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CAPACITARY ORLICZ SPACES, CALDERÓN PRODUCTS AND INTERPOLATION

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Abstract. These notes are devoted to the analysis on a capacity space, with capacities as substitutes of measures of the Orlicz function spaces. The goal is to study some aspects of the classical theory of Orlicz spaces for these spaces including the classical theory of interpolation.

1. Introduction. The purpose of this paper is to present some basic developments connected with properties of the capacitary Orlicz function spaces, defined on a capacity space instead of a measure space, and their interpolation theory. We also extend briefly the classical theory of Calderón products. It is our feeling that these developments deserve to be widely known. On the one hand they relate to important aspects of mathematical analysis and on the other hand they have a simple and basic character.

One of the main problems that we have when dealing with capacities is that we are forced to work with a non-additive integral, the Choquet integral, so that some basic properties, such as the dominated convergence theorem or Fubini's Theorem are not longer available.

In the literature, a capacity on a space Ω is usually supposed to be an increasing set function $C : \Sigma \to [0, \infty]$, with different properties depending on the context, and the Choquet integral is defined as

$$\int f\, dC := \int_0^\infty C\{f > t\}\, dt \qquad (f \geq 0),$$

where $\{f > t\} \in \Sigma$ for every $t > 0$.

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In many important examples of capacities the domain Σ is a σ-algebra. This is the case of the variational capacity, and those of the Fuglede [Fu] and Meyers [Me] of potential theory. They are countably subadditive set functions which include the Riesz and the Bessel capacities.

Orlicz spaces appear naturally. They have been recently studied in connection with potential theory, harmonic analysis, risk measures theory, variational problems, unilateral problems, PDE, etc. (see [A], [Ar], [BY], [Ci], et al.) Therefore, these new Orlicz spaces are of interest.

The organization of the paper is as follows: Section 2 is devoted to recall some basic facts and to study the quasi-normed capacitary function spaces, a new class of function spaces that extends the usual quasi-normed function spaces.

In Section 3 we study Calderón products of quasi-normed capacitary function spaces. In particular, we show that under quite general assumptions on the capacity, the Calderón product of a pair of capacitary Lebesgue spaces is a capacitary Lebesgue space.

Sections 4 and 5 are devoted to extend the classical theory of Orlicz spaces. First we define the capacitary Orlicz function spaces as usual but replacing the underlying measure by a capacity. Then, we study some of their properties as function spaces and, in particular, we show that the concavity of C and the continuity of φ give a Banach function space $L^\varphi(C)$ with the usual Luxemburg functional. Finally, in Section 5, we study their interpolation properties extending the interpolation method developed by Gustavsson and Peetre [GP].

As usual, $f \lesssim g$ means that $f \le cg$ for a certain constant $c > 0$, and $f \simeq g$ means that $f \lesssim g \lesssim f$.

2. Capacitary function spaces. Let (Ω, Σ) be a measurable space. Sets will always be assumed to be in Σ and functions in $L_0(\Omega)$, the set of all (equivalence classes of) real valued measurable functions on Ω, and $L_0(\Omega)^+$ the positive ones. As in [Ce, CMS], by a capacity C we mean a set function on Σ satisfying the following properties:

(a) $C(\emptyset) = 0$,
(b) $0 \le C(A) \le \infty$,
(c) $C(A) \le C(B)$ if $A \subset B$,
(d) Fatou: $C(A_n) \uparrow C(A)$ whenever $A_n \uparrow A$,
(e) quasi-subadditivity: $C(A \cup B) \le c(C(A) + C(B))$, where $c \ge 1$ is a constant.

If $c = 1$, we say that the capacity is subadditive.

By (Ω, Σ, C) we denote a capacity space. It plays the role of a measure space in the theory of Banach function spaces. In this setting, a property is said to hold quasi-everywhere (C-q.e. for short) if the exceptional set has zero capacity.

The relation $\{f + g > t\} \subset \{f > t/2\} \cup \{g > t/2\}$ shows that the Choquet integral, defined on nonnegative functions, is quasi-subadditive with constant $2c$,

$$\int (f+g)\, dC \le 2c\Big(\int f\, dC + \int g\, dC\Big).$$

The Choquet integral is subadditive on sets,

$$\int (\chi_A + \chi_B)\, dC \le \int \chi_A\, dC + \int \chi_B\, dC,$$

if and only if

$$C(A \cup B) + C(A \cap B) \le C(A) + C(B).$$

Then the Choquet integral is also subadditive on nonnegative simple functions as it was proved by Choquet in [Ch] (see also [CCM] or [Ce] for a direct elementary proof). In this case C is called concave.

From now on, let (Ω, Σ, C) be the underlying capacity space. Let $L_0(C)$ be the real vector space of all measurable functions, two functions being equivalent if they coincide C-q.e., endowed with the topology of the convergence in capacity on sets of finite capacity and with the lattice structure given by $f \le g$ meaning that $f(x) \le g(x)$ C-q.e.

A set $X \subset L_0(C)$ is a quasi-normed capacitary function space if $X = \{f \in L_0(C) : \varrho(f) < \infty\}$, where $\varrho : L_0(\Omega)^+ \to [0, \infty]$ satisfies:

- $\varrho(f) = 0 \Leftrightarrow f = 0$ q.e., $\varrho(f + g) \le k(\varrho(f) + \varrho(g))$ and $\varrho(\alpha f) = \alpha \varrho(f)$ for every $\alpha \in \mathbb{R}^+$,
- $f \le g$ (C-q.e.) implies $\varrho(f) \le \varrho(g)$,
- $C(A) < \infty$ implies $\varrho(\chi_A) < \infty$ and there exists $k_A > 0$ such that $\int \chi_B\, dC \le k_A \varrho(\chi_B)$ for every $B \subset A$, and
- if $\varrho(f) < \infty$, then $\{f > 0\}$ is C-sigma-finite, that is, $\{f > 0\} = \bigcup_{k=1}^{\infty} \Omega_k$ with $C(\Omega_k) < \infty$ $(k \in \mathbb{N})$.

We endow X with $\|f\|_X := \varrho(|f|)$, that does not depend on the representative. Then, X is Fatou if it satisfies (a) and (b) in Theorem 2.1.

THEOREM 2.1. *Let X be a quasi-normed capacitary function space. The following conditions are equivalent*:

(a) *If* $\sup_n \|f_n\|_X = M < \infty$, $f_n \to f$ C-q.e., *then* $f \in X$ *and* $\|f\|_X \le \liminf_n \|f_n\|_X$.
(b) *If* $0 \le f_n \uparrow f$ C-q.e., *then* $\lim_n \varrho(f_n) = \varrho(f)$.

