

CANTOR–BERNSTEIN THEOREMS FOR ORLICZ SEQUENCE SPACES

CARLOS E. FINOL

*Escuela de Matemáticas, Universidad Central de Venezuela
P.O.Box 48059, Caracas 1041-A, Venezuela
E-mail: carlos.finol@ciens.ucv.ve*

MARCOS J. GONZÁLEZ

*Departamento de Matemáticas, Universidad Simón Bolívar
Apartado 89000, Caracas 1080-A, Venezuela
E-mail: mago@usb.ve*

MAREK WÓJTOWICZ

*Instytut Matematyki, Uniwersytet Kazimierza Wielkiego
Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland
E-mail: mwojt@ukw.edu.pl*

Abstract. For two Banach spaces X and Y , we write $\dim_\ell(X) = \dim_\ell(Y)$ if X embeds into Y and vice versa; then we say that X and Y have the same linear dimension. In this paper, we consider classes of Banach spaces with symmetric bases. We say that such a class \mathcal{F} has the Cantor–Bernstein property if for every $X, Y \in \mathcal{F}$ the condition $\dim_\ell(X) = \dim_\ell(Y)$ implies the respective bases (of X and Y) are equivalent, and hence the spaces X and Y are isomorphic. We prove (Theorems 3.1, 3.3, 3.5) that the class of Orlicz sequence spaces generated by regularly varying Orlicz functions is of this type. This complements some results in this direction obtained earlier by S. Banach (Proposition 1.1), L. Drewnowski (Proposition 1.2), and M. J. Gonzalez, B. Sari and M. Wójtowicz (Theorem 1.4). Our theorems apply to large families of concrete Orlicz spaces.

1. Introduction. In what follows, we use the notation from the abstract.

The study of the comparison of the linear dimension between Banach spaces goes back to S. Banach himself (see [B, Chap. XII, p. 193]). For example, in [B, Chap. XII,

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Théorème 1, p. 194] it is proved that each closed subspace of c_0 and ℓ_p , $1 \leq p < \infty$, with strictly lower dimension is finite-dimensional. Banach also proved the following result (see [B, Chap. XII, Lemme, p. 202, and Théorèmes 4–6, p. 203]):

PROPOSITION 1.1. *If $L_p = L^p[0, 1]$ is isomorphic to a subspace of $L_q = L^q[0, 1]$, where $p > 1$, $q > 1$, then $q \leq p \leq 2$ or $2 \leq p \leq q$. Consequently,*

- (a) *If L^p and L^q have the same linear dimension, then $p = q$.*
- (b) *If $1 < p < 2 < q$, then neither L^p embeds isomorphically into L^q nor L^q embeds isomorphically into L^p .*
- (c) *If $1 < p \neq 2$, then L_2 has strictly lower linear dimension than L_p .*

If X and Y are Banach spaces having the same linear dimension, they need not be isomorphic. These remarks go back to S. Banach and S. Mazur [BM]: in 1932 they proved that $C[0, 1]$ and $C[0, 1] \oplus \ell_1$ have the same linear dimension, yet they are not isomorphic because their duals do not have the same linear dimension. We also mention that Plichko and Wójtowicz [PW] have constructed a Banach space X having the same linear dimension as its bidual X^{**} but not isomorphic to it. We conclude then that *the concept of linear dimension in general is stronger than the concept of linear isomorphism.*

In 1987, L. Drewnowski [D] proved the following remarkable result, which motivates our studies of a similar property for *countable* symmetric bases.

PROPOSITION 1.2. *Let X and Y be two nonseparable Banach spaces with uncountable symmetric bases $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\delta)_{\delta \in \Delta}$, respectively. Then the following conditions are equivalent:*

- (i) *X and Y have the same linear dimension,*
- (ii) *X and Y are isomorphic,*
- (iii) *the bases $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\delta)_{\delta \in \Delta}$ are equivalent, that is, there exists a bijection $f : \Gamma \rightarrow \Delta$ such that the linear map T determined by the condition $T(x_\gamma) = y_{f(\gamma)}$, for every $\gamma \in \Gamma$, extends to an isomorphism from X onto Y .*

Rodriguez-Salinas [R] studied “large” Orlicz spaces $\mathfrak{h}_\varphi(\Gamma)$ of this type, while Finol and Wójtowicz [FW2] considered linear dimension problems for long symmetric basic sequences.

In this paper, we are interested in indicating families of symmetric bases having the following property:

DEFINITION 1.3. Given a family of symmetric bases \mathcal{F} , we say that \mathcal{F} has the *Cantor–Bernstein property* whenever, for every (x_n) and (y_n) in \mathcal{F} spanning Banach spaces X and Y , respectively, X and Y have the same linear dimension if and only if (x_n) is equivalent to (y_n) . Then we also say that the family \mathcal{F}_0 , of the Banach spaces X that each member of the family \mathcal{F} spans, has *the Cantor–Bernstein property*.

One should not confuse the above property with the Schröder–Bernstein property discussed by Casazza [C]: *A Banach space X is said to have the Schröder–Bernstein property whenever for every complemented subspace Y of X for which X is also isomorphic to a complemented subspace of Y , the space X is isomorphic to Y .* The long-standing open

question of whether or not every Banach space possesses this property has been solved by T. Gowers [Gow] in the negative.

Let us notice that the following result in the class of Orlicz sequence spaces, given in [GSW, Theorem 5.1], holds.

THEOREM 1.4. *Let \mathcal{S} be either the family of all super-multiplicative Orlicz functions satisfying the Δ_2 -condition at zero or the family of all sub-multiplicative Orlicz functions. Then the family of all unit vector bases of Orlicz spaces ℓ_M such that $M \in \mathcal{S}$ has the Cantor–Bernstein property.*

Let us recall that a function f defined on the interval $[0,1]$ is said to be *sub-* [resp., *super-*] *multiplicative* whenever $f(st) \leq f(s)f(t)$ [resp., $f(st) \geq f(s)f(t)$], for every s, t in $[0, 1]$. The examples of such functions can be found in [FM], [FW] and [GSW].

REMARK 1.5. If we remove in Theorem 1.4 the assumption on Δ_2 -condition on M , then the statement remains true if we replace ℓ_M by \mathfrak{h}_M everywhere.

In order to illustrate the ideas that have motivated this investigation we provide a proof for Theorem 1.4, which is based on the geometry of Orlicz sequence spaces as developed by Lindberg [L], Lindenstrauss and Tzafriri [LT1] and, more recently, by Kamińska and Raynaud [KR]. Such a proof is different from the one given in [GSW], which is based on the notion of semi-homogeneity instead.

In 1971, Lindenstrauss and Tzafriri [LT, Theorem 2] defined a large family of Orlicz sequence spaces having (up to equivalence) a unique symmetric basis.

Our first result, stated as Theorem 3.1, asserts that such a family actually has the Cantor–Bernstein property.

Next we focus on finding applications for Theorem 3.1 to the framework of function theory.

