FUNCTION SPACES X
BANACH CENTER PUBLICATIONS, VOLUME 102
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

# TRACTABLE EMBEDDINGS OF BESOV SPACES INTO ZYGMUND SPACES, II

HANS TRIEBEL

*Institut für Mathematik, Friedrich-Schiller-Universität Jena*
*D-07737 Jena, Germany*
*E-mail: hans.triebel@uni-jena.de*

*Dedicated to the memory of Miroslav Krbec*

**Abstract.** The paper deals with dimension-controllable (tractable) embeddings of Besov spaces on $n$-dimensional cubes into Zygmund spaces.

**1. Introduction and theorem.** Let

$$|x|_q = \begin{cases} \Big( \sum_{j=1}^n |x_j|^q \Big)^{1/q} & \text{if } 0 < q < \infty, \\ \max_{j=1,\ldots,n} |x_j| & \text{if } q = \infty, \end{cases} \tag{1.1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $\mathbb{B}^n_q(r)$ be the related ball of radius $r > 0$ in $\mathbb{R}^n$, centered at the origin, hence

$$\mathbb{B}^n_q(r) = \big\{ x \in \mathbb{R}^n : \ |x|_q < r \big\}, \tag{1.2}$$

and let $|\mathbb{B}^n_q(r)|$ be its volume with respect to the Lebesgue measure in $\mathbb{R}^n$. Let

$$\mathbb{Q}^n = \big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \ 0 < x_j < 1 \big\} \tag{1.3}$$

be the unit cube in $\mathbb{R}^n$. Of course, $|\mathbb{Q}^n| = 1$. Let $B^s_{p,p}(\mathbb{Q}^n)$ with $1 < p < \infty$ and $s > 0$ be the usual Besov spaces in $\mathbb{Q}^n$, hence the restriction of $B^s_{p,p}(\mathbb{R}^n)$ to $\mathbb{Q}^n$.

THEOREM. *Let $1 < p < \infty$, $0 < s < 1/p$ and $0 < q \le \infty$. Then there are a constant $c > 0$ and a radius $r > 0$ such that*

$$\Big( \int_0^1 |\log t|^{sp} \, f^*(t)^p \, \mathrm{d}t \Big)^{1/p} \le c \, \Big| \int_{\mathbb{Q}^n} f(y) \, \mathrm{d}y \Big| + \Big( \int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x) - f(y)|^p}{|x-y|_q^{sp+n}} \, \frac{\mathrm{d}x \, \mathrm{d}y}{|\mathbb{B}^n_q(r)|} \Big)^{1/p} \tag{1.4}$$

*for all $n \in \mathbb{N}$ and all $f \in B^s_{p,p}(\mathbb{Q}^n)$.*

---

2010 *Mathematics Subject Classification*: Primary 46E35; Secondary 42B35.
*Key words and phrases*: Besov spaces, Zygmund spaces, logarithmic inequalities.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc102-0-16

Here $f^*(t)$ is the usual rearrangement of $f \in L_1(\mathbb{Q}^n)$, hence

$$f^*(t) = \inf\{\lambda \in [0,\infty) : \ \mu_f(\lambda) \le t\}, \qquad 0 \le t \le 1, \tag{1.5}$$

with

$$\mu_f(\lambda) = \big|\{x \in \mathbb{Q}^n : \ |f(x)| > \lambda\}\big|, \qquad \lambda \ge 0. \tag{1.6}$$

The left-hand side of (1.4) with $1 + |\log t|$ in place of $|\log t|$ is the standard (quasi-)norm of the *Zygmund space* $L_p(\log L)_s(\mathbb{Q}^n)$. We dealt in [13] with inequalities of type (1.4) for $q = 2$ (hence the usual Euclidean norm in $\mathbb{R}^n$). But there is a weak point in the arguments. We overlooked the influence of the tiny volume $|\mathbb{B}^n_2(r)|$. This will be corrected in the present paper and extended from $q = 2$ to $0 < q \le \infty$.