Proof. To prove that (a) implies (b), take $0 \le f_n \uparrow f$ C-q.e. If $\varrho(f) < \infty$, then $\varrho(f) = \|f\|_X \le \lim_n \|f_n\|_X = \varrho(f_n)$ by (a) and $\varrho(f_n) \le \varrho(f)$ $(n \in \mathbb{N})$. So $\lim_n \varrho(f_n) = \varrho(f)$. If $\varrho(f) = \infty$, since $f_n \uparrow f$ C-q.e., necessarily $\lim_n \varrho(f_n) = \infty$ because $\sup_n \varrho(f_n) = M < \infty$ would imply $f \in X$ by (a).

To prove the converse, suppose that (b) holds and that $\sup_n \|f_n\|_X = M < \infty$ and $f_n \to f$ C-q.e. Define $g_n := \inf_{m \ge n} |f_m|$ $(n \in \mathbb{N})$, so $g_n \uparrow |f|$ C-q.e. and $\|f\|_X = \varrho(|f|) = \lim_n \varrho(g_n)$. Since $g_n \le |f_m|$ for every $m \ge n$, it follows that $\varrho(g_n) \le \inf_{m \ge n} \varrho(|f_m|)$ and then $\|f\|_X \le \lim_n \inf_{m \ge n} \varrho(|f_m|) = \liminf_n \|f_n\|_X$. ■

Conditions (a) and (b) are called the Fatou conditions. If they hold, then we say that X has the Fatou property.

THEOREM 2.2. *Any quasi-normed capacitary function space X on (Ω, Σ, C) is continuously imbedded in* $\mathrm{L}_0(C)$.

Proof. It is sufficient to prove that the condition $\|f_n\|_X \to 0$ for $\{f_n\}_{n\in\mathbb{N}} \subset X$ implies $f_n \to 0$ in capacity on any set Ω_0 of finite capacity.

Assume the contrary, so that there exist a set Ω_0 with $0 < C(\Omega_0) < \infty$ and a positive number ε such that for some subsequence f_{n_k}, the inequality $|f_{n_k}(t)| > \varepsilon$ is satisfied on a set $\Omega_k \subset \Omega_0$ with capacity $C(\Omega_k) > \delta > 0$, for all $k = 1, 2, \ldots$ Then $\varepsilon\chi_{\Omega_k}(t) \le |f_{n_k}(t)|$ and so $\varepsilon\|\chi_{\Omega_k}\|_X \le \|f_{n_k}\|_X$. Since $C(\Omega_0) < \infty$ we have

$$\frac{\varepsilon}{C_X}\int \chi_{\Omega_k}\, dC \le \varepsilon\|\chi_{\Omega_k}\|_X \le \|f_{n_k}\|_X,$$

and if $k \to \infty$, it follows that $\lim_k C(\Omega_k) = 0$, which is impossible. Hence $f_n \to 0$ in capacity on any set of finite capacity. ■

3. Calderón products of quasi-normed capacitary function spaces. From now on, let X_0 and X_1 be quasi-normed capacitary function spaces and $\alpha \in (0,1)$. The *Calderón product* of X_0 and X_1, denoted by $X = X_0^{1-\alpha}X_1^{\alpha}$, is the class of all $f \in L_0(C)$ such that

$$|f(t)| \le \lambda|f_0(t)|^{1-\alpha}|f_1(t)|^{\alpha} \qquad (t \in \Omega) \tag{1}$$

for some $\lambda > 0$, and each $f_0 \in X_0$ and $f_1 \in X_1$ with $\|f_0\|_{X_0} \le 1$, $\|f_1\|_{X_1} \le 1$.

We endow X with $\|f\|_X := \inf \lambda$, where the infimum runs over all λ satisfying (1). Note that $\{f \ne 0\}$ is C-sigma-finite and if

$$\varrho_\alpha(f) := \begin{cases} \|f\|_X & \text{if } f \in X, \\ \infty & \text{if } f \notin X, \end{cases} \tag{2}$$

then $X = \{f \in L_0(C) : \varrho_\alpha(f) < \infty\}$. We can also write for $f \ge 0$,

$$\varrho_\alpha(f) = \inf\{\lambda > 0 : f \le \lambda f_0^{1-\alpha} f_1^{\alpha},\ f_i \ge 0,\ \|f_i\|_{X_i} \le 1,\ i = 0,1\},$$

and note that ϱ_α satisfies all the required properties to define a quasi-normed capacitary function space with $\|f\|_X = \varrho_\alpha(|f|)$.

Indeed, we just follow the usual arguments but recalling that given sequences convergent to zero in X_0 and X_1, respectively, by Theorem 2.2 and [CMS, Theorem 5] they converge to zero in capacity on any $A \subset \{f \ne 0\}$ of finite capacity. Hence, by passing to subsequences, they can be supposed to be convergent to zero C-q.e. on A. Then the proof follows.

We may canonically associate to X a couple of spaces in the following way:

(a) $X_0 \cap X_1$ consists of the elements common to X_0 and X_1. The quasi-norm is introduced by

$$\|f\|_{X_0\cap X_1} = \max\{\|f\|_{X_0}, \|f\|_{X_1}\} \quad (x \in X_0 \cap X_1),$$

(b) $X_0 + X_1$ denotes the set of elements of the form $x = u + v$, where $u \in X_0$, $v \in X_1$, and it is equipped with the quasi-norm

$$\|x\|_{X_0+X_1} = \inf\{\|u\|_{X_0} + \|v\|_{X_1}\},$$

where the infimum is taken over all elements $u \in X_0$, $v \in X_1$ whose sum is equal to x.

PROPOSITION 3.1. $X_0^{1-\alpha}X_1^\alpha$ *satisfies*

$$X_0 \cap X_1 \hookrightarrow X_0^{1-\alpha}X_1^\alpha \hookrightarrow X_0 + X_1.$$

Proof. The first embedding follows as usual.