DEFINITION 1.6 (see [M, p. 11] and [BGT, Section 2.3, p. 83]). A φ -function M (i.e., M is increasing and continuous on $[0, \infty)$ with $M(0) = 0$ and $\lim_{s \rightarrow \infty} M(s) = \infty$), is said to be *regularly varying at zero* (resp., *at infinity*), whenever the limit $f(s) = \lim_{t \rightarrow 0^+} \frac{M(ts)}{M(t)}$ (resp., $\lim_{t \rightarrow \infty} \frac{M(ts)}{M(t)}$) exists and is *nonzero*, for each $s > 0$. For example, $M_p(t) := t^p$ is regularly varying for every $p > 0$. If the limit $f(s) = \lim_{t \rightarrow 0^+} \frac{M(ts)}{M(t)}$ (resp., $\lim_{t \rightarrow \infty} \frac{M(ts)}{M(t)}$) exists and

$$f(t) = M_\infty(t) := \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1, \end{cases}$$

then the function M is called *rapidly varying at zero* (resp., *at infinity*). Notice that $\lim_{p \rightarrow \infty} M_p(t) = M_\infty(t)$, for every $t \geq 0$.

In Theorem 3.3 we establish that the family of Orlicz sequence spaces generated by regularly (or rapidly) varying Orlicz functions has, in fact, the Cantor–Bernstein property.

In Theorem 3.5 we strengthen Theorem 3.3 by showing that the strong Cantor–Bernstein property established in Theorem 3.3 is hereditary: *the family of symmetric basic*

sequences in an Orlicz sequence space generated by a regularly varying Orlicz function has the Cantor–Bernstein property.

Finally, in the last section, we show how our Theorems 3.1, 3.3 and 3.5 apply to large families of concrete Orlicz sequence spaces. For example, Theorem 3.3 applies to the family of Orlicz functions $M_{p,a,b}$, having each principal value $x^p |\log(bx)|^a$, studied in [GSW] (see Corollary 5.1).

2. Preliminaries. We follow the notation and terminology used in the monographs by Lindenstrauss and Tzafriri [LT1], and Singer [S]. For the convenience of the reader we recall some definitions.

The bases (x_n) and (y_n) of Banach spaces X and Y , respectively, are said to be *equivalent* whenever there exists a linear isomorphism T from X onto Y such that $Tx_n = y_n$, for every $n \geq 1$. Equivalently, there exists a positive constant K such that, for every integer $N \geq 1$ and every scalar sequence $(\alpha_n)_{n=1}^\infty$,

$$\frac{1}{K} \left\| \sum_{n=1}^N \alpha_n x_n \right\| \leq \left\| \sum_{n=1}^N \alpha_n y_n \right\| \leq K \left\| \sum_{n=1}^N \alpha_n x_n \right\|.$$

A basis (x_n) of a Banach space X is said to be *symmetric* if every permutation of (x_n) is equivalent to (x_n) . An unconditional basis which is equivalent to each of its subsequences is called *subsymmetric*. Every symmetric basis is subsymmetric. We shall use the fact that *every subsymmetric basis is semi-normalized* (that is, the sequence of norms is bounded and bounded away from zero).

A basis is said to be *perfectly homogeneous* if it is equivalent to each of its normalized block basic sequences. *A basis (x_n) is perfectly homogeneous if and only if (x_n) is equivalent to the unit vector basis of either the Banach space c_0 or some of the Banach spaces ℓ_p , with $1 \leq p < \infty$* (see Zippin [Z]; cf. [LT]).

The examples of Banach spaces having symmetric bases are provided by Orlicz sequence spaces.

An *Orlicz function* M is a non-negative, non-decreasing, convex function defined on $[0, \infty)$ and satisfying $M(0) = 0$. Let $\mathbb{R}^\mathbb{N}$ denote the space of all scalar sequences. We define the function $\varrho_M : \mathbb{R}^\mathbb{N} \rightarrow [0, \infty]$ by the formula $\varrho_M(a) = \sum M(|a_n|)$, where $a = \{a_n\}$ is a scalar sequence. The *Orlicz sequence space* ℓ_M is defined as the linear set

$$\ell_M := \{a \in \mathbb{R}^\mathbb{N} : \varrho_M(a/\lambda) < \infty \text{ for some } \lambda > 0\},$$

and equipped with the norm $\|a\|_M := \inf\{\lambda > 0 : \varrho_M(a/\lambda) \leq 1\}$ it becomes a Banach space. By \mathfrak{h}_M we denote the closed subspace of ℓ_M defined as

$$\mathfrak{h}_M := \{a \in \mathbb{R}^\mathbb{N} : \varrho_M(a/\lambda) < \infty \text{ for all } \lambda > 0\},$$

where the unit vectors form a symmetric basis of \mathfrak{h}_M (see e.g. [LT1, Proposition 4.a.2., p. 138]).

If $M(t) > 0$ for all $t > 0$ then M is called *non-degenerate*. An Orlicz function M is degenerate if and only if the unit vector basis of \mathfrak{h}_M is equivalent to the unit vector basis of c_0 (see [L, Proposition 2.14]).

The function M fulfils the so-called Δ_2 -condition at 0 whenever there exist constants $K, t_0 > 0$ such that $0 < M(2t) \leq K \cdot M(t)$, for every $t \in [0, t_0]$. In this case, the spaces ℓ_M and \mathfrak{h}_M coincide, M is necessarily non-degenerate and the unit vectors form a boundedly complete basis for $\mathfrak{h}_M = \ell_M$. More precisely, we have the following (see [LT1, Proposition 4.a.4]):

PROPOSITION 2.1. *Let M be an Orlicz function M . Then the following four conditions are equivalent:*

- (i) *The function M fulfils the Δ_2 -condition at zero.*
- (ii) *$\ell_M = \mathfrak{h}_M$.*
- (iii) *The unit vectors form a boundedly complete symmetric basis of \mathfrak{h}_M .*
- (iv) *The space ℓ_M is separable.*

Let M, N be two Orlicz functions. We say that M dominates N at 0 if there exist constants $a, t_0 > 0$, such that $M(at) \geq N(t)$ for every $t \in [0, t_0]$. The functions are equivalent at 0 if M dominates N and N dominates M . The following statement is well-known [LT1, Proposition 4.a.5].

PROPOSITION 2.2. *Let M, N be two Orlicz functions. Then the following conditions are equivalent:*

- (i) *The unit vector bases of \mathfrak{h}_M and \mathfrak{h}_N are equivalent.*
- (ii) *M and N are equivalent, that is, there exist constants $a, b > 0$ and t_0 such that $M(at) \leq N(t) \leq M(bt)$, for every $0 < t \leq t_0$.*
- (iii) *There exist constants $A, B, a, b > 0$ and $t_0 > 0$ such that $AM(at) \leq N(t) \leq BM(bt)$, for every $0 < t \leq t_0$.*

The properties of regularly varying measurable functions and regularly varying φ -functions at infinity were studied by Matuszewska [M] (see also Bingham, Goldie and Teugels [BGT]). Let us notice that a φ -function $M(t)$ is regularly varying at zero if and only if $\widetilde{M}(t) := M(t^{-1})^{-1}$ is regularly varying at infinity. Indeed, for every $s, u > 0$, we have the equality

$$f(u) = \lim_{s \rightarrow \infty} \frac{\widetilde{M}(su)}{\widetilde{M}(s)} = \lim_{s \rightarrow \infty} \frac{M(1/s)}{M(1/su)} \stackrel{1/t=su}{=} \lim_{t \rightarrow 0^+} \frac{M(tu)}{M(t)},$$

whenever either one of the limits involved exists. The same remarks also apply to rapidly varying functions. This fact allows us to translate the statement (at infinity) already established in [M, BGT] to the present case (at zero).

