (A_0, A_1)_{\theta,s} \hookrightarrow (A_0, A_1)_{p;K}$. Therefore, we conclude that $(A_0, A_1)_{\theta,s} = (A_0, A_1)_{\mu,s}$ for any $0 < \theta \ne \mu < 1$, which is impossible (see [16, Theorem 3.1]).

Concerning Theorem 3.2, there is no similar result for K-spaces. In order to show it, we establish first an auxiliary result. For $n \in \mathbb{N}$, let ℓ_q^n be the space $\mathbb{K}^n$ with the ℓ_q-norm, and if $(\omega_j)_{j=1}^n$ is a positive n-tuple, write $\ell_q^n(\omega_j)$ for the corresponding weighted ℓ_q^n-space. We put $\ell_q^n(n^{1/q})$ for the space $\ell_q^n(\omega_j)$ if $\omega_j = n^{1/q}$ for $1 \le j \le n$.

LEMMA 3.7. *Let $n \in \mathbb{N}$ and $1 \le q \le \infty$. Then*

$$\ell_1^n(j2^{-j}) \hookrightarrow (\ell_1^n, \ell_1^n(2^{-j}))_{q;K}, \quad \ell_\infty^n(n^{1/q}) \hookrightarrow (\ell_\infty^n, \ell_\infty^n(2^j))_{q;K}$$

and the norms of the embeddings can be bounded from above with constants independent of n.

Proof. By [14, Remark 3.3] and [8, Lemma 7.2], we have $(\ell_1^n, \ell_1^n(2^{-j}))_{1;K} = \ell_1^n(j2^{-j})$ with equivalence of norms where the constants do not depend on n. Hence (2) implies that $\ell_1^n(j2^{-j}) \hookrightarrow (\ell_1^n, \ell_1^n(2^{-j}))_{q;K}$.

To prove the second embedding of the statement, note that $(\ell_\infty^n, \ell_\infty^n(2^j))_{q;K} = (\ell_\infty^n(2^j), \ell_\infty^n)_{q;K}$ and that

$$K(t, \xi; \ell_\infty^n(2^j), \ell_\infty^n) = \max_{1 \le j \le n} \min(2^j, t)|\xi_j|.$$

Hence, using again [14, Remark 3.3], we obtain

$$\|\xi\|^q_{(\ell_\infty^n, \ell_\infty^n(2^j))_{q;K}} \sim \sum_{m=1}^{\infty} 2^{-mq} \max_{1 \le j \le n} \min(2^{jq}, 2^{mq})|\xi_j|^q$$
$$= \sum_{m=1}^{n} \max_{1 \le j \le n} \min(2^{(j-m)q}, 1)|\xi_j|^q + \sum_{m=n+1}^{\infty} 2^{-mq} \max_{1 \le j \le n} \min(2^{jq}, 2^{mq})|\xi_j|^q = S_1 + S_2$$

where the constants in the equivalence do not depend on n. Next we estimate S_2. Let $k \le n$, we obtain

$$S_2 = \sum_{m=n+1}^{\infty} 2^{-mq} \max_{1\le j\le n} 2^{jq}|\xi_j|^q = \frac{2^{-(n+1)q}}{1-2^{-q}} \max_{1\le j\le n} 2^{jq}|\xi_j|^q$$
$$\sim \max_{1\le j\le n} 2^{(j-n)q}|\xi_j|^q \le \max_{1\le j\le n} \min(1, 2^{(j-k)q})|\xi_j|^q \le S_1.$$

Consequently,

$$\|\xi\|^q_{(\ell^n_\infty, \ell^n_\infty(2^j))_{q;K}} \sim \sum_{m=1}^{n} \max_{1\le j\le n} \min(2^{(j-m)q}, 1)|\xi_j|^q$$
$$\le \sum_{m=1}^{n} \max_{1\le j\le n} |\xi_j|^q = \max_{1\le j\le n} n|\xi_j|^q = \|\xi\|^q_{\ell_\infty(n^{1/q})}. \quad ■$$