Let $f \in X_0^{1-\alpha}X_1^\alpha$. Then, if $|f(t)| \le \lambda |f_0(t)|^{1-\alpha}|f_1(t)|^\alpha$, with f_0, f_1 and $\lambda > 0$ satisfying the required conditions, then

$$|f(t)| \le \lambda\{(1-\alpha)|f_0(t)| + \alpha|f_1(t)|\}$$

and then

$$\|f\|_{X_0+X_1} \lesssim \lambda\{(1-\alpha)\|f_0\|_{X_0+X_1} + \alpha\|f_1\|_{X_0+X_1}\} \le \lambda$$

which implies that $f \in X_0 + X_1$ and $\|f\|_{X_0+X_1} \lesssim \|f\|_{X_0^{1-\alpha}X_1^\alpha}$. ■

THEOREM 3.2. *The space* $X_0^{1-\alpha}X_1^\alpha$ *is complete.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence satisfying $\sum_n \|f_n\|_X < \infty$. Given $\epsilon > 0$, we can find $\lambda_n > 0$, $f_{0,n} \in X_0$ and $f_{1,n} \in X_1$ with norms less than one, and $\lambda_n \le \|f_n\|_X + \frac{\epsilon}{2^n}$ such that $|f_n(t)| \le \lambda_n |f_{0,n}(t)|^{1-\alpha}|f_{1,n}(t)|^\alpha$. Then

$$\sum_n |f_n(t)| \le \Lambda \cdot \sum_n \Big(\frac{\lambda_n}{\Lambda}|f_{0,n}(t)|\Big)^{1-\alpha}\Big(\frac{\lambda_n}{\Lambda}|f_{1,n}(t)|\Big)^\alpha, \quad \text{where } \Lambda = \sum_n \lambda_n.$$

By Corollary 1.2.10 of [S] (see [CMS, Theorem 2]) applied with $\frac{1}{p} = 1-\alpha$ and $\frac{1}{q} = \alpha$ to $\bar{f}_n^p := \frac{\lambda_n}{\Lambda}|f_{0,n}|$ and $g_n := \big(\frac{\lambda_n}{\Lambda}|f_{1,n}|\big)^\alpha$,

$$\begin{aligned}\sum_n |f_n| &\le k\Lambda \cdot \Big(\sum_n \bar{f}_n^p\Big)^{1/p}\Big(\sum_n g_n^q\Big)^{1/q} \\ &= k \cdot \Lambda \cdot \Big(\sum_n \frac{\lambda_n}{\Lambda}|f_{0,n}|\Big)^{1-\alpha}\Big(\sum_n \frac{\lambda_n}{\Lambda}|f_{1,n}|\Big)^\alpha.\end{aligned}$$

As the functions in brackets are defined C-q.e. belonging to X_0 and X_1, then $\sum_n |f_n| \in X$. If $f := \sum_n f_n$, then $f \in X$, $\|f\|_X \le k\sum_n \|f_n\|_X$.

Applying this inequality to $f(\cdot) - \sum_{n=1}^N f_n(\cdot) = \sum_{N+1}^\infty f_n(\cdot)$ and letting $N \to \infty$, we see that $\lim_{N\to\infty} \sum_{n=1}^N f_n = f$ C-q.e. ■

THEOREM 3.3. *Let* $0 < p_0, p_1 \le \infty$, $\alpha \in (0,1)$ *and* $\frac{1}{p} = \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$. *Then*

$$L^{p_0}(C)^{1-\alpha}L^{p_1}(C)^\alpha = L^p(C)$$

with equivalent quasi-norms (*or equal norms in the normed case*).

Proof. Let $X_i = L^{p_i}(C)$ $(i = 0,1)$ and $f \in X_0^{1-\alpha}X_1^\alpha$, and suppose that $|f(t)| \le \lambda|f_0(t)|^{1-\alpha}|f_1(t)|^\alpha$ as in (1). By applying Corollary 1.2.10 of [S] with conjugate exponents $\frac{p_0}{(1-\alpha)p}$ and $\frac{p_1}{\alpha p}$, it follows that

$$\begin{aligned}\int_\Omega |f|^p\, dC &\le \int_\Omega \lambda^p |f_0|^{(1-\alpha)p}|f_1|^{\alpha p}\, dC \\ &\lesssim \lambda^p \|f_0\|_{X_0}^{(1-\alpha)p}\|f_1\|_{X_1}^{\alpha p} \le \lambda^p,\end{aligned}$$

from which we obtain $L^{p_0}(C)^{1-\alpha}L^{p_1}(C)^\alpha \hookrightarrow L^p(C)$. The opposite embedding follows trivially. ■

Recall that the *distribution function* C_f (see [CMS]) of $f \in L_0(\Omega)$ is defined by

$$C_f(t) := C\{|f| > t\} \qquad (t > 0)$$

and the *decreasing rearrangement* f_C^* of f as

$$f_C^\star(x) = \inf\{t > 0 : C\{|f| > t\} \le x\} \qquad (x > 0).$$

Define $f^{**} := \frac{1}{t}\int_0^t f_C^\star(s)\,ds$, which is decreasing and $f_C^\star \le f^{**}$ (see [S]).

Then, two functions g and $\widetilde{g}$ are called *equicapacitable* on Ω if

$$C\{x \in \Omega : |g(x)| > \lambda\} = C\{x \in \Omega : |\widetilde{g}(x)| > \lambda\} \qquad (\lambda > 0).$$

For a quasi-Banach lattice X, we define

$$X^* := \{f \in L_0(C) : f^{**} \in X\}, \qquad \|f\|_{X^*} = \|f^{**}\|_X.$$

Then X^* is a vector space and $f_n \uparrow f$ C-q.e. implies $\|f_n\|_{X^*} \uparrow \|f\|_{X^*}$.

In this capacitary setting the relation between $(X_0^*)^{1-\alpha}(X_1^*)^\alpha$ and $X^* = (X_0^{1-\alpha}X_1^\alpha)^*$ for $0 < \alpha < 1$, X_0 and X_1 be Banach lattices can be partially analyzed. Let $f \in (X_0^*)^{1-\alpha}(X_1^*)^\alpha$. The embedding

$$(X_0^*)^{1-\alpha}(X_1^*)^\alpha \hookrightarrow (X_0^{1-\alpha}X_1^\alpha)^*$$

follows as usual.

The proof of $(X_0^{1-\alpha}X_1^\alpha)^* \hookrightarrow (X_0^*)^{1-\alpha}(X_1^*)^\alpha$ can be done under some additional conditions. The function f_C^* is related to $C_f(t)$ as follows:

$$C_f[f_C^*(t)] \ge t, \quad f_C^*[C_f(t)] \ge t \quad (t > 0). \tag{3}$$

Then

$$f_C^*\{C_f[|f(x)|]\} \ge |f(x)|. \tag{4}$$

Consider the *Hardy operators* P and Q defined as

$$(Pf)(t) := \frac{1}{t}\int_0^t f(s)\,ds, \quad (Qf)(t) := \int_t^\infty \frac{f(s)}{s}\,ds.$$

If $g \ge 0$, then it is well-known that

$$Q(Pg)(t) = (Pg)(t) + (Qg)(t) \qquad (t > 0).$$

On the other hand, if $g_1, g_2 \ge 0$, then by the Hölder inequality,

$$Q(g_1^{1-\alpha}g_2^\alpha) \le 2c(Qg_1)^{1-\alpha}(Qg_2)^\alpha.$$

Now we are ready to show that if $f \in X$ with finite norm and P and Q are bounded in X_0 and X_1, then the desired result holds. Indeed, let c be a bound for the norms of P and Q in X_0 and X_1. Suppose that $f^{**}(\cdot) \le \lambda g_1(\cdot)^{1-\alpha}g_2(\cdot)^\alpha$ for $f \in X$. Define

$$h_1 = \frac{1}{c^2}\,Qg_1, \ h_2 = \frac{1}{c^2}\,Qg_2, \ h_i(0) = \infty, \ h_i(+\infty) = \lim_{t\to\infty} h_i(t) \quad (i = 1, 2).$$

Then $f_C^* \le Qf^{**} \le \lambda Q(g_1^{1-\alpha}g_2^\alpha) \le 2c^3\lambda h_1^{1-\alpha}h_2^\alpha$ since $f^{**} = Pf_C^*$.