For example, the following test for an Orlicz function to be regularly varying, in terms of its (right or left) derivative (which is also given implicitly in the proof of the Corollary to Theorem 2 in [LT]) is true:

PROPOSITION 2.3. *Let M be a non-degenerate Orlicz function such that the limit $p = \lim_{t \rightarrow 0^+} \frac{tM'(t)}{M(t)}$ exists and is finite. Then M is a regularly varying function with index p . If, on the other hand, $\lim_{t \rightarrow 0^+} \frac{tM'(t)}{M(t)} = \infty$, then M is a rapidly varying function.*

The geometry of Orlicz sequence spaces has been developed in [L], [LT1] and, more recently, in [KR]. We follow the approach suggested by Kamińska and Raynaud [KR].

An Orlicz function M can be regarded as an element of the cube $[0, \infty]^{[0, \infty]}$, where $[0, \infty]$ is the one-point compactification of $[0, \infty)$. By the Tychonoff theorem, $[0, \infty]^{[0, \infty]}$ is a compact Hausdorff space under the product topology, which coincides with the topology of pointwise convergence. Given a non-degenerate Orlicz function M and $\lambda > 0$, we denote by M_λ the non-degenerate Orlicz function $M_\lambda(t) := \frac{M(\lambda t)}{M(\lambda)}$, for every $t \in [0, \infty)$. Let us consider the following subsets of $[0, \infty]^{[0, \infty]}$, for $0 < \Lambda \leq \infty$: set

$$C_{M, \Lambda}^0 := \text{conv } E_{M, \Lambda}^0 \quad (1)$$

where $E_{M, \Lambda}^0 := \{M_\lambda : 0 < \lambda < \Lambda\}$, and set

$$E_{M, \Lambda} := \overline{E_{M, \Lambda}^0} = \overline{\{M_\lambda : 0 < \lambda < \Lambda\}}, \quad E_M := \bigcap_{\Lambda > 0} E_{M, \Lambda} \quad (2)$$

$$C_{M, \Lambda} := \overline{C_{M, \Lambda}^0} = \overline{\text{conv } E_{M, \Lambda}}, \quad C_M = \bigcap_{\Lambda > 0} C_{M, \Lambda}. \quad (3)$$

where the closure is taken with respect to the product topology. Then the closed subsets $C_{M, \Lambda}$ and $E_{M, \Lambda}$ of $[0, \infty]^{[0, \infty]}$ are non-empty compact sets and, consequently, the intersections C_M and E_M are non-empty and compact as well. Since the evaluation at each point $t \in [0, \infty)$ is an open continuous surjection from $[0, \infty]^{[0, \infty]}$ onto $[0, \infty]$ (that is, for every $t \in [0, \infty]$, the projection on the t -th coordinate $f \mapsto f(t)$ is a continuous open surjection), we have the following tests which we state here for future references.

LEMMA 2.4. *Let M be a non-degenerate Orlicz function.*

- (a) *If $N \in E_{M, \lambda}$ then, for every fixed $t > 0$, there exists a sequence $\{\lambda_n\} \subset (0, \lambda)$ such that $\lim_{n \rightarrow \infty} \frac{M(\lambda_n t)}{M(\lambda_n)} = N(t)$.*
- (b) *If $N \in C_{M, \lambda}$ then, for every fixed $t > 0$, there exists a sequence of functions $\{M_n\} \subset C_{M, \lambda}^0$ such that $N(t) = \lim_{n \rightarrow \infty} M_n(t)$.*

Notice that every function N in $C_{M, \infty}$ is a Young function (i.e., a non-negative, non-decreasing, convex function from $[0, \infty)$ into $[0, \infty]$) fixing at least three points in $[0, \infty]$, that is, $N(0) = 0$, $N(1) = 1$ and $N(\infty) = \infty$. Hence, N is finite at least on $[0, 1]$. Since N is convex, it is continuous on $(0, 1)$. Also, the inequality $0 \leq \frac{M(\lambda t)}{M(\lambda)} \leq t$ implies that $0 \leq N(t) \leq t$, and hence N is continuous at 0, whence on $[0, 1)$. Consequently, $N \in C[0, s]$, for every $0 < s < 1$. In other words, N belongs to $C[0, 1)$.

As we have mentioned above, the set $C_{M, 1}$ consists entirely of Young functions. It turns out that, for each function $N \in C_{M, 1}$, the associated Orlicz sequence space \mathfrak{h}_N is isomorphic to some subspace of ℓ_M (see [LT1, Theorem 4.a.8]). The converse is also true and is based on the following result (see [LT1, Proposition 4.a.7]):

PROPOSITION 2.5 (K. Lindberg). *Let \mathfrak{h}_M be an Orlicz sequence space, and let $(u_n)_{n=1}^\infty$ be a normalized block basic sequence of the unit vector basis of \mathfrak{h}_M . Then there exists a subsequence $(u_{n_k})_{k=1}^\infty$ of $(u_n)_{n=1}^\infty$, which is equivalent to the unit vector basis of \mathfrak{h}_N , for some (possibly degenerate) Orlicz function $N \in C_{M, 1}$. Moreover, the function N is the pointwise limit of a sequence $\{M_k\} \subset C_{M, 1}$, where $M_k \in C_{M, \|u_{n_k}\|_\infty}^0$, for every $k \geq 1$.*

Let M be a non-degenerate Orlicz function. Let us define a map $f_M : \mathcal{I} \rightarrow E_{M,1}$, where $\mathcal{I} := (0, 1]$, as follows: for every $\lambda \in (0, 1]$, we set $f_M(\lambda)(t) = \frac{M(\lambda t)}{M(\lambda)}$, for every $t \in \mathbb{R}^+$. Then the map f_M is continuous on $(0, 1]$, with respect to the topology of pointwise convergence in $E_{M,1}$, that is, the topology induced on $E_{M,1}$ by the product topology τ in $[0, \infty]^{[0, \infty)}$. Since $E_{M,1}$ is a compact set, we may extend the function $f_M(\lambda)$ to a continuous function F_M defined on the Stone–Čech compactification of \mathcal{I} , denoted by $\beta\mathcal{I}$. The advantage of choosing this compactification is that it guarantees the extension F_M exists and is onto (i.e. surjective). That is, *the range of F_M is all of $E_{M,1}$* . For each $\omega \in \beta\mathcal{I}$, we let $F_M(\omega) = M_\omega$. On the other hand, for every $\lambda \in (0, 1]$, each $f_M(\lambda)$ is a normalized non-degenerate Orlicz function and belongs to the Fréchet space $C[0, 1]$ (its topology is generated by the sequence of seminorms $p_k(f) := \sup_{t \in [0, k/(k+1)]} |f(t)|$, $k = 1, 2, \dots$).