The above theorem is a fractional version of so-called *logarithmic Sobolev inequalities* dealing mainly with corresponding assertions for the classical Sobolev spaces $W^1_p(\mathbb{R}^n)$, $1 < p < \infty$, their restrictions to domains, including $\mathbb{Q}^n$, and also with the Gauss measure in place of the Lebesgue measure. We described in [13] some of the key assertions and gave related references. This will not be repeated here since the present paper should be considered as the direct continuation of [13]. But to illustrate the situation we mention the following assertion which may be found in [10, Section 7.1, pp. 169–171], based on [9], and which is closely related to the above topic: *Let* $1 < p < \infty$*. Then there is a constant* $c > 0$ *such that*

$$\Big(\int_0^1 |\log t|^{p/2} \, f^*(t)^p \, \mathrm{d}t\Big)^{1/p} \le c\, \|f \,|\, L_p(\mathbb{Q}^n)\| + c\, \|\nabla f \,|\, L_p(\mathbb{Q}^n)\| \tag{1.7}$$

*for all* $f \in W^1_p(\mathbb{R}^n)$ *with* $\operatorname{supp} f \subset \overline{\mathbb{Q}^n}$ *where* $c$ *is independent of* $n \in \mathbb{N}$*, but may depend on* $p$. Furthermore in [10] there are some assertions with $\mathbb{B}^n_q = \mathbb{B}^n_q(1)$ in place of $\mathbb{Q}^n$. One may also consult [5, 6, 7, 8] and the most recent paper [4] where one finds modifications of (1.7) and further references.

In Section 2.1 we collect some preliminaries: the volume of $\mathbb{B}^n_q(r)$, Whitney decompositions and some basic assertions about $B^s_{p,p}(\mathbb{Q}^n)$. Afterwards we prove in Section 2.2 the above theorem following closely [13].

## 2. Preliminaries and proofs

**2.1. Preliminaries.** We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ be the collection of all integers. Let $\mathbb{R}^n$ be Euclidean $n$-space, $n \in \mathbb{N}$, and $\mathbb{R} = \mathbb{R}^1$. Let $\Omega$ be a domain in $\mathbb{R}^n$. Then $L_p(\Omega)$ with $1 \le p < \infty$ is the usual Banach space of all complex-valued Lebesgue-measurable functions such that

$$\|f \,|\, L_p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \, \mathrm{d}x\Big)^{1/p}$$

is finite. We need $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{Q}^n$ where $\mathbb{Q}^n$ is the unit cube (1.3), and $\Omega = \mathbb{B}^n_q(r)$ with $\mathbb{B}^n_q(r)$ as in (1.1), (1.2). Let $f^*(t)$ with $0 < t \le 1$ be the usual rearrangement of $f \in L_1(\mathbb{Q}^n)$ given by (1.5), (1.6). The Zygmund space $L_p(\log L)_a(\mathbb{Q}^n)$ with $1 < p < \infty$ and $a \in \mathbb{R}$ consists of all $f \in L_1(\mathbb{Q}^n)$ such that

$$\|f \,|\, L_p(\log L)_a(\mathbb{Q}^n)\| = \Big(\int_0^1 \big(1 + |\log t|\big)^{ap} \, f^*(t)^p \, \mathrm{d}t\Big)^{1/p}$$

is finite. The standard references are [1, 2]. We rely on [3] and the specific properties considered in [13]. This will not be repeated here with exception of some assertions needed later on.

Let $|\mathbb{B}^n_q(r)|$ be the volume of $\mathbb{B}^n_q(r)$ according to (1.2) with respect to the Lebesgue measure. Obviously,

$$|\mathbb{B}^n_q(r)| = r^n\, |\mathbb{B}^n_q| \qquad \text{where} \quad \mathbb{B}^n_q = \mathbb{B}^n_q(1). \tag{2.1}$$

The volume $|\mathbb{B}^n_q|$ is surely known since a long time. But we have no suitable reference at hand. This may justify to insert the following calculation in appropriate modification of [3, pp. 97–98]. Let $\Gamma(t)$, $t>0$, be the usual $\Gamma$-function.

