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GENERALIZED FRACTIONAL INTEGRALS ON CENTRAL MORREY SPACES AND GENERALIZED λ-CMO SPACES

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Dedicated to Professor Yoshihiro Mizuta in celebration of his 65th birthday

Abstract. We introduce the generalized fractional integrals $\tilde{I}_{\alpha,d}$ and prove the strong and weak boundedness of $\tilde{I}_{\alpha,d}$ on the central Morrey spaces $B^{p,\lambda}(\mathbb{R}^n)$. In order to show the boundedness, the generalized λ-central mean oscillation spaces $\Lambda_{p,\lambda}^{(d)}(\mathbb{R}^n)$ and the generalized weak λ-central mean oscillation spaces $W\Lambda_{p,\lambda}^{(d)}(\mathbb{R}^n)$ play an important role.

1. Introduction. In 1989, Y. Chen and K. Lau [3] and J. García-Cuerva [6] introduced the central mean oscillation spaces $\mathrm{CMO}^p(\mathbb{R}^n)$, $1 \le p < \infty$. These spaces contain $B^p(\mathbb{R}^n)$ modulo constants. The spaces $B^p(\mathbb{R}^n)$ were introduced by A. Beurling [2], together with their preduals $A^p(\mathbb{R}^n)$, so-called the Beurling algebras. Further, in 1994, J. García-Cuerva and M. J. Herrero [7] defined $B_{p,q}(\mathbb{R}^n)$ and $\Lambda_{p,q}(\mathbb{R}^n)$, $1 \le p < \infty$ and $0 < q \le 1$, which special cases where $q = 1$, give $B^p(\mathbb{R}^n)$ and $\mathrm{CMO}^p(\mathbb{R}^n)$, respectively. Later, in 2000, J. Alvarez, M. Guzmán-Partida and J. Lakey [1] introduced the non-homogeneous central Morrey spaces $B^{p,\lambda}(\mathbb{R}^n)$ and the λ-central mean oscillation spaces $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, $1 \le p < \infty$ and $\lambda \in \mathbb{R}$. Here note that when $\lambda = n(1/q - 1)$ and $0 < q \le 1$, $B^{p,\lambda}(\mathbb{R}^n)$ gives $B_{p,q}(\mathbb{R}^n)$, and that $B^{p,\lambda}(\mathbb{R}^n)$ is the non-homogeneous Herz space $K_{p,\infty}^{-n/p-\lambda}(\mathbb{R}^n)$ (cf. H. Feichtinger [4] and C. Herz [10]).

On the other hand, for the fractional integrals I_α, $0 < \alpha < n$, which are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

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the following results are well-known: For $0 < \alpha < n$, $1 \le p < n/\alpha$ and $1/q = 1/p - \alpha/n$,

(i) $I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n), \quad 1 < p < n/\alpha;$
(ii) $I_\alpha : L^1(\mathbb{R}^n) \to WL^q(\mathbb{R}^n), \quad p = 1.$

Remark 1.1. In the above, (i) is the famous Hardy–Littlewood–Sobolev theorem, which is due to G. H. Hardy and J. E. Littlewood [8, 9] for the 1-dimensional case and S. L. Sobolev [18] for the n-dimensional case, and (ii) belongs to A. Zygmund [19].

Afterwards, Z. W. Fu, Y. Lin and S. Z. Lu [5] proved that for $0 < \alpha < n$, $1 < p < n/\alpha$, $-n/p + \alpha \le \lambda + \alpha = \mu < 0$ and $1/q = 1/p - \alpha/n$,

$$I_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to B^{q,\mu}(\mathbb{R}^n). \tag{1.1}$$

Furthermore, in [11, 12], we introduced $\mathrm{CMO}^p_q(\mathbb{R}^n)$ and $W\mathrm{CMO}^p_q(\mathbb{R}^n)$, $1 \le p < \infty$ and $0 < q \le 1$, and for the modified fractional integrals $\tilde{I}_\alpha$, which are defined by