Since $X = C[0, 1]$ is locally convex, the space X^* of continuous linear functionals on X separates the points in $C[0, 1]$. Moreover, the collection of $\{f_M(\lambda) : \lambda \in (0, 1]\}$ is equicontinuous on $[0, s)$, for every $0 < s < 1$, and bounded by the function $\text{Id}(t) = t$. Hence, on one hand, the restriction $N|_{[0,1]}$ belongs to $C[0, 1]$, for every $N \in E_{M,1}$, and, on the other hand, $E_{M,1}|_{[0,1]}$ is a compact subset. Indeed, if we recall that $C[0, 1]$ is metrizable, we only need to verify that $E_{M,1}|_{[0,1]}$ is sequentially compact, but this follows from the fact that if $f_n : I \rightarrow \mathbb{R}$ is a sequence of convex functions converging to a finite limit function f on I , then f is convex and, moreover, the convergence is uniform on any closed subinterval of I° , the interior of I (see [RV], Theorem D, p. 16) which implies that $f \in C([0, 1])$. Consequently, by the Krein–Smulyan Theorem, we have the following result which is included implicitly in the proof of [LT1, Theorem 4.a.8]:

PROPOSITION 2.6. *Let M be a non-degenerate Orlicz function. Then $N \in C_{M,1} = \text{conv } E_{M,1}$ if and only if there exists a Borel probability measure μ over $\beta\mathcal{I}$ such that $N(t) = \int_{\beta\mathcal{I}} M_\omega(t) d\mu(\omega)$, for every $t \in [0, 1)$.*

3. Main results. Let X be a Banach space with a subsymmetric basis (x_n) . We say that

SCB The basis (x_n) has the *Strong Cantor–Bernstein property* (SCB property, for short) whenever, for every Banach space Y with a subsymmetric basis (y_n) , X and Y have the same linear dimension if and only if (x_n) is equivalent to (y_n) .

USB The space X is said to have *unique subsymmetric basis* (USB, for short) if, for every Banach space Y having a subsymmetric basic sequence (y_n) , the isomorphism of the spaces X and Y implies that (x_n) is equivalent to (y_n) .

Notice that *the family of all subsymmetric basic sequences in X has the Cantor–Bernstein property if and only if every subsymmetric basic sequence in X has the strong Cantor–Bernstein property*. Notice also that the SCB property implies the USB property. The problem of uniqueness of symmetric basis have received some attention in the past (see, e.g. [ACL], [CL1], [LT1], [Tz]).

Our main results report on certain families of Orlicz sequence spaces having the Cantor–Bernstein property.

The following theorem strengthens Theorem 2 from [LT].

THEOREM 3.1. *Let M be an Orlicz function fulfilling the Δ_2 -condition at zero and having the additional property that C_M does not contain any function equivalent to M . Then the corresponding unit vector basis of the Orlicz sequence space ℓ_M has the Strong Cantor–Bernstein property and, consequently, it is (up to equivalence) the unique symmetric basis for ℓ_M .*

REMARK 3.2. If, in Theorem 3.1, we remove the assumption on the Δ_2 -condition, then the statement remains true if we replace ℓ_M by \mathfrak{h}_M everywhere.

The examples of Orlicz functions satisfying the hypotheses of Theorem 3.1 are provided by regularly (or rapidly) varying functions which are not equivalent to any multiplicative function. These are the Orlicz functions for which the set C_M reduces to a singleton (see Lemma 4.3 below).

As a consequence of Theorem 3.1, we obtain the following:

THEOREM 3.3. *Let M and N be two regularly varying Orlicz functions. Then the following four conditions are equivalent:*

- (i) $\dim_\ell(\ell_M) = \dim_\ell(\ell_N)$.
- (ii) ℓ_M and ℓ_N are isomorphic.
- (iii) The unit vector bases of ℓ_M and ℓ_N are equivalent.
- (iv) there are constants $A, B, a, b, t_0 > 0$ such that

$$AN(at) \leq M(t) \leq BN(bt), \quad \text{for every } 0 < t \leq t_0.$$

REMARK 3.4. If, in Theorem 3.3, we replace the assumption on M and N to be regularly varying by the assumption to be rapidly varying, then the statement remains true if we replace ℓ_M and ℓ_N , respectively, by \mathfrak{h}_M and \mathfrak{h}_N everywhere.

Theorem 3.3 shows us that, in a class of Orlicz sequence spaces, the linear dimension can be expressed in the terms of equivalence of functions.

Once we have established these results, by using a well-known representation theorem for Orlicz functions in $C_{M,1}$, it will be relatively easy to prove that every function in $C_{M,1}$ is regularly varying whenever M itself is. Consequently, we will be able to prove the following:

THEOREM 3.5. *Let M be an Orlicz function which is equivalent to a regularly (resp., rapidly) varying function. Then, for every $N \in C_{M,1}$, the unit vector basis of ℓ_N (resp., \mathfrak{h}_M) has the Strong Cantor–Bernstein property.*

In particular, for every $N \in C_{M,1}$, the space ℓ_N (resp., \mathfrak{h}_M) has (up to equivalence) a unique symmetric basis.

4. The proofs. The following lemma allows us to reduce the problem of the comparison of linear dimensions of two Banach spaces having subsymmetric bases to the study of the equivalence relation between one of the bases and its ‘large’ subsymmetric block basic sequences.

LEMMA 4.1. *Let X and Y be two Banach spaces having subsymmetric bases (x_n) and (y_n) , respectively. Then the following conditions are equivalent:*

- (a) *The space X and Y have the same linear dimension.*
- (b) *The basis (x_n) is equivalent to a normalized block vector basis of (y_n) and, vice versa, the basis (y_n) is equivalent to a normalized block vector basis of (x_n) .*

We only sketch the proof. Assume (a). First recall that every subsymmetric basis is semi-normalized. According to [BP1, Theorem 3] and [BP2], either (x_n) is equivalent to the unit vector basis of ℓ_1 , or some subsequence of (x_n) , and hence (x_n) itself, is equivalent to a normalized unit vector basis of (y_n) . Due to the symmetry of the assumptions, a similar assertion holds for (y_n) and (x_n) interchanged. Therefore, if either (x_n) or (y_n) is equivalent to the unit vector basis of ℓ_1 , then they are equivalent to each other because the unit vector basis of ℓ_1 is perfectly homogeneous. The remaining case is the statement of (b).

In order to identify trivial cases, we will need the following particular case of Zippin's theorem which is a direct consequence of [GSW, Theorem 3.1].

LEMMA 4.2. *Let M be a non-degenerate Orlicz function with $M(1) = 1$. The following three conditions are equivalent:*

- (a) *The unit vector basis of \mathfrak{h}_M is perfectly homogeneous.*
- (b) *The function M is equivalent to a multiplicative Orlicz function.*
- (c) *There exists a constant $D \geq 1$ such that for every $s, t \in [0, 1/D]$*

$$M\left(\frac{st}{D}\right) \leq M(s)M(t) \leq M(Dst).$$

In the case M fulfils the Δ_2 -condition, conditions (a)–(c) are equivalent to:

- (c') *There exists a constant $C \geq 1$ such that, for every $s, t \in [0, 1]$,*

$$\frac{M(st)}{C} \leq M(s)M(t) \leq CM(st).$$

We shall also apply the following fact which allows us to compute the set C_M when M is either regularly varying or rapidly varying (see the hypotheses with those of Lemma 4.4 below).

LEMMA 4.3. *Let M be a non-degenerate Orlicz function such that the limit $f(t) = \lim_{s \rightarrow 0^+} \frac{M(st)}{M(s)}$ exists, for every $t \geq 0$. Then*

$$C_M := \bigcap_{0 < \lambda \leq 1} C_{M, \lambda} = \{M_p(t)\},$$

where $p = \lim_{s \rightarrow 0^+} \frac{\log M(s)}{\log s}$.