Define now $f_1(\cdot) := h_1\{C_f(|f(\cdot)|)\}$ and $f_2(\cdot) := h_2\{C_f(|f(\cdot)|)\}$. Since $|f|$ and f_C^* are equicapacitable, then f_i is equicapacitable with $h_i\{C_f(f_C^*)\}$ $(i = 1, 2)$. Hence, $(f_i)_C^* = h_i\{C_f(f_C^*)\}$ at all points of continuity of $(f_i)_C^*$. (3) and the non-increasing character of h_i imply $(f_i)^*_C \le h_i$, except perhaps at the discontinuity points of $(f_i)^*_C$. Then $f_i^{**} \le \frac{1}{c^2} PQg_i$. The boundedness of P and Q gives $f_i \in X^*_{i-1}$, $i = 1, 2$, but

$$|f| \le 2c^3 \lambda h_1 \{C_f\{|f|\}\}^{1-\alpha} h_2 \{C_f\{|f|\}\}^{\alpha} = 2c^3 \lambda f_1^{1-\alpha} f_2^{\alpha} \quad \text{by (4).}$$

Then, the conclusion follows.

REMARK 3.4. In particular, for $1/p = (1-\alpha)/p_0 + \alpha/p_1$,

$$L^p(C)^* \hookrightarrow \big((L^{p_0}(C))^*\big)^{1-\alpha} \big((L^{p_1}(C))^*\big)^{\alpha} \hookrightarrow \big(L^{p_0}(C)^{1-\alpha} L^{p_1}(C)^{\alpha}\big)^*,$$

where $L^{p_0}(C)^{1-\alpha} L^{p_1}(C)^{\alpha} = L^p(C)$ (see Theorem 3.3).

4. Capacitary Orlicz spaces. From now on, $\varphi : [0,\infty) \to [0,\infty]$ is an *unbounded increasing function*, $\varphi(0) = 0$, which is neither identically zero nor identically infinite.

Define the *Orlicz class* $P_C(\varphi)$ as the set of all $f \in L_0(\Omega)$ for which

$$M^{\varphi}(f) := \rho_{\varphi}(f) = \int_{\Omega} \varphi(|f|)\, dC < \infty.$$

Then

$$L^{\varphi}(C) := \{f \in L_0(\Omega) : \|f\|_{\varphi} < \infty\},$$

where

$$\|f\|_{\varphi} := \inf\{\lambda > 0 : M^{\varphi}(\lambda^{-1} f) \le 1\}.$$

The space $L^{\varphi}(C)$ is called a *capacitary Orlicz function space*.

DEFINITION 4.1. A function H on $[0,\infty)$ (or on a linear space) is called *quasi-convex* with constant $\beta \ge 1$, if

$$H(\lambda x + (1-\lambda)y) \le \beta \big\{\lambda H(x) + (1-\lambda) H(y)\big\} \quad \text{for } 0 \le \lambda \le 1 \text{ and } x, y > 0.$$

Let us observe that the quasi-subadditivity of the Choquet integral implies that M^{φ} is quasi-convex when φ is. We say that φ satisfies the Δ_2*-condition* if there exist $s_0, c > 0$ such that

$$\varphi(2s) \le c\varphi(s) < \infty \quad (s_0 \le s < \infty).$$

Let C be a finite capacity and φ a quasi-convex function with the Δ_2-condition. Then, as usual, $P_C(\varphi)$ is a linear subspace of $L_0(\Omega)$.

PROPOSITION 4.2. $f = 0 \quad C$-q.e. $\Leftrightarrow M^{\varphi}(kf) \le 1$ $(k > 0)$.

Proof. If $f = 0$ C-q.e., then $M^{\varphi}(kf) = 0$ $(k > 0)$. Conversely, suppose that $M^{\varphi}(kf) \le 1$ $(k > 0)$, but for some $\epsilon > 0$, $|f| \ge \epsilon$ on $E \subset \Omega$ with $C(E) > 0$. Then

$$M^{\varphi}(kf) = \int_{\Omega} \varphi(k|f|)\, dC \ge \int_E \varphi(\epsilon k)\, dC = C(E)\varphi(\epsilon k).$$

Since $\varphi(s) \uparrow \infty$ as $s \uparrow \infty$, we obtain a contradiction. ∎

Note that $L^p(C)$ is an Orlicz space since, if $\varphi(t) = t^p$, then

$$\|f\|_{\varphi} := \inf\Big\{\lambda > 0 : \frac{1}{\lambda^p} \int_{\Omega} |f(x)|^p\, dC \le 1\Big\}$$

and $L^\varphi(C) = L^p(C)$ with $\|f\|_{L^\varphi(C)} = \|f\|_{L^p(C)}$, for any $p \in (0,\infty)$. It is complete also when $0 < p < 1$ although in that case φ is not convex. It is a *p-convex function*, where a function $\varphi : [0,\infty) \to [0,\infty)$ is called *s-convex* (resp. *(s)-convex*) $(0 < s \le 1)$ if

$$\varphi(\alpha t_1 + \beta t_2) \le \alpha^s \varphi(t_1) + \beta^s \varphi(t_2) \text{ for each } t_1, t_2 \in [0,\infty)$$

and all $\alpha, \beta \ge 0$ such that $\alpha^s + \beta^s = 1$ (resp., such that $\alpha + \beta = 1$).

Any convex function is 1-convex and every (s)-convex function is s-convex. The converse is false, $\varphi(t) = t^p$ $(0 < p < 1)$ is not (p)-convex.

From now on, if nothing else is said, φ will be any s-convex function and $0 < s \le 1$. Define

$$L_\varphi(C) := \{f : \lim_{\lambda\to 0^+} \rho_\varphi(\lambda f) = 0\}.$$

Trivially, $L_\varphi(C) \subset L^\varphi(C)$.

Modular spaces were first defined by H. Nakano in 1950 (see [Nak]) on vector lattices. Independently, another version was introduced by J. Musielak and W. Orlicz around 1959 (see [MO1] and [MO]).