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y)\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{Q_1}(y)}{|y|^{n-\alpha}}\Big)\, dy, \tag{1.2}$$

where χ_{Q_1} is the characteristic function of Q_1, proved the following: For $0 < \alpha < 1$, $1 \le p_1 < n/\alpha$, $n/(n+1-\alpha) < q_1 \le 1$, $1/p_2 = 1/p_1 - \alpha/n$ and $1/q_2 = 1/q_1 + \alpha/n$,

(i) $\tilde{I}_\alpha : B^{p_1}_{q_1}(\mathbb{R}^n) \to \mathrm{CMO}^{p_2}_{q_2}(\mathbb{R}^n), \quad 1 < p_1 < n/\alpha;$
(ii) $\tilde{I}_\alpha : B^{1}_{q_1}(\mathbb{R}^n) \to W\mathrm{CMO}^{p_2}_{q_2}(\mathbb{R}^n), \quad p_1 = 1.$

Here $B^p_q(\mathbb{R}^n) = B_{p,q}(\mathbb{R}^n)$ and $\mathrm{CMO}^p_q(\mathbb{R}^n)$ is the special case of $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, where $\lambda = n(1/q - 1)$ and $0 < q \le 1$.

Recently, in [15] (cf. [13]), in order to unify $B^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, we introduced the B_σ-function spaces, $0 \le \sigma < \infty$, and showed several B_σ-Morrey–Campanato estimates for I_α and $\tilde{I}_\alpha$. From one of these estimates we obtained the following corollary: For $0 < \alpha < n$, $1 < p < n/\alpha$, $1 \le q \le pn/(n - p\alpha)$ and $-n/p + \alpha \le \lambda + \alpha = \mu < 1$,

$$\tilde{I}_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to \mathrm{CMO}^{q,\mu}(\mathbb{R}^n). \tag{1.3}$$

As we stated above, for $0 < \alpha < n$ and $1 < p < n/\alpha$, when $-n/p \le \lambda < -\alpha$, I_α is well-defined and bounded for $B^{p,\lambda}(\mathbb{R}^n)$ and when $-n/p \le \lambda < 1 - \alpha$, $\tilde{I}_\alpha$ is well-defined and bounded for $B^{p,\lambda}(\mathbb{R}^n)$ (see (1.1) and (1.3), respectively). Therefore, in this paper for the whole of λ such that $-n/p \le \lambda < \infty$, we investigate the boundedness of I_α for $B^{p,\lambda}(\mathbb{R}^n)$. In order to do so, first we introduce the "new" function spaces, the special cases of which are $\Lambda_{p,q}(\mathbb{R}^n)$. Next we define the modification of I_α which is well-defined for $B^{p,\lambda}(\mathbb{R}^n)$, when $\lambda \ge 1 - \alpha$, and show the strong estimate of this modification of I_α for $B^{p,\lambda}(\mathbb{R}^n)$, using the above "new" function spaces. Moreover we also observe the weak estimate of this modification of I_α for the critical case $B^{1,\lambda}(\mathbb{R}^n)$.

We note that the same results in this paper still hold for the homogeneous versions of the function spaces.

2. Generalized λ-CMO spaces. First we explain the notation used in the present paper. We use the symbol $A \lesssim B$ to denote that there exists a constant $C > 0$ such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. For $r > 0$, by Q_r, we mean the following:

$$Q_r = \{y \in \mathbb{R}^n : |y| < r\} \quad \text{or} \quad Q_r = \{y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max_{1 \le i \le n} |y_i| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote the Lebesgue measure of G by $|G|$ and the characteristic function of G by χ_G. Further, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a measurable set $G \subset \mathbb{R}^n$ with $|G| > 0$, let

$$f_G = \fint_G f(y)\,dy = \frac{1}{|G|}\int_G f(y)\,dy,$$

and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Next we state the definition of the non-homogeneous central Morrey space $B^{p,\lambda}(\mathbb{R}^n)$ (see [1] and [5]).