For the proof of Lemma 4.3 we will need the following useful fact. Its proof is straightforward, and we state it here for future reference.

LEMMA 4.4. *Let M be an Orlicz function such that the limit $\lim_{s \rightarrow 0^+} \frac{M(st)}{M(s)} = f(t)$ exists. Then we have the following mutually exclusive possibilities:*

- (a) *either $f(s) > 0$, for some $0 < s < 1$, in which case M is a regularly varying function and $f(t) = t^p$, $t \geq 0$, where $p = \lim_{s \rightarrow 0^+} \frac{sM'(s)}{M(s)}$,*
- (b) *or $f(s) = 0$, for some $0 < s < 1$, in which case M is rapidly varying, that is, $f(t) = M_\infty(t)$, $t \geq 0$, where $M_\infty(t)$ denotes the pointwise limit of $M_p(t) = t^p$ as $p \rightarrow \infty$.*

Proof of Lemma 4.3. First recall that $C_{M,\lambda}$ is a compact non-empty subset of the cube $[0, \infty]^{[0, \infty)}$ (endowed with the product topology), consisting entirely of Young functions. Consequently, $C_M = \bigcap_{0 < \lambda \leq 1} C_{M,\lambda}$ is a compact non-empty subset of $[0, \infty]^{[0, \infty)}$ as well and, moreover, it contains a Young function $N \in C_M$. We claim that N equals f , that is, that $C_M = \{f\}$. Let us fix $t \geq 0$ and let us prove that $N(t) = f(t)$.

Indeed, on one hand, since the limit $\lim_{s \rightarrow 0^+} \frac{M(ts)}{M(s)} = f(t)$ exists for every $\varepsilon > 0$, there is $s_0 > 0$ such that

$$\left| f(t) - \frac{M(st)}{M(s)} \right| < \varepsilon \quad (4)$$

for every $0 < s < s_0$.

On the other hand (since $N \in C_M \subset C_{M,s_0}$) according to Lemma 2.4, for every fixed $t > 0$, there is a sequence of functions $\{M_n\} \subset C_{M,s_0}^0$ such that $N(t) = \lim_{n \rightarrow \infty} M_n(t)$. From the definition of C_{M,s_0}^0 (see (1)) it follows that, for each $n \geq 1$, there exist $\alpha_1, \dots, \alpha_m \geq 0$, with $\alpha_1 + \dots + \alpha_m = 1$, and $s_1, \dots, s_m \in (0, s_0)$ such that

$$M_n(t) = \sum_{k=1}^m \alpha_k \frac{M(s_k t)}{M(s_k)}. \quad (5)$$

Hence we obtain

$$\begin{aligned} |f(t) - M_n(t)| &= \left| f(t) - \sum_{k=1}^m \alpha_k \frac{M(s_k t)}{M(s_k)} \right| = \left| \sum_{k=1}^m \alpha_k \left[f(t) - \frac{M(s_k t)}{M(s_k)} \right] \right| \\ &\leq \sum_{k=1}^m \alpha_k \left| f(t) - \frac{M(s_k t)}{M(s_k)} \right| < \sum_{k=1}^m \alpha_k \varepsilon = \varepsilon; \end{aligned}$$

by taking limits as $n \rightarrow \infty$, we get that $|f(t) - N(t)| < \varepsilon$, for every $t \geq 0$. Since $\varepsilon > 0$ is arbitrary, we deduce that $N(t) = f(t)$ as claimed.

We have thus proved that $C_M = \{f(t)\}$. Finally, according to Lemma 4.4, either

- (a) $f(s) > 0$, for some $0 < s < 1$; in this case M is a regularly varying function and $f(t) = t^p$, for every $t \geq 0$, where $p = \lim_{s \rightarrow 0^+} \frac{sM'(s)}{M(s)}$, or
- (b) $f(s) = 0$, for some $0 < s < 1$, and then M is rapidly varying, that is, $f(t) = M_\infty(t)$ for every $t \geq 0$, where $M_\infty(t)$ denotes the pointwise limit of $M_p(t) = t^p$ as $p \rightarrow \infty$.

This concludes the proof of Lemma 4.3. ■

4.1. Proof of Theorem 1.4. The proof of Theorem 1.4 is based partially on the lemma below; its proof follows the proof of the theorem.

LEMMA 4.5. *Let M be a non-degenerate sub- [resp., super-] multiplicative Orlicz function. Then M dominates [resp., is dominated by] every function in $C_{M,1}$.*

Proof of Theorem 1.4. Let M and N be both sub-multiplicative Orlicz functions. Suppose the spaces \mathfrak{h}_M and \mathfrak{h}_N have the same linear dimension. Let (e_n^M) and (e_n^N) denote the unit vector bases of \mathfrak{h}_M and \mathfrak{h}_N , respectively. According to Lemma 4.1, (e_n^M) is equivalent to a normalized block basis of (e_n^N) and, vice versa, (e_n^N) is equivalent to a normalized block basis of (e_n^M) . If, for example, M is degenerate then (e_n^M) is equivalent to the unit vector basis of c_0 . But the latter basis is perfectly homogeneous, so is (e_n^M) , that is, the basis (e_n^M) is equivalent to each of its normalized block bases and in particular to (e_n^N) . Summing up, we conclude that, if either M or N is degenerate, the bases (e_n^M) and (e_n^N) are equivalent.

Assume now that the both functions, M and N , are non-degenerate. Then Lemma 4.5 implies that M dominates every function in $C_{M,1}$. On the other hand, according to Proposition 2.5 and the fact that (e_n^N) is symmetric, there exists $N_1 \in C_{M,1}$ such that (e_n^N) is equivalent to the unit vector basis of \mathfrak{h}_{N_1} and now using Proposition 2.2 we conclude that N is equivalent to N_1 . This proves that M dominates N . Since the assumptions are symmetric on M and N , we also see that N dominates M and hence these functions are equivalent. By using Proposition 2.2 again, we deduce that (e_n^M) and (e_n^N) are equivalent also in this case. ■

Proof of Lemma 4.5. Suppose, for example, that M is sub-multiplicative, that is, $M(st) \leq M(s)M(t)$, for every $s, t \in [0, 1]$. Let $N \in C_{M,1}$. According to Proposition 2.6, there exists a Borel probability measure μ over $\beta\mathcal{I}$ such that

$$N(t) = \int_{\beta\mathcal{I}} M_\omega(t) d\mu(\omega),$$

for every $t \in [0, 1]$. On the other hand, since $M_\omega(t)$ is a pointwise limit of convex combinations of Orlicz functions of the form $M_s(t) := \frac{M(st)}{M(s)}$, we have $M_\omega(t) \leq M(t)$, for every $t \in [0, 1]$. Since this inequality holds for every $\omega \in \beta\mathcal{I}$, we also have

$$N(t) = \int_{\beta\mathcal{I}} M_\omega(t) d\mu(\omega) \leq \int_{\beta\mathcal{I}} M(t) d\mu(\omega) = M(t),$$

for every $[0, 1]$. Thus M dominates N . This completes the proof for the sub-multiplicative case. The proof of the super-multiplicative case is similar. ■

4.2. Proof of Theorem 3.1. In the proof of Theorem 3.1, we shall use two lemmas below; their proofs follow the proof of the theorem.