Let X be a real vector space on $L_0(\Omega)$. A functional $\rho : X \to [0,\infty]$ is called a *modular* if it satisfies the following conditions:

(a) $\rho(x) = 0 \Longleftrightarrow x = 0$,
(b) $\rho(-x) = \rho(x)$ for each $x \in X$,
(c) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for $x, y \in X$, $\alpha, \beta \ge 0$ such that $\alpha + \beta = 1$.

It is a *pseudo-modular* if $\rho(0) = 0$ and it satisfies (b) and (c), and the pseudo-modular ρ is said to be *s-convex* if ρ is an s-convex function.

PROPOSITION 4.3. *If C is a concave capacity, then ρ_φ is an s-convex pseudo-modular on $L_0(\Omega)$.*

Proof. By observing that φ is an s-convex function and C concave. ∎

THEOREM 4.4. *If ρ is an s-convex pseudo-modular in $L_\varphi(C)$, then $L_\varphi(C) = L^\varphi(C)$ and a norm can be defined on $L_\varphi(C)$ as follows*

$$\|f\|_{\varphi,s} := \inf\Big\{\lambda > 0 : \rho_\varphi\Big(\frac{f}{\lambda^{1/s}}\Big) \le 1\Big\}.$$

Proof. If $f \in L^\varphi(C)$, then $\rho_\varphi(\lambda_0 f) < \infty$ for some $\lambda_0 > 0$. Hence, if $0 < \lambda < \lambda_0$, then

$$\rho_\varphi(\lambda f) = \rho_\varphi\Big(\frac{\lambda}{\lambda_0}\,\lambda_0 f\Big) = \rho_\varphi\Big(\frac{\lambda}{\lambda_0}\,(\lambda_0 f) + \Big(1 - \frac{\lambda}{\lambda_0}\Big)0\Big) \le \Big(\frac{\lambda}{\lambda_0}\Big)^s \rho_\varphi(\lambda_0 f) \to 0$$

as $\lambda \to 0$, so that $f \in L_\varphi(C)$.

Now, let us show that $\|\cdot\|_{\varphi,s}$ is a norm. By a direct proof $\|f\|_{\varphi,s} = 0$ if and only if $f = 0$ C-q.e. and $\|\lambda f\|_{\varphi,s} = |\lambda|^s \|f\|_{\varphi,s}$ for all $\lambda \in \mathbb{R}$.

Let $u, v > 0$ such that $\|f\|_{\varphi,s} < u$, $\|g\|_{\varphi,s} < v$. It follows that

$$\begin{aligned}\rho_\varphi\Big(\frac{f+g}{(u+v)^{1/s}}\Big) &= \rho_\varphi\Big(\frac{u^{1/s}}{(u+v)^{1/s}}\,\frac{f}{u^{1/s}} + \frac{v^{1/s}}{(u+v)^{1/s}}\,\frac{g}{v^{1/s}}\Big) \\ &\le \frac{u}{u+v}\,\rho_\varphi\Big(\frac{f}{u^{1/s}}\Big) + \frac{v}{u+v}\,\rho_\varphi\Big(\frac{g}{v^{1/s}}\Big) \le 1,\end{aligned}$$

and thus, $\|f+g\|_{\varphi,s} \le \|f\|_{\varphi,s} + \|g\|_{\varphi,s}$. ∎

Thus, if C is concave, then $L_\varphi(C) = L^\varphi(C)$ and $\|\cdot\|_{\varphi,s}$ is a norm. In this case, $L_\varphi(C)$ is called a *capacitary s-convex Orlicz function space.*

REMARK 4.5. $\|f\|_{\varphi,s} = \|f\|_\varphi^s$ since

$$\inf\Big\{(u^{1/s})^s > 0 : \rho_\varphi\Big(\frac{f}{u^{1/s}}\Big) \le 1\Big\} = \Big[\inf\Big\{\lambda > 0 : \rho_\varphi\Big(\frac{f}{\lambda}\Big) \le 1\Big\}\Big]^s = \|f\|_\varphi^s.$$

By Theorem 4.4 and Remark 4.5, if ρ is an s-convex pseudo-modular in $L_\varphi(C)$, then $\|\cdot\|_\varphi$ is a quasi-norm in $L^\varphi(C)$ since, for $f, g \in L_0(\Omega)$,

$$\|f+g\|_\varphi = (\|f+g\|_{\varphi,s})^{1/s} \le 2^{1/s}(\|f\|_{\varphi,s}^{1/s} + \|g\|_{\varphi,s}^{1/s}) = 2^{1/s}(\|f\|_\varphi + \|g\|_\varphi).$$

PROPOSITION 4.6. *$\|\cdot\|_\varphi$ is a quasi-norm on $L^\varphi(C)$.*

Proof. Observe that, since φ is s-convex, we have

$$\varphi(a^{1/s}t) = \varphi\big(a^{1/s}t + (1-a)^{1/s}0\big) \le a\varphi(t) \quad (0 < a < 1)$$

and hence, $\varphi(\lambda t) \le \lambda^s \varphi(t)$ ($\lambda \in (0,1)$). Then, the first two properties of a quasi-norm follow.

Moreover, let $f, g \in L^\varphi(C)$ and take $u^{1/s} > \|(2c)^{1/s} f\|_\varphi$ and $v^{1/s} > \|(2c)^{1/s} g\|_\varphi$. By the quasi-subadditivity, we have for $\theta := \frac{u}{u+v}$,

$$\begin{aligned} M^\varphi\Big(\frac{f+g}{(u+v)^{1/s}}\Big) &\le \int_\Omega \Big(\theta\varphi\Big(\frac{|f|}{u^{1/s}}\Big) + (1-\theta)\varphi\Big(\frac{|g|}{v^{1/s}}\Big)\Big)\, dC \\ &\le \int_\Omega \Big(\frac{\theta}{2c}\varphi\Big(\frac{(2c)^{1/s}|f|}{u^{1/s}}\Big) + \frac{1-\theta}{2c}\varphi\Big(\frac{(2c)^{1/s}|g|}{v^{1/s}}\Big)\Big)\, dC \\ &\le \theta M^\varphi\Big(\frac{(2c)^{1/s}f}{u^{1/s}}\Big) + (1-\theta)M^\varphi\Big(\frac{(2c)^{1/s}g}{v^{1/s}}\Big) \le 1. \end{aligned}$$

The assertion follows since $\|f+g\|_\varphi \le (u+v)^{1/s} \le 2^{1/s}(u^{1/s} + v^{1/s})$. ■

THEOREM 4.7. *Under the same conditions,*

(i) $\|f_k - f\|_{\varphi,s} \overset{k\to\infty}{\longrightarrow} 0$ *if and only if* $\rho_\varphi(\lambda(f_k - f)) \overset{k\to\infty}{\longrightarrow} 0$ $(\lambda > 0)$.
(ii) $\{f_k\}_k$ *is a Cauchy sequence in* $L^\varphi(C)$ *with respect to* $\|\cdot\|_{\varphi,s}$ *if and only if* $\rho_\varphi(\lambda(f_k - f_l)) \overset{k,l\to\infty}{\longrightarrow}$ *for all* $\lambda > 0$.