Proposition 2.1.

(i) *Let $n \in \mathbb{N}$ and $0<q\le\infty$. Then*

$$|\mathbb{B}^n_q| = 2^n\, \frac{\Gamma\big(1+1/q\big)^n}{\Gamma\big(1+n/q\big)}\,. \tag{2.2}$$

(ii) *Let $n\in\mathbb{N}$ and $0<q<\infty$. Then there is a function $\theta:(0,\infty)\to\mathbb{R}$ with $0<\theta(t)<1/12$ for all $t>0$ such that*

$$|\mathbb{B}^n_q| = 2^n (2\pi)^{(n-1)/2}\, \big(\tfrac{1}{q}\big)^{n/2-1/2}\, n^{-n/q-1/2} \exp\big(nq\theta\big(\tfrac{1}{q}\big) - \tfrac{q}{n}\theta\big(\tfrac{n}{q}\big)\big). \tag{2.3}$$

*Proof.* Obviously $|\mathbb{B}^n_\infty| = 2^n$. Let $0<q<\infty$. Then one has by (1.1) and $t_j = x_j^q$, $x_j>0$,

$$\begin{aligned} |\mathbb{B}^n_q| = \int_{|x|_q\le 1} \mathrm{d}x &= \big(\tfrac{2}{q}\big)^n \int_{\sum_{k=1}^n t_k\le 1, t_k>0} \prod_{j=1}^n t_j^{1/q-1}\,\mathrm{d}t \\ &= \big(\tfrac{2}{q}\big)^n\, \Gamma\big(\tfrac{1}{q}\big)^n\, \Gamma\big(\tfrac{n}{q}\big)^{-1} \int_0^1 \tau^{n/q-1}\,\mathrm{d}\tau, \end{aligned} \tag{2.4}$$

where we used [3, p. 98] with a reference to [14, 12.5]. Then (2.2) follows from (2.4) and $\Gamma(1+t) = t\Gamma(t)$, $t>0$. We apply Stirling's formula

$$\Gamma(t) = e^{-t}\, t^{t-1/2}\, (2\pi)^{1/2}\, e^{\theta(t)/t}, \qquad 0<t<\infty,$$

according to [14, 12.33] to

$$|\mathbb{B}^n_q| = \frac{2^n}{nq^{n-1}}\, \frac{\Gamma\big(1/q\big)^n}{\Gamma\big(n/q\big)}\,, \qquad 0<q<\infty.$$

This proves (2.3). ∎

Remark 2.2. For fixed $q$ with $0<q<\infty$ there are two real numbers $c_1$ and $c_2$ such that for all $n\in\mathbb{N}$,

$$|\mathbb{B}^n_q| = C_{n,q} n^{-n/q} \qquad \text{with} \quad 2^{c_1 n} \le C_{n,q} \le 2^{c_2 n}. \tag{2.5}$$

This estimate will be used later on.

As a second preparation we need a modification of the well-known Whitney decomposition of an open set $\Omega$ in $\mathbb{R}^n$. We follow closely [11, pp. 167–168] and adapt the arguments given there to our needs. Let $F$ be a non-empty closed set in $\mathbb{R}^n$ and let $\Omega = \mathbb{R}^n\setminus F$.

Let $\mathbb{Q}^n$ be the unit cube according to (1.3) and

$$Q^n_{j,m} = 2^{-j}m + 2^{-j}\mathbb{Q}^n, \qquad j \in \mathbb{Z}, \quad m \in \mathbb{Z}^n,$$

where $\mathbb{Z}^n$ is the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $0 < q \le \infty$ and

$$\operatorname{dist}_q\big(Q^n_{j,m}, F\big) = \inf\big\{|x-y|_q : \ x \in Q^n_{j,m}, \ y \in F\big\}.$$

Furthermore,

$$\operatorname{diam}_q\big(Q^n_{j,m}\big) = \sup\big\{|x-y|_q : \ x \in Q^n_{j,m}, \ y \in Q^n_{j,m}\big\} = 2^{-j} n^{1/q}. \tag{2.6}$$