COUNTEREXAMPLE 3.8. Take any $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1 + 1/r$, consider the couples $\overline{A} = (\ell^n_1, \ell^n_1(2^{-j}))$, $\overline{B} = (\ell^n_\infty, \ell^n_\infty(2^j))$, $\overline{C} = (\mathbb{K}, \mathbb{K})$, and let R be the bilinear operator defined by $R((\xi_j), (\eta_j)) = \sum_{j=1}^n \xi_j\eta_j$. It is easy to check that $R : (A_0 + A_1) \times (B_0+B_1) \to C_0+C_1$ is bounded, and the restrictions $R : A_j \times B_j \to C_j$ are also bounded, with norm 1 for $j = 0, 1$. If the bilinear theorem $K \times K \to K$ were true, using Lemma 3.7 there would be some $M < \infty$ such that

$$\big\|R : \ell^n_1(j2^{-j}) \times \ell^n_\infty(n^{1/q}) \to \mathbb{K}\big\| \le M$$

for every $n \in \mathbb{N}$. Take $\xi = (0, \dots, 0, 2^n/n)$ and $\eta = (0, \dots, 0, n^{-1/q})$. Since $\|\xi\|_{\ell^n_1(j2^{-j})} = 1$, $\|\eta\|_{\ell^n_\infty(n^{1/q})} = 1$ and $R(\xi, \eta) = 2^n/n^{1+1/q}$, it follows that $2^n/n^{1+1/q} \le M$ for every $n \in \mathbb{N}$ which is impossible.

4. Norm estimates. In this final section we compare norm estimates for bilinear operators with the norms of linear operators interpolated by the limiting methods. We start with an auxiliary result.

LEMMA 4.1. *Let $\overline{E} = (\mathbb{K}, \mathbb{K})$. Then $\overline{E}_{1;J} = \mathbb{K}$ and $\|\cdot\|_{1;J}$ coincides with $|\cdot|$.*

Proof. If $\lambda \in \mathbb{K}$, we can take the representation $\lambda = \sum_{m=-\infty}^{\infty} v_m$ with $v_m = 0$ for $m \ne 0$ and $v_0 = \lambda$. It follows that $\|\lambda\|_{1;J} \le |\lambda|$. Conversely, given any J-representation $\lambda = \sum_{m=-\infty}^{\infty} \lambda_m$ of λ, we have

$$|\lambda| \le \sum_{m=-\infty}^{\infty} |\lambda_m| \le \sum_{m=-\infty}^{\infty} \max(1, 2^{-m})J(2^m, \lambda_m).$$

Hence, $|\lambda| \le \|\lambda\|_{1;J}$. ■

Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(\overline{A}, \overline{B})$. Put $\overline{E} = (\mathbb{K}, \mathbb{K})$, and define the bilinear operator R by $R(\lambda, a) = \lambda Ta$. The operator R is bounded from $(\mathbb{K} + \mathbb{K}) \times (A_0 + A_1)$ into $B_0 + B_1$, and restrictions $R : \mathbb{K} \times A_j \to B_j$ are also bounded. It follows from Lemma 4.1 that for any $1 \le q \le \infty$ we have

$$\|T\|_{\mathcal{L}(\overline{A}_{q;K}, \overline{B}_{q;K})} = \big\|R : \overline{E}_{1;J} \times \overline{A}_{q;K} \to \overline{B}_{q;K}\big\|.$$

Whence, norm estimates for interpolated bilinear operators cannot be better than corresponding estimates for interpolated linear operators.

For the real method, it is well-known that if $T \in \mathcal{L}(\overline{A}, \overline{B})$ then

$$\|T\|_{\overline{A}_{\theta,q},\overline{B}_{\theta,q}} \leq \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}$$

(see, for example, [2, Theorem 3.1.2]). For limiting real methods, this estimate is no longer true. If $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, it was proved in [8, Theorem 7.9] that

$$\|T\|_{\overline{A}_{q;K},\overline{B}_{q;K}} \leq M\|T\|_{A_1,B_1}\left[1 + \max\left\{0, \log \frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}}\right\}\right],$$

where M does not depend on T, $\overline{A}$ or $\overline{B}$. However, for general couples, even this weaker estimate fails as has been shown by the authors in [14, Counterexample 3.6]. Next we establish two results which complement those of [14] and illustrate the poor norm estimates that are fulfilled for the limiting methods. Subsequently, we work with the continuous norm $\|\cdot\|_{\overline{A}_{q;K}}$ of the limiting K-space.