Proof. If $\rho_\varphi(\lambda f_k) \overset{k\to\infty}{\longrightarrow} 0$, then there exists $k_\lambda \in \mathbb{N}$ such that

$$\rho_\varphi\Big(\frac{f_k}{(\lambda^{-s})^{1/s}}\Big) \le 1 \quad \text{for each } k \ge k_\lambda \text{ and } \lambda > 0.$$

Hence, $\|f_k\|_{\varphi,s} \le \frac{1}{\lambda^s}$ for all $k \ge k_\lambda$, $\lambda > 0$, and so $\|f_k\|_{\varphi,s} \overset{k\to\infty}{\longrightarrow} 0$.

Conversely, if $\|f_k\|_{\varphi,s} \overset{k\to\infty}{\longrightarrow} 0$, then given $\epsilon > 0$, there exists $k_{\lambda,\epsilon} \in \mathbb{N}$ such that $\rho_\varphi(\frac{\lambda f_k}{\epsilon^{1/s}}) \le 1$ for all $k \ge k_{\lambda,\epsilon}$ and

$$\begin{aligned} \rho_\varphi(\lambda f_k) &= \int_\Omega \varphi\Big(\epsilon^{1/s}\Big(\frac{\lambda|f_k|}{\epsilon^{1/s}}\Big)\Big)\, dC \\ &\le \int_\Omega \Big(\epsilon\varphi\Big(\frac{\lambda|f_k|}{\epsilon^{1/s}}\Big) + (1-\epsilon)\varphi(0)\Big)\, dC = \epsilon\rho_\varphi\Big(\frac{\lambda f_k}{\epsilon^{1/s}}\Big). \end{aligned}$$

Hence, $\rho_\varphi(\lambda f_k) \to 0$ as $k \to 0$ for any $\lambda > 0$. ■

Theorem 4.8. *Let C be a concave capacity and φ an increasing convex function. Then $(L^\varphi(C), \|\cdot\|_\varphi)$ is a Banach function space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence for $\|\cdot\|_\varphi$ and $x_0 := \sup\{x \in \mathbb{R} : \varphi(x) = 0\}$. Then, $0 \le x_0 < \infty$ since $\{\varphi = 0\}$ is relatively compact.

Since by Remark 4.5 it follows that Theorem 4.7 holds also for $\|\cdot\|_\varphi$, then there exists $k_{mn} \ge 0$ such that

$$\int_\Omega \varphi\big(k_{mn}|f_n - f_m|\big)\, dC \le 1 \qquad (m, n \in \mathbb{N}).$$

First note that $A_{mn} := \big\{k_{mn}|f_n - f_m| > x_0\big\} \in \Sigma$ is at most σ-finite. Indeed, defining $B_k := \big\{k_{mn}|f_n - f_m| > x_0 + k^{-1}\big\}$ $(k \in \mathbb{N})$, we have $A_{mn} = \bigcup_{k=1}^\infty B_k$, where $C(B_k) < \infty$ for all k since

$$C(B_k)\varphi(x_0 + k^{-1}) = \int_{B_k} \varphi(x_0 + k^{-1})\, dC \le \int_{B_k} \varphi\big(k_{mn}|f_n - f_m|\big)\, dC \le 1.$$

Therefore, each A_{mn} is σ-finite and so is $A := \bigcup_{m,n\ge 1} A_{mn}$.

On A^c, $k_{mn}|f_n - f_m| \le x_0$ and then $|f_n - f_m| \to 0$ uniformly. Hence, there exists $g_0 \in L_0(A^c)$ such that $f_n \to g_0$ and $|g_0| \le x_0$ on A^c.

Temporarily, write Ω for A. If B satisfies $C(B) < \infty$, then

$$C\big(B \cap \{|f_n - f_m| \ge \epsilon|\}\big) \le \frac{1}{\varphi(k_{mn}\epsilon)} \int_\Omega \varphi\big(k_{mn}|f_n - f_m|\big)\, dC \le \frac{1}{\varphi(k_{mn}\epsilon)}\,.$$

Since $k_{mn} \to \infty$ and $\epsilon > 0$ is fixed, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in capacity on B. Then, by [CMS, Theorem 5], there is a subsequence pointwise convergent on B to some $\widetilde{f}$, and on $\bigcup_k B_k$ since $C(B_k) < \infty$ $(k \in \mathbb{N})$. Then, there exists $\{f_{n_i}\}_{i\in\mathbb{N}}$ such that $f_{n_i} \to \widetilde{f}$ C-q.e.

Let $f := \widetilde{f}\chi_A + g_0\chi_{A^c}$. Hence, $f_{n_i} \to f$ C-q.e., and by Cauchy, $\|f_n\|_\varphi \to \rho$. Then, by the Fatou property, $f \in L^\varphi(C)$ and by continuity,

$$\varphi\big(|f_{n_i} - f_{n_j}|k\big) \to \varphi\big(|f - f_{n_j}|k\big) \ C\text{-q.e. as } i \to \infty \qquad (k \ge 0).$$

Then, if $n_0 \ge 1$ is chosen such that $n_i, n_j \ge n_0$ implies $k_{n_i n_j} \ge k$,

$$\int_\Omega \varphi\big(k|f_{n_i} - f_{n_j}|\big)\, dC \le \int_\Omega \varphi\big(k_{n_i n_j}|f_{n_i} - f_{n_j}|\big)\, dC \le 1.$$

Letting $n_i \to \infty$, we have $\|f - f_{n_j}\|_\varphi \le k^{-1}$ and the result then follows. ■

In general, the continuity property of an s-convex function is needed for the completeness of the capacitary s-convex Orlicz function space because s-convex functions are not always continuous.

Example 4.9. Let $0 < s < 1$ and $k > 1$. Define for $u \in \mathbb{R}_+$,

$$f(u) = \big\{u^{s/(1-s)} \text{ if } 0 \le u \le 1,\ ku^{s/(1-s)} \text{ if } u \ge 1\big\}.$$

Then $f \ge 0$, discontinuous at $u = 1$, s-convex not (s)-convex.