Proposition 2.3. *Let $0 < q \le \infty$ and let $F$ be a non-empty closed set in $\mathbb{R}^n$, $F \ne \mathbb{R}^n$. Let $\Omega = \mathbb{R}^n \setminus F$. Then there are subsets $\mathbb{Z}^n_j \subset \mathbb{Z}^n$, $j \in \mathbb{Z}$ (which may be empty), such that*

$$\Omega = \bigcup_{j\in\mathbb{Z}, m\in\mathbb{Z}^n_j} \overline{Q^n_{j,m}} \tag{2.7}$$

*with*

$$Q^n_{j_1,m_1} \cap Q^n_{j_2,m_2} = \emptyset \quad \text{if } j_1, j_2 \in \mathbb{Z}, \ m_j \in \mathbb{Z}^n_j, \ (j_1, m_1) \ne (j_2, m_2), \tag{2.8}$$

*(pairwise disjoint) and*

$$\operatorname{diam}_q(Q^n_{j,m}) \le \operatorname{dist}_q(Q^n_{j,m}, F) \le 2^{1+\max(1,1/q)} \operatorname{diam}_q(Q^n_{j,m}) \tag{2.9}$$

*if $j \in \mathbb{Z}$, $m \in \mathbb{Z}^n_j$.*

*Proof.* Let for some $c > 0$ (chosen later on)

$$\Omega_j = \big\{x \in \mathbb{R}^n : \ c\,2^{-j} < \operatorname{dist}_q(x, F) \le c\,2^{-j+1}\big\}, \qquad j \in \mathbb{Z}.$$

Obviously $\Omega = \bigcup_{j\in\mathbb{Z}} \Omega_j$. Let $M_j$ be the collection of all $m \in \mathbb{Z}^n$ such that $\Omega_j \cap \overline{Q^n_{j,m}} \ne \emptyset$. Then we wish to show that $c > 0$ can be chosen such that

$$\Omega = \bigcup_{j\in\mathbb{Z}, m\in M_j} \overline{Q^n_{j,m}}$$

with (2.9). It is sufficient to justify (2.9). Let $m \in M_j$ and $x \in \Omega_j \cap \overline{Q^n_{j,m}}$. Recall that $(a+b)^q \le a^p + b^p$ and $(a^q + b^q)^{1/q} \le 2^{1/q-1}(a+b)$ if $a > 0$, $b > 0$ and $0 < q \le 1$. Then one has

$$\begin{aligned} \operatorname{dist}_q(Q^n_{j,m}, F) &\le \operatorname{dist}_q(x, F) \le c\,2^{-j+1}, \\ 2^{\max(1/q,1)-1} \operatorname{dist}_q(Q^n_{j,m}, F) &\ge \operatorname{dist}_q(x, F) - 2^{\max(1/q,1)-1} \operatorname{diam}_q(Q^n_{j,m}) \\ &\ge c\,2^{-j} - n^{1/q}\,2^{-j}\,2^{\max(1/q,1)-1} \end{aligned} \tag{2.10}$$

and hence

$$\operatorname{dist}_q(Q^n_{j,m}, F) \ge c\,2^{1-\max(1/q,1)}\,2^{-j} - n^{1/q}\,2^{-j}. \tag{2.11}$$

Then (2.9) follows from (2.10), (2.11) with $c = 2^{\max(1/q,1)} n^{1/q}$ and (2.6). Finally one eliminates cubes $Q^n_{j',m'}$, $m' \in M_{j'}$, with $Q^n_{j',m'} \subset Q^n_{j,m}$ for some $j < j'$, $m \in M_j$, by the same standard covering arguments as in [11, p. 168]. This reduces $M_j$ to some $\mathbb{Z}^n_j$ and proves (2.8) in addition to (2.7), (2.9). ∎