The first lemma deals with a natural stability property of the relating the constructions C_M and $C_{M,1}$ (see (3)).

LEMMA 4.6. *Let M be a non-degenerate Orlicz function.*

- (a) *For every $N \in C_M$, we have $C_{N,1} \subset C_M$.*
- (b) *For every $N \in C_{M,1}$, we have $C_N \subset C_M$.*

LEMMA 4.7. *Let M be a non-degenerate Orlicz function. Then every $N \in C_{M,1} \setminus C_M$ dominates M .*

In particular, if M is equivalent at zero to a regularly varying function with index p (see Proposition 2.3), then every $N \in C_{M,1}$ which is not equivalent to t^p dominates M .

Proof of Theorem 3.1. Let M be an Orlicz function for which M is not equivalent to any function in C_M . We need to prove that for every Banach space X having a symmetric basis (x_n) , the following conditions are equivalent:

- (i) $\dim_\ell(\mathfrak{h}_M) = \dim_\ell(X)$.
- (ii) The spaces \mathfrak{h}_M and X are isomorphic.
- (iii) The unit vector basis of \mathfrak{h}_M is equivalent to (x_n) .

This will suffice because in the case M fulfils the Δ_2 -condition we have $\mathfrak{h}_M = \ell_M$ (by virtue of the equivalence between (i) and (ii) of Proposition 2.1).

Also, it is enough to prove that (i) \Rightarrow (iii), because the implications (iii) \Rightarrow (ii) \Rightarrow (i) are obvious.

Let $\dim_\ell(\mathfrak{h}_M) = \dim_\ell(X)$. According to Lemma 4.1, the basis (x_n) is equivalent to a normalized block basis of the unit vector basis of \mathfrak{h}_M . By Proposition 2.5, there exists an Orlicz function $N \in C_{M,1}$ such that the basis (x_n) is equivalent to the unit vector basis of the space \mathfrak{h}_N , and hence $X = \mathfrak{h}_N$. We thus have $\dim_\ell(\mathfrak{h}_M) = \dim_\ell(\mathfrak{h}_N)$ with $N \in C_{M,1}$. By Proposition 2.5 again and the equivalence between (i) and (ii) in Proposition 2.2, we have $M \in C_{N,1}$ up to equivalence of functions. We claim that $N \notin C_M$. Otherwise, by Lemma 4.6, part (a), we would have $M \in C_{N,1} \subset C_M$, up to equivalence, but this contradicts our assumption that M is not equivalent to any function in C_M . Therefore, N is *not* equivalent to any function in C_M whence, by Lemma 4.7, N dominates M .

The case $M \in C_N$ is not possible either because $C_N \subset C_M$. Hence, by using Lemma 4.7 again, M dominates N .

Summing up, M and N are equivalent, as claimed. ■

Proof of Lemma 4.6. We notice first that $C_{N,1}$ is the smallest closed convex set containing N and invariant under the continuous semi-flow $(F(t), \lambda) \mapsto \frac{F(\lambda t)}{F(\lambda)}$. Since C_M has the same properties (closed, convex, $N \in C_M$ and is invariant under the semi-flow), it contains the smallest one $C_{N,1}$. This proves part (a).

For the proof of part (b), notice first that, according to Proposition 2.6, there exists a regular Borel probability measure μ defined on $\beta\mathcal{I}$ such that $N(t) = \int_{\beta\mathcal{I}} M_\omega(t) d\mu(\omega)$, for every $t \in [0, 1]$. Then part (b) follows from the fact that $C_{N,\lambda/2} \subset C_{M,\lambda}$, for every $0 < \lambda \leq 1$. Indeed, since $N(t) = \int_{\beta\mathcal{I}} M_\omega(t) d\mu(\omega)$, for every $t \in [0, 1]$, we have $N(\lambda t) = \int_{\beta\mathcal{I}} M_\omega(\lambda t) d\mu(\omega)$ and $N(\lambda) = \int_{\beta\mathcal{I}} M_\omega(\lambda) d\mu(\omega)$, for every $t, \lambda \in [0, 1]$.

Therefore, since the relation $d\nu(\omega) := \frac{1}{N(\lambda)} M_\omega(\lambda) d\mu(\omega)$ defines a probability measure on $\beta\mathcal{I}$ such that

$$\begin{aligned} \frac{N(\lambda t)}{N(\lambda)} &= \frac{1}{N(\lambda)} \int_{\beta\mathcal{I}} M_\omega(\lambda t) d\mu(\omega) \\ &= \frac{1}{N(\lambda)} \int_{\beta\mathcal{I}} \frac{M_\omega(\lambda t)}{M_\omega(\lambda)} M_\omega(\lambda) d\mu(\omega) = \int_{\beta\mathcal{I}} \frac{M_\omega(\lambda t)}{M_\omega(\lambda)} d\nu(\omega), \end{aligned}$$

and $\frac{M_\omega(\lambda t)}{M_\omega(\lambda)} \in E_{M,\lambda}$, by using again Proposition 2.6, we conclude that $\frac{N(\lambda t)}{N(\lambda)} \in C_{M,\lambda}$. ■

Proof of Lemma 4.7. By virtue of Proposition 2.6, $N(t) = \int_{\beta I_0} M_\omega(t) d\mu(\omega)$, for every $t \in [0, 1)$, for some probability Borel measure on βI_0 , the Stone–Čech compactification of the interval $(0, 1]$. Then, $N \in C_M$ if and only if μ is concentrated on $\beta I_0 \setminus I_0$. Otherwise, there exists an interval $[\lambda_1, \lambda_2] \subset I_0$, such that $\mu([\lambda_1, \lambda_2]) > 0$. On the other hand, if $M(\lambda_0) = \max_{\lambda \in [\lambda_1, \lambda_2]} \{M(\lambda)\}$, then

$$N(t) = \int_{\beta I_0} M_\omega(t) d\mu(\omega) \geq \int_{\lambda_1}^{\lambda_2} \frac{M(\lambda t)}{M(\lambda)} d\mu(\lambda) \geq \mu([\lambda_1, \lambda_2]) \frac{M(\lambda_1 t)}{M(\lambda_0)},$$

and hence $N(t) \geq AM(at)$, for every $0 < t \leq t_0$, where $A = \frac{\mu([\lambda_1, \lambda_2])}{M(\lambda_0)}$ and $a = \lambda_1$.

The last assertion is immediate from the observation that for the case where M is equivalent to a regularly varying function with index of regularity p (see Proposition 2.3) then, according to Lemma 4.3, every function of C_M is equivalent to t^p . ■

4.3. Proof of Theorem 3.3. The first part of the proof of Theorem 3.3 is based on the lemma below; its proof follows the proof of the theorem.

LEMMA 4.8. *Every regularly varying Orlicz function at zero fulfils the Δ_2 -condition at zero.*

Proof of Theorem 3.3. Suppose M is a non-degenerate Orlicz function such that the limit $f(t) = \lim_{s \rightarrow 0^+} \frac{M(st)}{M(s)}$ exists. If M is a regularly varying Orlicz function at zero then, according to Lemma 4.8, it fulfils the Δ_2 -condition at zero and, by virtue of Proposition 2.1, $\ell_M = \mathfrak{h}_M$. Otherwise, M is rapidly increasing and does not fulfil the Δ_2 -condition at zero. Now we consider two cases:

Case 1. M is not equivalent to a multiplicative Orlicz function.