Theorem 4.10. *Let φ be a continuous s-convex, or an increasing convex function. Then $(L^\varphi(C), \|\cdot\|_\varphi)$ is a quasi-Banach function space.*

Proof. For all $\lambda, \eta > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho_\varphi(\lambda(f_n - f_m)) < \eta \quad (m, n \geq N).$$

Thus, defining $A_{n,m} := \{x \in \Omega : \lambda|f_n(x) - f_m(x)| \geq \epsilon\}$ $(\varepsilon > 0)$ we have

$$C(A_{n,m})\varphi(\epsilon) \leq \rho_\varphi(\lambda(f_n - f_m)) < \eta \qquad (m, n \geq N).$$

Then, by [CMS, Theorem 5], $\{\lambda f_n\}_{n\in\mathbb{N}}$ is convergent in capacity to a function λf and contains a subsequence $\{\lambda f_{n_k}\}_{k\in\mathbb{N}}$ convergent to λf C-q.e. in Ω. By continuity,

$$\varphi(\lambda|f_n(x) - f_{n_k}(x)|) \to \varphi(\lambda|f_n(x) - f(x)|) \ C\text{-q.e. in } \Omega$$

and, by the Fatou property, it follows that

$$\rho_\varphi(\lambda(f_n - f)) \leq \liminf_{k\to\infty} \rho_\varphi(\lambda(f_n - f_{n_k})) < \eta \quad (n \geq N).$$

Thus $\|f_n - f\|_\varphi \to 0$ as $n \to \infty$, and $f \in L^\varphi(C)$. ∎

EXAMPLE 4.11. Let (Ω, Σ, μ) be a measure space, and $\psi(t) := t^{1-p}$ $(0 < p < 1)$ which is concave and continuous. Then $C_\psi(A) := \psi(\mu(A))$ defines a concave Fatou capacity (see [Ce]). Hence, if for instance $\varphi(t) = t^2$, then $L^\varphi(C_\psi)$ is a Banach function space with $\|\cdot\|_\varphi$.

Nevertheless, if $\varphi(t) := t^p$, then $L^\varphi(C)$ defined by the condition $\|f\|_\varphi < \infty$ is a *capacitary p-convex Orlicz function space.*

5. Interpolation of capacitary s-convex spaces

DEFINITION 5.1. Let φ be a positive function on $\mathbb{R}_+$ such that, for every $\lambda \in \mathbb{R}_+$ there exists a constant $\bar{C} = C(\lambda)$ such that $\varphi(\lambda x) \leq \bar{C}\varphi(x)$. Then, φ is of *lower type* p_0 and *upper type* p_1 when

$$\varphi(\lambda x) \leq \bar{C} \max(\lambda^{p_0}, \lambda^{p_1})\varphi(x).$$

Assume further that φ is continuous increasing with $\varphi(\mathbb{R}_+) = \mathbb{R}_+$ so that, φ^{-1} exists and is continuous increasing too. Then, if φ is of type (p_0, p_1) with $p_0 > 0$, then φ^{-1} is of type (p_1^{-1}, p_0^{-1}) (see [GP]).

Every s-convex function is of positive lower type since for all $\alpha > 0$, if we take $\beta = (1 - \alpha^s)^{1/s}$ and $y = 0$, it follows that

$$\varphi(\alpha x) = \varphi(\alpha x + \beta 0) \leq \alpha^s\varphi(x) + \beta^s\varphi(0) = \alpha^s\varphi(x).$$

A positive function ρ on $\mathbb{R}_+$ is *quasi-concave* when it is equivalent to a concave one, and it is pseudo-concave if and only if for a suitable $\bar{C}$

$$\rho(\lambda x) \leq \bar{C} \max(1, \lambda)\rho(x). \tag{5}$$

The class of functions satisfying (5) will be denoted by $\mathfrak{B}(C)$ (see [Pe]).

REMARK 5.2. Let us introduce $R(x, y) = x\rho(y/x)$. Then $\rho \in \mathfrak{B}(1)$ if and only if R is non-decreasing in each variable separately. In fact, it fulfils in the strong sense $x < x'$ and $y < y' \Rightarrow R(x, y) < R(x', y')$.

Given $\rho \in \mathfrak{B}(1)$, it follows for any positive sequences $\{x_\eta\}_\eta, \{y_\eta\}_\eta$,

$$\sum R(x_\eta, y_\eta) \leq 2R\Big(\sum x_\eta, \sum y_\eta\Big).$$

Definition 5.3. A function $\rho : X \to [0,\infty]$ is called a *quasi-modular* if it satisfies the following properties:

(a) $\rho(x) = 0 \Longleftrightarrow x = 0$,
(b) $\rho(\lambda x) \le \rho(x)$ if $|\lambda| \le 1$, $\rho(-x) = \rho(x)$,
(c) $\lim_{\lambda\to 0} \rho(\lambda x) = 0$ if $\rho(x) < \infty$,
(d) $\rho((x+y)/h) \le k(\rho(x)+\rho(y))$ for certain constants h and k.

From now on, let φ, φ_0 and φ_1 be continuous increasing functions on $\mathbb{R}_+$ such that $\varphi, \varphi_i((0,\infty)) = (0,\infty)$, $i = 0,1$, and $\varphi(0) = 0$. It follows by similar techniques to the ones in Theorem 4.10 that $(L^\varphi(C), \|\cdot\|_\varphi)$ is a quasi-Banach function space when φ is of positive lower type.

Proposition 5.4. *Assume that φ is of positive lower type and it can be expressed by $\varphi^{-1} = \varphi_0^{-1}\rho\big(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\big)$ with ρ quasi-concave. If*

$$\int_\Omega \varphi_i(|a_i|)\,dC \le C_i, \quad i = 0,1, \qquad |a| \le |a_0|\rho\Big(\frac{|a_1|}{|a_0|}\Big),$$

then

$$\int_\Omega \varphi(|a|)\,dC \le 2c(C_0 + C_1),$$

where c is the subadditivity constant associated with the capacity.

Proof. Following [GP], put $b_i := \varphi_i(|a_i|)$, $i = 0,1$, and $b = b_0 + b_1$. We see that $\varphi_0^{-1}, \varphi_1^{-1}$ are increasing, $b_0 \le b$ and $b_1 \le b$. So that $\varphi_0^{-1}(b_0) \le \varphi_0^{-1}(b)$, $\varphi_1^{-1}(b_1) \le \varphi_1^{-1}(b)$ and by Remark 5.2,

$$|a| \le R(|a_0|,|a_1|) = R\big(\varphi_0^{-1}(b_0), \varphi_1^{-1}(b_1)\big) \le R\big(\varphi_0^{-1}(b), \varphi_1^{-1}(b)\big) = \varphi^{-1}(b).$$

The positive lower type of φ and the quasi-subadditivity,

$$\int_\Omega \varphi(|a|)\,dC \le 2c\Big\{\int_\Omega \varphi_0(|a_0|)\,dC + \int_\Omega \varphi_1(|a_1|)\,dC\Big\} \le 2c(C_0 + C_1). \qquad \blacksquare$$

Remark 5.5. Let us interpret the last proposition. Let X_0, X_1 be two rearrangement invariant (r.i. for short)[1] quasi-Banach function spaces, a capacity space (Ω, Σ, C), and ρ be a quasi-concave function. Introduce $X = X_0\rho(\frac{X_1}{X_0})$ as the space of those $h \in L_0(\Omega)$ for which one can find $\widetilde{C}$ and $a_0 \in X_0$ and $a_1 \in X_1$ such that

$$|h| \le \widetilde{C}|a_0|\rho\Big(\frac{|a_1|}{|a_0|}\Big).$$

We equip X with $\|\cdot\|_X = \inf_{\widetilde{C}} \widetilde{C}$. Then, it follows similarly to Theorem 4.10 that $\|\cdot\|_X$ is a quasi-norm and X is a quasi-Banach space.