Definition 2.1. For $1 \le p < \infty$ and $-n/p \le \lambda < \infty$,

$$B^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{B^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{B^{p,\lambda}} = \sup_{r\ge 1} \frac{1}{r^\lambda}\Big(\fint_{Q_r} |f(y)|^p\,dy\Big)^{1/p}.$$

Now we introduce the "new" function spaces, i.e., the generalized λ-central mean oscillation spaces $\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$ (see [7] and [14]; cf. [17]).

Definition 2.2. For $1 \le p < \infty$, $d \in \mathbb{N}_0$ and $-n/p \le \lambda < d+1$, the function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ will be said to *belong to the generalized λ-central mean oscillation* (λ-CMO) *space* $\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$ if and only if for every $r \ge 1$, there is a polynomial $P^d_r f$ of degree at most d such that

$$\|f\|_{\Lambda^{(d)}_{p,\lambda}} = \sup_{r\ge 1} \frac{1}{r^\lambda}\Big(\fint_{Q_r} |f(y) - P^d_r f(y)|^p\,dy\Big)^{1/p} < \infty.$$

Also we define the generalized weak λ-CMO spaces $W\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$.

Definition 2.3. For $1 \le p < \infty$, $d \in \mathbb{N}_0$ and $-n/p \le \lambda < d+1$, the function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ will be said to *belong to the generalized weak* λ-CMO *space* $W\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$ if and only if for every $r \ge 1$, there is a polynomial $P^d_r f$ of degree at most d such that

$$\|f\|_{W\Lambda^{(d)}_{p,\lambda}} = \sup_{r\ge 1} \frac{1}{r^\lambda}\Big(\frac{1}{|Q_r|}\sup_{t>0} t^p\big|\{y \in Q_r : |f(y) - P^d_r f(y)| > t\}\big|\Big)^{1/p} < \infty.$$

Identifying functions which differ by a polynomial of degree at most d, a.e., we see that $\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$ is a Banach space and $W\Lambda^{(d)}_{p,\lambda}(\mathbb{R}^n)$ is a complete quasi-normed space.

Remark 2.1. We note that particularly

$$\Lambda^{(0)}_{p,\lambda}(\mathbb{R}^n) = \mathrm{CMO}^{p,\lambda}(\mathbb{R}^n) \quad\text{and}\quad W\Lambda^{(0)}_{p,\lambda}(\mathbb{R}^n) = W\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n).$$

Here $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$ and $W\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$, so-called the λ-CMO space and the weak λ-CMO space, are defined by

$$\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{CMO}^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{\mathrm{CMO}^{p,\lambda}} = \sup_{r\ge 1} \frac{1}{r^\lambda}\left(\fint_{Q_r} |f(y) - f_{Q_r}|^p\,dy\right)^{1/p},$$

and

$$W\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{W\mathrm{CMO}^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{WCMO^{p,\lambda}} = \sup_{r\geq 1} \frac{1}{r^\lambda}\Big(\frac{1}{|Q_r|}\sup_{t>0} t^p\big|\{y\in Q_r : |f(y)-f_{Q_r}|>t\}\big|\Big)^{1/p},$$

respectively (see [1]; cf. [11]).

Remark 2.2 (Remark 6.2 of [17]). For $1 \leq p < \infty$, $d \in \mathbb{N}_0$, $-n/p \leq \lambda < d+1$ and $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, we have

$$\|f\|_{\Lambda^{(d)}_{p,\lambda}} \sim \sup_{r\geq 1} \inf_{P\in\mathcal{P}^d(\mathbb{R}^n)} \frac{1}{r^\lambda}\Big(\fint_{Q_r} |f(y)-P(y)|^p\,dy\Big)^{1/p}$$

and

$$\|f\|_{W\Lambda^{(d)}_{p,\lambda}} \sim \sup_{r\geq 1} \inf_{P\in\mathcal{P}^d(\mathbb{R}^n)} \frac{1}{r^\lambda}\Big(\frac{1}{|Q_r|}\sup_{t>0} t^p\big|\{y\in Q_r : |f(y)-P(y)|>t\}\big|\Big)^{1/p},$$

where $\mathcal{P}^d(\mathbb{R}^n)$ is the set of all polynomials having degree at most d.