Proposition 4.2. *For any $s, t > 0$, there exist Banach couples $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ and an operator $T \in \mathcal{L}(\overline{A}, \overline{B})$ such that $\|T\|_{A_0,B_0} = s$, $\|T\|_{A_1,B_1} = t$ and*

$$\|T\|_{\overline{A}_{\infty;K},\overline{B}_{\infty;K}} = \max(s, t).$$

Proof. Let $B_0 = B_1 = \mathbb{K}$ with the usual norm $|\cdot|$. Take $A_0 = A_1 = \mathbb{K}$ normed with $\|\lambda\|_{A_0} = s^{-1}|\lambda|$ and $\|\lambda\|_{A_1} = t^{-1}|\lambda|$, respectively, and put $T\lambda = \lambda$. It is clear that $\|T\|_{A_0,B_0} = s$ and $\|T\|_{A_1,B_1} = t$. Since

$$\|\lambda\|_{\overline{B}_{\infty;K}} = 2K(1, \lambda; B_0, B_1) = 2|\lambda|$$

and

$$\|\lambda\|_{\overline{A}_{\infty;K}} = 2K(1, \lambda; A_0, A_1) = 2\min(s^{-1}, t^{-1})|\lambda|,$$

we derive

$$\|T\|_{\overline{A}_{\infty;K},\overline{B}_{\infty;K}} = \max(s, t),$$

as desired. ∎

We close the paper with the case $q < \infty$.

Theorem 4.3. *Let $1 \leq q < \infty$. Then*

$$\sup\left\{\|T\|_{\overline{A}_{q;K},\overline{B}_{q;K}} : \|T\|_{A_0,B_0} \leq s,\ \|T\|_{A_1,B_1} \leq t\right\} \sim \max(s, t),$$

where the supremum is taken over all Banach pairs $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ and all $T \in \mathcal{L}(\overline{A}, \overline{B})$ satisfying the stated conditions.

Proof. According to [12, Corollary 1.7],

$$\sup\{\|T\|_{\overline{A}_{q;K},\overline{B}_{q;K}} : \|T\|_{A_0,B_0} \leq s, \|T\|_{A_1,B_1} \leq t\} \sim sg(t/s),$$

where

$$g(\tau) = \sup_{\alpha\in(0,\infty)} \frac{\left\|\frac{\min(1,\alpha\tau\cdot)}{\max(1,\cdot)}\right\|_{L_q((0,\infty),dt/t)}}{\left\|\frac{\min(1,\alpha\cdot)}{\max(1,\cdot)}\right\|_{L_q((0,\infty),dt/t)}} = \sup_{\alpha\in(0,\infty)} C_{\alpha,\tau}.$$

Let us compute g. We start with the case $1/\tau < 1$. Then

$$\sup_{\alpha\in(0,\infty)} C_{\alpha,\tau} = \max\Big(\sup_{0<\alpha<1/\tau} C_{\alpha,\tau}, \sup_{1/\tau\le\alpha<1} C_{\alpha,\tau}, \sup_{\alpha\ge 1} C_{\alpha,\tau}\Big).$$

Let $0<\alpha<1/\tau$. Then

$$\begin{aligned}\Big(\int_0^\infty \Big[\frac{\min(1,\alpha\tau t)}{\max(1,t)}\Big]^q \frac{dt}{t}\Big)^{1/q} &= \Big(\int_0^1 (\alpha\tau t)^q \frac{dt}{t} + \int_1^{1/\alpha\tau} (\alpha\tau)^q \frac{dt}{t} + \int_{1/\alpha\tau}^\infty t^{-q}\frac{dt}{t}\Big)^{1/q} \\ &= \big(2/q - \log(\alpha\tau)\big)^{1/q}\alpha\tau,\end{aligned}$$