By Lemma 4.3, every function N in C_M is equivalent to t^p , so N is multiplicative. Since, by assumption, M is not equivalent to any multiplicative Orlicz function, M is not equivalent to any function in C_M . Consequently, M fulfils the conditions of Theorem 3.1, and hence the equivalences between (i)–(iii) follow. The remaining equivalences are obtained from Proposition 2.2.

Case 2. M is equivalent to a multiplicative Orlicz function.

In this case, according to Lemma 4.2 the unit vector basis (e_n^M) of ℓ_M is perfectly homogeneous. Hence, by Lemma 4.1, for every Banach space with a symmetric basis (x_n) having the same linear dimension as ℓ_M , the basis (x_n) is equivalent to a normalized block basis of (e_n^M) and hence to (e_n^M) itself, because the last one is perfectly homogeneous. This proves again that conditions (i)–(iii) are equivalent. ■

Proof of Lemma 4.8. According to part (a) of Lemma 4.4, for every regularly varying Orlicz function M the limit $p = \lim_{s \rightarrow 0^+} \frac{sM'(s)}{M(s)}$ exists and is finite. On the other hand, for every non-degenerate Orlicz function M , $\sup_{0 < s \leq 1} \frac{sM'_+(s)}{M(s)}$ is finite if and only if M fulfils the Δ_2 -condition at zero (see e.g. [LT1, p. 140]). Consequently, M fulfils the Δ_2 -condition at zero. ■

4.4. Proof of Theorem 3.5. We first remark that, in the proof of Theorem 3.3, we have actually proved the following stronger statement:

(*) *Let M be an Orlicz function such that the limit $f(t) = \lim_{s \rightarrow 0^+} \frac{M(st)}{M(s)}$ exists. Then, the unit vector basis of \mathfrak{h}_M has the SCB property. Consequently, the space \mathfrak{h}_M has the USB property.*

According to Lemma 4.3, we have $C_M = \{M_p(t)\}$ for $p = \lim_{s \rightarrow 0^+} \frac{\log M(s)}{\log s}$. Now, using Lemma 4.7, we obtain $C_N \subset C_M = \{M_p(t)\}$, for every Orlicz function $N \in C_{M,1}$. This shows that every function in $C_{M,1}$ fulfils the assumption that the limit $M_p(t) = \lim_{s \rightarrow 0^+} \frac{N(st)}{N(s)}$ exists. Consequently, by property (*), the unit vector basis of \mathfrak{h}_M has the SCB property.

If M is regularly varying at zero so is N , because $C_N = \{M_p(t)\}$ (recall that C_N is non-empty in general), and hence $\lim_{s \rightarrow 0^+} \frac{M(st)}{M(s)} = M_p(t)$. Now, according to Lemma 4.8, N fulfils the Δ_2 -condition at zero. Hence, by Proposition 2.1, $\mathfrak{h}_N = \ell_N$. By (*) again, the unit vector basis of ℓ_N has the SCB property.

If M is *not* regularly varying at zero, it is rapidly increasing (that is, p must be ∞), and hence every $N \in C_{M,1}$ is rapidly increasing, too. By using (*), we deduce that the unit vector basis of \mathfrak{h}_N has the SCB property.

This completes the proof of Theorem 3.5. ■

5. Applications to concrete examples. In this section we give a few examples of classes of Orlicz functions fulfilling the assumptions of Theorem 3.1.

In the first example, taken from [GSW], we consider Orlicz functions $M_{a,b,p}(x)$ equal to $x^p |\log(bx)|^a$ near zero. The cases $b = 1 = a$ and $a > 0$ were studied by Lindberg [L, Example 3.4] and Lindenstrauss and Tzafriri [LT1, Example 4.c.1].

Let a, b, p be fixed real numbers, with $a \neq 0$, $b > 0$, and $p > 1$. Let $M_{a,b,p}$ be the function defined on the interval $[0, 1/b)$ by the formulas $M_{a,b,p}(0) = 0$, and

$$M_{a,b,p}(x) = x^p |\log(bx)|^a \quad \text{for } x \neq 0.$$

We have

$$\lim_{x \rightarrow 0^+} M_{a,b,p}(x) = \lim_{x \rightarrow 0^+} x^p |\log(bx)|^a \stackrel{x=1/y}{=} \lim_{y \rightarrow \infty} \left(\frac{\log(y/b)}{y^{p/a}} \right)^a = 0.$$

That is, $M_{a,b,p}(0^+) = 0$. On the other hand, for $x > 0$, the first derivatives of $M_{a,b,p}$ is computed by using that $|\log(bx)| = -\log(bx)$, for $x < 1/b$:

$$\begin{aligned} M'_{a,b,p}(x) &= (x^p |\log(bx)|^a)' \\ &= px^{p-1} |\log(bx)|^a - x^p a |\log(bx)|^{a-1} \frac{1}{x} \\ &= x^{p-1} |\log(bx)|^{a-1} (p |\log(bx)| - a) = M_{a-1,b,p-1}(x)(p |\log(bx)| - a). \end{aligned}$$

By using this formula, the second derivative is computed as follows:

$$\begin{aligned} M''_{a,b,p}(x) &= (M_{a-1,b,p-1}(x)(p |\log(bx)| - a))' \\ &= (M_{a-1,b,p-1}(x))' (p |\log(bx)| - a) + M_{a-1,b,p-1}(x) (p |\log(bx)| - a)' \\ &= M_{a-2,b,p-2}(x) ((p-1) |\log(bx)| - a + 1) (p |\log(bx)| - a) - \frac{p}{x} M_{a-1,b,p-1}(x). \end{aligned}$$

Now, by noticing that $p \frac{1}{x} M_{a-1,b,p-1}(x) = px^{p-2} |\log(bx)|^{a-1}$, we obtain

$$M''_{a,b,p}(x) = A(a, b, p, x) \left[\left(p + \frac{a}{\log(bx)} \right) \left(p - 1 + \frac{a-1}{\log(bx)} \right) + \frac{p}{\log(bx)} \right],$$

where $A(a, b, p, x) = (\log(bx))^2 M_{a-2,b,p-2}(x)$. Thus, *the both derivatives are positive on an interval $[0, t_0]$ with $t_0 \in (0, 1/b)$, and hence $M_{a,b,p}$ extends—linearly on $[t_0, \infty)$ —to an Orlicz function, denoted further also by $M_{a,b,p}$.* Now, for $x \in (0, t_0)$, we have

$$\frac{xM'_{a,b,p}(x)}{M_{a,b,p}(x)} = p - \frac{a}{|\log(bx)|},$$

whence $\lim_{x \rightarrow 0} \frac{xM'_{a,b,p}(x)}{M_{a,b,p}(x)} = p$. According to Proposition 2.3, the function $M_{a,b,p}$ is regularly varying near zero. Moreover, $M_{a,b,p}$ is not equivalent at 0 to a power function. Indeed,

$$\lim_{x \rightarrow 0^+} \frac{M_{a,b,p}(x)}{x^\alpha} = \lim_{x \rightarrow 0^+} \frac{x^p |\log(bx)|^a}{x^\alpha} = \lim_{x \rightarrow 0^+} x^{p-\alpha} |\log(bx)|^a = \begin{cases} 0 & \text{if } \alpha \leq p, \\ \infty & \text{if } \alpha > p. \end{cases}$$

By virtue of Theorem 3.5, each of the functions mentioned above generates an Orlicz sequence space for which every symmetric basic sequence enjoys the Strong Cantor–Bernstein property and, consequently, spans an Orlicz sequence space having a unique symmetric basis.