3. Generalized fractional integrals. Let $0 < \alpha < n$, $1 \leq p < \infty$ and $-n/p \leq \lambda < \infty$. Now under the condition $\lambda + \alpha \geq 1$ we consider the boundedness of fractional integrals I_α on $B^{p,\lambda}(\mathbb{R}^n)$. Then, in general, $I_\alpha f$ is not necessarily well-defined. Therefore we modify the definition of fractional integrals I_α and introduce the following definition of generalized fractional integrals $\tilde{I}_{\alpha,d}$.

Definition 3.1. For $0 < \alpha < n$ and $d \in \mathbb{N}_0$, we define the generalized fractional integral (of order α), i.e., $\tilde{I}_{\alpha,d}$, as follows: For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\tilde{I}_{\alpha,d}\, f(x) = \int_{\mathbb{R}^n} f(y)\Big\{K_\alpha(x-y) - \Big(\sum_{\{l:|l|\leq d\}} \frac{x^l}{l!}(D^l K_\alpha)(-y)\Big)(1-\chi_{Q_1}(y))\Big\}\,dy,$$

where

$$K_\alpha(x) = \frac{1}{|x|^{n-\alpha}}$$

and for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n_0$, $|l| = l_1 + l_2 + \ldots + l_n$, $x^l = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$, and D^l is the partial derivative of order l, i.e.,

$$D^l = (\partial/\partial x_1)^{l_1}(\partial/\partial x_2)^{l_2}\cdots(\partial/\partial x_n)^{l_n}.$$

Note that in particular

$$\tilde{I}_{\alpha,0} = \tilde{I}_\alpha$$

(see (1.2) above) and that $\tilde{I}_{\alpha,d}(|f|) \not\equiv \infty$ on $\mathbb{R}^n$, if

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n-\alpha+d+1}}\,dy < \infty$$

(cf. Y. Mizuta [16]). If $I_\alpha f$ is well-defined, then $\tilde{I}_{\alpha,d} f$ is also well-defined and $I_\alpha f - \tilde{I}_{\alpha,d} f$ is a polynomial of degree at most d.

Then our results for a generalized fractional integral $\tilde{I}_{\alpha,d}$ are the following strong and weak estimates on $B^{p,\lambda}(\mathbb{R}^n)$.

Theorem 3.1. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $d \in \mathbb{N}_0$, $-n/p + \alpha \leq \lambda + \alpha = \mu < d+1$ and $q = pn/(n-p\alpha)$, i.e., $1/q = 1/p - \alpha/n$. Then $\tilde{I}_{\alpha,d}$ is bounded from $B^{p,\lambda}(\mathbb{R}^n)$ to $\Lambda^{(d)}_{q,\mu}(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that*

$$\|\tilde{I}_{\alpha,d} f\|_{\Lambda^{(d)}_{q,\mu}} \leq C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).$$

THEOREM 3.2. *Let $0 < \alpha < n$, $d \in \mathbb{N}_0$, $-n + \alpha \le \lambda + \alpha = \mu < d + 1$ and $q = n/(n - \alpha)$, i.e., $1/q = 1 - \alpha/n$. Then $\tilde{I}_{\alpha,d}$ is bounded from $B^{1,\lambda}(\mathbb{R}^n)$ to $W\Lambda^{(d)}_{q,\mu}(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that*

$$\|\tilde{I}_{\alpha,d} f\|_{W\Lambda^{(d)}_{q,\mu}} \le C\|f\|_{B^{1,\lambda}}, \quad f \in B^{1,\lambda}(\mathbb{R}^n).$$

In the above theorems, if $d = 0$, then we get the following strong and weak estimates for $\tilde{I}_\alpha$.