and

$$\begin{aligned}\Big(\int_0^\infty \Big[\frac{\min(1,\alpha t)}{\max(1,t)}\Big]^q \frac{dt}{t}\Big)^{1/q} &= \Big(\int_0^1 (\alpha t)^q \frac{dt}{t} + \int_1^{1/\alpha} \alpha^q \frac{dt}{t} + \int_{1/\alpha}^\infty t^{-q}\frac{dt}{t}\Big)^{1/q} \\ &= \big(2/q - \log(\alpha)\big)^{1/q}\alpha,\end{aligned}$$

so

$$\begin{aligned}\sup_{0<\alpha<1/\tau} C_{\alpha,\tau} &= \sup_{0<\alpha<1/\tau} \tau\Big[\frac{2/q-\log(\alpha\tau)}{2/q-\log\alpha}\Big]^{1/q} = \sup_{0<\alpha<1/\tau}\tau\Big[\frac{2/q-\log\alpha-\log\tau}{2/q-\log\alpha}\Big]^{1/q} \\ &= \sup_{0<\alpha<1/\tau}\tau\Big[1-\frac{\log\tau}{2/q-\log\alpha}\Big]^{1/q} = \tau.\end{aligned}$$

Now, let $1/\tau\le\alpha<1$. Then

$$\begin{aligned}\Big(\int_0^\infty \Big[\frac{\min(1,\alpha\tau t)}{\max(1,t)}\Big]^q \frac{dt}{t}\Big)^{1/q} &= \Big(\int_0^{1/\alpha\tau} (\alpha\tau t)^q \frac{dt}{t} + \int_{1/\alpha\tau}^1 \frac{dt}{t} + \int_1^\infty t^{-q}\frac{dt}{t}\Big)^{1/q} \\ &= \big(2/q + \log(\alpha\tau)\big)^{1/q},\end{aligned}$$

so in this case

$$C_{\alpha,\tau} = \Big[\frac{2/q+\log\alpha+\log\tau}{\alpha^q(2/q-\log\alpha)}\Big]^{1/q}.$$

We have

$$\frac{\partial C^q_{\alpha,\tau}}{\partial\alpha}(\alpha,\tau)=0 \iff \log\alpha = \frac{-q\log\tau\pm\sqrt{q^2\log^2\tau+4q\log\tau}}{2q}.$$

Since $\log\tau>0$, one of the roots is positive, and the other one is less than or equal to

$$\frac{-q\log\tau - q\log\tau}{2q} = \log(1/\tau).$$

This implies that the derivative does not change its sign on the interval $1/\tau\le\alpha<1$. Since

$$\frac{\partial C^q_{\alpha,\tau}}{\partial\alpha}(1,\tau) = \frac{\alpha^{q-1}}{(\alpha^q(2/q-\log\alpha))^2}\log 1/\tau < 0,$$

we deduce that $C_{\alpha,\tau}$ is decreasing on $[1/\tau,1]$, and therefore,

$$\sup_{1/\tau\le\alpha<1} C_{\alpha,\tau} = \tau\Big[\frac{2/q-\log\tau+\log\tau}{2/q+\log\tau}\Big]^{1/q} = \tau\Big[1-\frac{\log\tau}{2/q+\log\tau}\Big]^{1/q}.$$

In the case $\alpha \geq 1$, we have

$$\Big(\int_0^\infty \Big[\frac{\min(1,\alpha t)}{\max(1,t)}\Big]^q \frac{dt}{t}\Big)^{1/q} = \Big(\int_0^{1/\alpha} (\alpha t)^q \frac{dt}{t} + \int_{1/\alpha}^1 \frac{dt}{t} + \int_1^\infty t^{-q}\frac{dt}{t}\Big)^{1/q}$$
$$= (2/q + \log\alpha)^{1/q},$$

so

$$\sup_{\alpha\geq 1} C_{\alpha,\tau} = \sup_{\alpha\geq 1}\Big[\frac{2/q+\log(\alpha\tau)}{2/q+\log\alpha}\Big]^{1/q} = \sup_{\alpha\geq 1}\Big[1+\frac{\log\tau}{2/q+\log\alpha}\Big]^{1/q} = \Big[1+\frac{q\log\tau}{2}\Big]^{1/q}.$$

Therefore,

$$\sup_{0<\alpha<\infty} C_{\alpha,\tau} = \max\Big(\tau, \tau\Big(1-\frac{\log\tau}{2/q+\log\tau}\Big)^{1/q}, \Big(1+\frac{q\log\tau}{2}\Big)^{1/q}\Big).$$