If $\rho = \rho_\alpha$ in (2), then X is the *Calderón product* $X_0^{1-\alpha}X_1^\alpha$ in (1).

Let φ_i be continuous increasing functions on $\mathbb{R}_+$ and $X_i = L^{\varphi_i}(C)$, $i = 0,1$. It follows that

$$L^{\varphi_0}(C)\rho\Big(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\Big) \hookrightarrow L^\varphi(C), \quad \varphi^{-1} = \varphi_0^{-1}\rho\Big(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\Big). \tag{6}$$

[1] X is r.i. if the following is satisfied: for every $f \in X$ if g measurable with $\mu_f = \mu_g$, then $g \in X$ and $\|f\|_X = \|g\|_X$, where $\mu_f(\lambda) := \mu\{x : |f(x)| > \lambda\}$, $\lambda > 0$.

At this point it is natural to study the converse embedding.

Consider the same interpolation method as in [GP]. Let $\bar{X} = (X_0, X_1)$ be a quasi-Banach couple and ρ a quasi-concave function.

$$\langle X_0, X_1, \varrho\rangle = \{a \in \Sigma(\bar{X}) : \text{ there exists } u = \{u_\nu\}_{\nu\in\mathbb{Z}},\ u_\nu \in \Delta(\bar{X}) \text{ such that (7) and (8) are satisfied}\},$$

where for an absolute constant $\widehat{C}$,

$$a = \sum_{\nu\in\mathbb{Z}} u_\nu \text{ with convergence in } \Sigma(\bar{X}), \tag{7}$$

for all finite $F \subset \mathbb{Z}$ and every real sequence $\{\xi_\nu\}_{\nu\in F}$, $|\xi_\nu| \le 1$ we have

$$\Big\|\sum_{\nu\in F} \frac{\xi_\nu u_\nu}{\rho(2^\nu)}\Big\|_{X_0} \le \widehat{C}, \quad \Big\|\sum_{\nu\in F} \frac{2^\nu \xi_\nu u_\nu}{\rho(2^\nu)}\Big\|_{X_1} \le \widehat{C}. \tag{8}$$

We equip $\langle \bar{X}, \rho\rangle = \langle X_0, X_1, \rho\rangle$ with the quasi-norm

$$\|a\|_{\langle \bar{X},\rho\rangle} = \inf \widehat{C}.$$

Then, if ρ is of lower type 0 and upper type 1, $\langle \bar{X}, \rho\rangle$ is complete.

From now on, assume that φ_0 and φ_1 have positive lower type. If $\rho \in \mathfrak{B}(1)$ and φ is defined by $\varphi^{-1} = \varphi_0^{-1}\rho(\frac{\varphi_1^{-1}}{\varphi_0^{-1}})$, then $L^\varphi(C), L^{\varphi_0}(C)$ and $L^{\varphi_1}(C)$ are quasi-Banach spaces (see [GP]).

Theorem 5.6. *If one of the functions φ_0, φ_1, say φ_0, has finite upper type and $\rho \in \mathfrak{B}(1)$, then φ defined by $\varphi^{-1} = \varphi_0^{-1}\rho(\frac{\varphi_1^{-1}}{\varphi_0^{-1}})$ satisfies $L^\varphi(C) \hookrightarrow \langle \overline{L^\varphi(C)}, \rho\rangle$.*

Proof. It follows similarly to [GP, Theorem 7.1]. ∎

The converse is unknown for us. Let us just comment that we do not have a capacitary version of Fubini's theorem.

Theorem 5.7. *Under the same conditions $L^\varphi(C) \hookrightarrow L^{\varphi_0}(C)\rho(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)})$.*

Proof. Let $f \in L^\varphi(C)$ with norm less than one and $\psi(t) := \varphi_0(\frac{|f|}{\rho(t)}) - \varphi_1(\frac{t|f|}{\rho(t)})$. By hypothesis, ψ is decreasing, continuous, $\lim_{t\to 0}\psi(t) > 0$ and $\lim_{t\to\infty}\psi(t) < 0$. Thus, there exists a unique t such that $\psi(t) = 0$. Let us denote this unique t by the same symbol t. Defining $x = \frac{|f|}{\rho(t)}$ and $y = \frac{t|f|}{\rho(t)}$, since $\psi(t) = 0$, we have $\varphi_0(x) = \varphi_1(y)$. Moreover, $\varphi^{-1}(\varphi_0(x)) = |f|$. Thus

$$\int_\Omega \varphi_0\Big(\frac{|f|}{\rho(t)}\Big)\, dC = \int_\Omega \varphi(|f|)\, dC \le 1,$$

and we can write $|f|$ as an element in $L^{\varphi_0}(C)\rho\big(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\big)$. ∎

In particular, (6) together with Theorem 5.7 recover Theorem 3.3.

Corollary 5.8. *Assume that φ_0 and φ_1 have finite upper type. Define $\varphi^{-1} = \varphi_0^{-1}\rho\big(\frac{\varphi_1^{-1}}{\varphi_0^{-1}}\big)$ for ρ being a quasi-concave function in $\mathfrak{B}(1)$. Then*

$$L^\varphi(C) = L^{\varphi_0}(C)\rho\Big(\frac{L^{\varphi_1}(C)}{L^{\varphi_0}(C)}\Big) \hookrightarrow \langle \overline{L^\varphi(C)}, \rho\rangle.$$

THEOREM 5.9. *Let φ_0 be of finite upper type such that $\varphi_0((0,\infty)) = (0,\infty)$. Define $\varphi^{-1} = \varphi_0^{-1}\rho(\frac{1}{\varphi_0^{-1}})$ for $\rho \in \mathfrak{B}^+(1)$ being quasi-concave. Then*

$$L^{\varphi}(C) = L^{\varphi_0}(C)\rho(\frac{L^{\infty}(C)}{L^{\varphi_0}(C)}) \hookrightarrow \langle L^{\varphi_0}(C), L^{\infty}(C), \rho\rangle.$$

Proof. See [GP, Theorem 9.1]. ∎

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