COROLLARY 3.3 (cf. Theorem 2.3 of [15]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $-n/p + \alpha \le \lambda + \alpha = \mu < 1$ and $q = pn/(n - p\alpha)$, i.e., $1/q = 1/p - \alpha/n$. Then $\tilde{I}_\alpha$ is bounded from $B^{p,\lambda}(\mathbb{R}^n)$ to $\mathrm{CMO}^{q,\mu}(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that*

$$\|\tilde{I}_\alpha f\|_{\mathrm{CMO}^{q,\mu}} \le C\|f\|_{B^{p,\lambda}}, \quad f \in B^{p,\lambda}(\mathbb{R}^n).$$

COROLLARY 3.4. *Let $0 < \alpha < n$, $-n + \alpha \le \lambda + \alpha = \mu < 1$ and $q = n/(n - \alpha)$, i.e., $1/q = 1 - \alpha/n$. Then $\tilde{I}_\alpha$ is bounded from $B^{1,\lambda}(\mathbb{R}^n)$ to $W\mathrm{CMO}^{q,\mu}(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that*

$$\|\tilde{I}_\alpha f\|_{W\mathrm{CMO}^{q,\mu}} \le C\|f\|_{B^{1,\lambda}}, \quad f \in B^{1,\lambda}(\mathbb{R}^n).$$

4. Proofs of the theorems. First of all, we show that $\tilde{I}_{\alpha,d} f$ is well-defined.

LEMMA 4.1. *Let $0 < \alpha < n$, $1 \le p < \infty$, $d \in \mathbb{N}_0$ and $-n/p + \alpha \le \lambda + \alpha < d + 1$. Then for $f \in B^{p,\lambda}(\mathbb{R}^n)$, $\tilde{I}_{\alpha,d} f$ is well-defined.*

Proof. Let $f \in B^{p,\lambda}(\mathbb{R}^n)$, $r \ge 1$ and $x \in Q_r$, and let

$$\begin{aligned}\tilde{I}_{\alpha,d} f(x) &= \tilde{I}_{\alpha,d}(f\chi_{Q_{2r}})(x) + \tilde{I}_{\alpha,d}(f(1 - \chi_{Q_{2r}}))(x) \\ &= I_\alpha(f\chi_{Q_{2r}})(x) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!} \int_{Q_{2r}\setminus Q_1} f(y)(D^l K_\alpha)(-y)\,dy \\ &\quad + \int_{\mathbb{R}^n\setminus Q_{2r}} f(y)\Big(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}(D^l K_\alpha)(-y)\Big)\,dy. \end{aligned} \tag{4.1}$$

Since $f\chi_{Q_{2r}} \in L^p(\mathbb{R}^n)$, the first term is well-defined. The second term is also well-defined, since $(D^l K_\alpha)(\chi_{Q_{2r}} - \chi_{Q_1}) \in L^{p'}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$. Here we note that the second term is a polynomial of degree at most d. For the third term, the integral converges absolutely by virtue of Lemma 4.2, which is shown in the proof of Theorem 3.1 below, and so the present term is well-defined.

Further, since for $1 \le s < r$,

$$f\chi_{Q_{2s}} + f(1 - \chi_{Q_{2s}}) = f\chi_{Q_{2r}} + f(1 - \chi_{Q_{2r}}),$$

it follows that for $x \in Q_s \subset Q_r$,

$$\tilde{I}_{\alpha,d}(f\chi_{Q_{2s}})(x) + \tilde{I}_{\alpha,d}(f(1 - \chi_{Q_{2s}}))(x) = \tilde{I}_{\alpha,d}(f\chi_{Q_{2r}})(x) + \tilde{I}_{\alpha,d}(f(1 - \chi_{Q_{2r}}))(x).$$

This shows that $\tilde{I}_{\alpha,d} f$ is independent of Q_r containing x. Thus $\tilde{I}_{\alpha,d} f$ is well-defined on $\mathbb{R}^n$. ∎

In the proofs of Theorems 3.1 and 3.2, the following two lemmas are important.