On the other hand, by applying Theorem 3.3 we obtain the following:

COROLLARY 5.1. *Let a_1, a_2, b_1, b_2, p_1 and p_2 be real numbers, such that $b_1, b_2 > 0$, and $p_1, p_2 > 1$. Let M_{a_1, b_1, p_1} and M_{a_2, b_2, p_2} be defined as above. Set $M_1 = M_{a_1, b_1, p_1}$ and $M_2 = M_{a_2, b_2, p_2}$. Then the following conditions are equivalent:*

- (i) $\dim_\ell(\ell_{M_1}) = \dim_\ell(\ell_{M_2})$.
- (ii) *The Orlicz sequence spaces ℓ_{M_1} and ℓ_{M_2} are isomorphic.*
- (iii) *The unit vector bases of ℓ_{M_1} and ℓ_{M_2} are equivalent.*
- (iv) *There are constants $A, B, t_0 > 0$, such that $AM_2(t) \leq M_1(t) \leq BM_2(t)$, for every $0 < t \leq t_0$.*
- (v) $p_1 = p_2$ and $a_1 = a_2$.

An example of another rapidly varying Orlicz function is $M(t) := \frac{e-1}{e^{1/t^2}-1}$, for $t > 0$, and $M(0) := 0$. Indeed, the function $N(t) = \frac{e^{t^2}-1}{e-1}$, $t > 0$, is convex, and $M(t) = 1/N(1/t)$, for every $t > 0$. Since the function $u \mapsto 1/u$ is convex and decreasing, we obtain that $N(1/t)$ is convex and decreasing, and hence $M(t) = 1/N(1/t)$ is convex and increasing, whence Orlicz. Finally, for every $t > 0$,

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{M(st)}{M(s)} & \stackrel{1/u=st}{=} \lim_{u \rightarrow \infty} \frac{N(ut)}{N(u)} = \lim_{u \rightarrow \infty} \frac{e^{(ut)^2} - 1}{e^{u^2} - 1} \\ & = \lim_{u \rightarrow \infty} \frac{2ut^2 e^{(ut)^2}}{2ue^{u^2}} = \lim_{u \rightarrow \infty} t^2 e^{(t^2-1)u^2} = M_\infty(t), \end{aligned}$$

that is, the limit equals 0 for every $t \in (0, 1)$, 1 for $t = 1$, and ∞ for every $t > 1$.

Now, according to Theorem 3.5, even though M does not fulfil the Δ_2 -condition at zero, and consequently, ℓ_M is not separable, still the Orlicz sequence space \mathfrak{h}_M has

the property that every symmetric basic sequence enjoys the Strong Cantor–Bernstein property and, consequently, spans an Orlicz sequence space having a unique symmetric basis. We thus have obtained:

COROLLARY 5.2. *Let $M(t) := \frac{e^{-1}}{e^{1/t^2}-1}$. Then the family of all symmetric basic sequences in the Orlicz sequence space \mathfrak{h}_M has the Cantor–Bernstein property.*

The function L constructed by K. Lindberg [L, Example 3.11], defined for each $p > 1 + \sqrt{2}$ by the formula

$$L(t) = t^{p+\sin(\log|\log t|)},$$

is an Orlicz function on some neighborhood of $t = 0$, which can be extended linearly to the whole \mathbb{R}^+ . In 1971, Lindenstrauss and Tzafriri [LT] proved that L is an Orlicz function fulfilling the Δ_2 -condition which is not equivalent to any function in C_L (that is, L fulfils the assumptions of Theorem 3.1). They then concluded that *the Orlicz space ℓ_L has a unique symmetric basis* (see also [LT1, Example 4.c.2]). From Theorem 3.1 we obtain a much stronger result:

COROLLARY 5.3. *Let L be the function defined above. Then the unit vector basis of the Orlicz sequence space ℓ_L has the strong Cantor–Bernstein property.*

The stronger property enjoyed by the Orlicz space ℓ_M in Theorem 3.5 may be informally called *the hereditary strong Cantor–Bernstein property*, or briefly *HSCB property*. It is also enjoyed by ℓ_L , where L is the Lindberg function, although Theorem 3.1 is not enough to establish this fact. More precisely, we have the following:

COROLLARY 5.4. *Let L be the function defined above. Then the family of all symmetric basic sequences in the Orlicz sequence space ℓ_L has the Cantor–Bernstein property.*

The proof of the corollary is based on the following property:

Let L be the Orlicz function defined above, and let E_L denote the set defined in (2). Then

(L) E_L is the set $\{t^q : p - \sqrt{2} \leq q \leq p + \sqrt{2}\}$.

For the proof of property (L), see [LT1, Example 4.c.2].

We claim that *every function in $N \in C_L$ is super-multiplicative*. Indeed, set $\alpha := p - \sqrt{2}$, $\beta := p + \sqrt{2}$ and observe that, by virtue of Proposition 2.6 and property (L), the function N can be represented as $N(t) = \int_{\alpha}^{\beta} t^q d\mu(q)$, and is such that the function

$$u \mapsto \frac{\log N(e^{-u})}{u} = \log \left(\int_{\alpha}^{\beta} (e^{-s})^u d\mu(s) \right)^{1/u}$$

is increasing, because $\left(\int_{\alpha}^{\beta} (e^{-s})^u d\mu(s) \right)^{1/u}$ are means with respect to the probability measure μ (see, e.g. [MPF, Chap. VI, Sec. 2, Corollary 1]). Therefore, by [HP, Theorem 7.2.4], the function $u \mapsto \log N(e^{-u})$ is super-additive, and hence $t \mapsto N(t)$ is super-multiplicative.

Suppose now that $N_1, N_2 \in C_{L,1}$ are such that $\dim_{\ell}(\mathfrak{h}_{N_1}) = \dim_{\ell}(\mathfrak{h}_{N_2})$. Then, according to Lemma 4.1, the unit vector bases of \mathfrak{h}_{N_1} and \mathfrak{h}_{N_2} are each one equivalent to a normalized block basis with respect the other. On the other hand, now by virtue of Proposition 2.5, this implies that $N_1 \in C_{N_2,1}$ and vice versa: $N_2 \in C_{N_1,1}$. If $N_2 \in C_{N_1}$

then, by Lemma 4.6, $N_1, N_2 \in C_{N_1} \subset C_L$, and hence both the functions are supermultiplicative. Now from Theorem 1.4 and Theorem 2.2 it follows that N_1 is equivalent to N_2 . A similar conclusion we obtain in the case $N_1 \in C_{N_2}$. Otherwise, $N_1 \in C_{N_2,1} \setminus C_{N_2}$ and $N_2 \in C_{N_1,1} \setminus C_{N_1}$, which means, according to Lemma 4.7, that N_1 dominates N_2 and N_2 dominates N_1 , that is, N_1 and N_2 are equivalent.

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