It is easy to check that the second value in the maximum is less than or equal to τ. To compare the last term, put $f(\tau) = 2\tau^q - 2 - q\log\tau = 2\tau^q - 2 - \log\tau^q$ for $\tau \geq 1$. We have $f(1) = 0$ and $f'(\tau) = q\tau^{-1}(2\tau^q - 1) > 0$, so $f(\tau) \geq 0$, and therefore $\tau \geq (1 + \frac{q\log\tau}{2})^{1/q}$, that is, if $1/\tau < 1$, we have $g(\tau) = \tau$.

Next consider the case $1/\tau \geq 1$. Then

$$\sup_{\alpha\in(0,\infty)} C_{\alpha,\tau} = \max\Big(\sup_{0<\alpha\leq 1} C_{\alpha,\tau}, \sup_{1<\alpha<1/\tau} C_{\alpha,\tau}, \sup_{\alpha\geq 1/\tau} C_{\alpha,\tau}\Big).$$

If $0 < \alpha \leq 1$, using what we already have, we get

$$\sup_{0<\alpha\leq 1} C_{\alpha,\tau} = \sup_{0<\alpha\leq 1} \tau\Big[\frac{2/q-\log(\alpha\tau)}{2/q-\log\alpha}\Big]^{1/q} = \sup_{0<\alpha\leq 1}\tau\Big[1-\frac{\log\tau}{2/q-\log\alpha}\Big]^{1/q}$$
$$= \tau\Big[1-\frac{q\log\tau}{2}\Big]^{1/q}.$$

If $1 < \alpha < 1/\tau$, we have

$$\sup_{1<\alpha<1/\tau} C_{\alpha,\tau} = \sup_{1<\alpha<1/\tau} \alpha\tau\Big[\frac{2/q-\log(\alpha\tau)}{2/q+\log\alpha}\Big]^{1/q},$$

and since

$$\frac{\partial C^q_{\alpha,\tau}}{\partial\alpha}(\alpha,\tau) = \frac{(\tau\alpha)^{q-1}\tau}{(2/q+\log\alpha)^2}\big(-q\log\alpha(\log\alpha - \log(1/\tau)) + \log(1/\tau)\big) > 0,$$

we obtain

$$\sup_{1<\alpha<1/\tau} C_{\alpha,\tau} = \Big[\frac{2/q}{2/q+\log(1/\tau)}\Big]^{1/q} = \Big[1+\frac{q\log\tau}{2-q\log\tau}\Big]^{1/q}.$$

Finally, if $\alpha \geq 1/\tau$, we have

$$\sup_{\alpha\geq 1/\tau} C_{\alpha,\tau} = \sup_{\alpha\geq 1/\tau}\Big[\frac{2/q+\log(\alpha\tau)}{2/q+\log\alpha}\Big]^{1/q} = \sup_{\alpha\geq 1/\tau}\Big[1+\frac{\log\tau}{2/q+\log\alpha}\Big]^{1/q} = 1,$$

and therefore

$$\sup_{0<\alpha<\infty} C_{\alpha,\tau} = \max\Big(\tau\Big[1-\frac{q\log\tau}{2}\Big]^{1/q}, \Big[1+\frac{q\log\tau}{2-q\log\tau}\Big]^{1/q}, 1\Big).$$

Clearly, the second term is less than or equal to 1. To estimate the first one, write $h(\tau) = \tau^q(1 - \frac{q}{2}\log\tau) - 1$ for $0 < \tau \leq 1$. We have $h(1) = 0$ and $h'(\tau) = \frac{q}{2}(1 - q\log\tau) > 0$, so $h(\tau) \leq 0$ whenever $0 < \tau \leq 1$. This yields that

$$\sup_{0<\alpha<\infty} C_{\alpha,\tau} = \max\Big(\tau\Big[1 - \frac{q\log\tau}{2}\Big]^{1/q}, \Big[1 + \frac{q\log\tau}{2 - q\log\tau}\Big]^{1/q}, 1\Big) = 1.$$

Consequently, $g(\tau) = \max(1, \tau)$, which completes the proof. ∎

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