Lemma 4.2 (Lemma 7.3 of [16]). *Let $x \in \mathbb{R}^n$, $0 < \alpha < n$ and $d \in \mathbb{N}_0$. If $y \in \mathbb{R}^n \setminus Q_{2|x|}$, then*

$$\Big|K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}(D^l K_\alpha)(-y)\Big| \le C\frac{|x|^{d+1}}{|y|^{n-\alpha+d+1}}\,.$$

Lemma 4.3. *Let $1 \le p < \infty$ and $\lambda \in \mathbb{R}$. If $\beta + \lambda < 0$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n\setminus Q_r} \frac{|f(y)|}{|y|^{n-\beta}}\,dy \le Cr^{\beta+\lambda}\|f\|_{B^{p,\lambda}} \quad \textit{for all } f \in B^{p,\lambda}(\mathbb{R}^n) \textit{ and } r \ge 1.$$

Proof. This lemma is proved by the same argument as the proof of Lemma 4.1 of [15].

Since $\beta + \lambda < 0$, it follows from Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus Q_r} \frac{|f(y)|}{|y|^{n-\beta}}\,dy &\lesssim \sum_{j=1}^{\infty} \frac{1}{(2^j r)^{n-\beta}} \int_{Q_{2^j r}\setminus Q_{2^{j-1} r}} |f(y)|\,dy \\ &\lesssim r^\beta \sum_{j=1}^{\infty} 2^{j\beta}\Big(\fint_{Q_{2^j r}} |f(y)|^p\,dy\Big)^{1/p} \lesssim r^{\beta+\lambda}\sum_{j=1}^{\infty}(2^{\beta+\lambda})^j\|f\|_{B^{p,\lambda}} \sim r^{\beta+\lambda}\|f\|_{B^{p,\lambda}},\end{aligned}$$

which concludes the proof. ∎

Proof of Theorem 3.1. Let $f \in B^{p,\lambda}(\mathbb{R}^n)$, $r \ge 1$ and $x \in Q_r$. Since $\tilde{I}_{\alpha,d} f$ is well-defined by Lemma 4.1, we prove only that

$$\|\tilde{I}_{\alpha,d} f\|_{\Lambda^{(d)}_{q,\mu}} \le C\|f\|_{B^{p,\lambda}}.$$

Now, in (4.1), putting

$$R^d_r f(x) = -\sum_{\{l:|l|\le d\}} \frac{x^l}{l!} \int_{Q_{2r}\setminus Q_1} f(y)(D^l K_\alpha)(-y)\,dy$$

and

$$J_{\alpha,d,r} f(x) = \int_{\mathbb{R}^n\setminus Q_{2r}} f(y)\Big(K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}\,(D^l K_\alpha)(-y)\Big)\,dy,$$

we have

$$\begin{aligned}&\Big(\int_{Q_r} \big|\tilde{I}_{\alpha,d} f(x) - R^d_r f(x)\big|^q\,dx\Big)^{1/q} \\ &\qquad \le \Big(\int_{Q_r} \big|I_\alpha(f\chi_{Q_{2r}})(x)\big|^q\,dx\Big)^{1/q} + \Big(\int_{Q_r} |J_{\alpha,d,r} f(x)|^q\,dx\Big)^{1/q} = I_1 + I_2. \qquad (4.2)\end{aligned}$$

To estimate I_1, we apply the strong (p,q) boundedness of I_α. Then

$$\begin{aligned}I_1 \le \|I_\alpha(f\chi_{Q_{2r}})\|_{L^q} \lesssim \|f\chi_{Q_{2r}}\|_{L^p} &\lesssim r^\lambda |Q_{2r}|^{1/p}\|f\|_{B^{p,\lambda}} \\ &\sim r^{\lambda+n/p}\|f\|_{B^{p,\lambda}} = r^{\mu+n/q}\|f\|_{B^{p,\lambda}}.\end{aligned}$$

Next we estimate I_2. Since it follows from Lemma 4.2 that for $x \in Q_r$ and $y \in \mathbb{R}^n\setminus Q_{2r}$,

$$\Big|K_\alpha(x-y) - \sum_{\{l:|l|\le d\}} \frac{x^l}{l!}(D^l K_\alpha)(-y)\Big| \lesssim \frac{|x|^{d+1}}{|y|^{n-\alpha+d+1}} \le \frac{r^{d+1}}{|y|^{n-\alpha+d+1}}\,,$$

we obtain by Lemma 4.3 and the assumption $\lambda + \alpha < d + 1$,

$$|J_{\alpha,d,r}f(x)| \lesssim r^{d+1} \int_{\mathbb{R}^n \setminus Q_{2r}} \frac{|f(y)|}{|y|^{n-\alpha+d+1}}\, dy \lesssim r^{\lambda+\alpha} \|f\|_{B^{p,\lambda}} = r^{\mu} \|f\|_{B^{p,\lambda}} \,. \tag{4.3}$$

Consequently

$$I_2 = \|J_{\alpha,d,r}f\|_{L^q(Q_r)} \lesssim r^{\mu}\|f\|_{B^{p,\lambda}} \cdot |Q_r|^{1/q} \sim r^{\mu+n/q}\|f\|_{B^{p,\lambda}}.$$

Thus we get

$$\begin{aligned} \|\tilde{I}_{\alpha,d}f\|_{\Lambda^{(d)}_{q,\mu}} &\lesssim \sup_{r\geq 1} \frac{1}{r^{\mu}} \Big(\fint_{Q_r} \big|\tilde{I}_{\alpha,d}f(y) - R^d_r f(y)\big|^q\, dy \Big)^{1/q} \\ &\lesssim \sup_{r\geq 1} \frac{1}{r^{\mu}} \Big(\frac{1}{|Q_r|} \Big)^{1/q} \cdot r^{\mu+n/q} \|f\|_{B^{p,\lambda}} \sim \|f\|_{B^{p,\lambda}} \,. \end{aligned}$$

This completes the proof. ■

Proof of Theorem 3.2. The proof of Theorem 3.2 is similar to that of Theorem 3.1. Therefore, in the same way as (4.2), it follows that for $f \in B^{1,\lambda}(\mathbb{R}^n)$, $r \geq 1$ and $x \in Q_r$,

$$\begin{aligned} &\sup_{t>0} (2t)^q \Big| \big\{ x \in Q_r : |\tilde{I}_{\alpha,d}f(x) - R^d_r f(x)| > 2t \big\} \Big| \\ &\quad \leq 2^q \Big\{ \sup_{t>0} t^q \big| \{ x \in Q_r : |I_\alpha(f\chi_{Q_{2r}})(x)| > t \} \big| + \sup_{t>0} t^q \big| \{ x \in Q_r : |J_{\alpha,d,r}f(x)| > t \} \big| \Big\} \\ &\quad = 2^q (I_1 + I_2). \end{aligned}$$

Then we have by using the weak $(1, q)$ boundedness of I_α,

$$I_1^{1/q} \lesssim r^{\mu+n/q}\|f\|_{B^{1,\lambda}}$$

and by (4.3),

$$I_2^{1/q} \lesssim r^{\mu+n/q}\|f\|_{B^{1,\lambda}}.$$

Thus

$$\|\tilde{I}_{\alpha,d}f\|_{W\Lambda^{(d)}_{q,\mu}} \lesssim \|f\|_{B^{1,\lambda}},$$

which shows the conclusion. ■

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