Finally we collect the needed properties for the special Besov spaces $B^s_{p,p}(\mathbb{Q}^n)$ with $1 < p < \infty$ and $0 < s < 1/p$. Recall that $B^s_{p,p}(\mathbb{Q}^n)$ is defined as the restriction of $B^s_{p,p}(\mathbb{R}^n)$

to $\mathbb{Q}^n$. This space can be equivalently normed by

$$\|f \,|\, B^s_{p,p}(\mathbb{Q}^n)\| \sim \|f \,|\, L_p(\mathbb{Q}^n)\| + \Big( \int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+n}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p}$$
$$\sim \Big| \int_{\mathbb{Q}^n} f(y) \, \mathrm{d}y \Big| + \Big( \int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+n}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p}.$$

In [13] we relied on expansions of $f \in B^s_{p,p}(\mathbb{Q}^n)$ in terms of Haar wavelets based on the corresponding characterizations in [12]. For our purpose it is not necessary to recall in detail the corresponding theory. It is sufficient to fix the following outcome. Let

$$\mathbb{Z}_j(\mathbb{Q}^n) = \big\{ m \in \mathbb{Z}^n : \; Q^n_{j,m} \subset \mathbb{Q}^n \big\}, \qquad j \in \mathbb{N}_0. \tag{2.12}$$

Proposition 2.4. *Let $1 < p < \infty$ and $0 < s < 1/p$. Then there is a constant $c > 0$ such that*

$$\int_0^1 |\log t|^{sp} \, f^*(t)^p \, \mathrm{d}t \le 2^{cn} \sum_{j=0}^{\infty} 2^{jsp+jn} \sum_{m \in \mathbb{Z}_j(\mathbb{Q}^n)} \int_{Q^n_{j,m} \times Q^n_{j,m}} |f(x)-f(y)|^p \, \mathrm{d}x \, \mathrm{d}y \tag{2.13}$$

*for all $n \in \mathbb{N}$ and all $f \in B^s_{p,p}(\mathbb{Q}^n)$ with $\int_{\mathbb{Q}^n} f(x) \, \mathrm{d}x = 0$.*

Remark 2.5. This assertion is covered by Proposition 2.5 and the arguments in the proofs of Proposition 3.1 and Theorem 3.3 in [13]. It seems to be quite natural to admit factors of type $2^{cn}$ in (2.13). Recall that the use of wavelets produces automatically a factor $2^n$. In [13] we assumed that this is also the case in all other calculations. We overlooked that the volume $|\mathbb{B}^n_2|$ according to (2.3), (2.5) does not fit in this scheme. It is the main reason of the present paper to argue more carefully in this respect.

**2.2. Proof of the theorem.** *Step 1.* It is sufficient to prove (1.4) for $f \in B^s_{p,p}(\mathbb{Q}^n)$ with $f_{\mathbb{Q}^n} = \int_{\mathbb{Q}^n} f(y) \, \mathrm{d}y = 0$. Otherwise one decomposes $f \in B^s_{p,p}(\mathbb{Q}^n)$ as $f = f_{\mathbb{Q}^n} + (f - f_{\mathbb{Q}^n})$ and incorporates the constant function $f_{\mathbb{Q}^n}$ afterwards resulting in the first term on the right-hand side of (1.4).

*Step 2.* We rely on the modification of Proposition 2.3 with

$$\mathbb{Q}^{2n} = \mathbb{Q}^n \times \mathbb{Q}^n = \big\{ (x,y) \in \mathbb{R}^{2n} : \; x \in \mathbb{Q}^n, \; y \in \mathbb{Q}^n \big\}$$

in place of $\mathbb{R}^n$ and

$$\mathbb{D}^n = \big\{ (z,z) \in \mathbb{Q}^{2n} : \; z \in \mathbb{Q}^n \big\}$$

in place of $F$. Then

$$\mathbb{Q}^{2n} \setminus \mathbb{D}^n = \bigcup_{J \in \mathbb{N}, M \in \mathbb{Z}'_J(\mathbb{Q}^{2n})} \overline{Q^{2n}_{J,M}} \tag{2.14}$$

is the counterpart of (2.7) where $\mathbb{Z}'_J(\mathbb{Q}^{2n})$ is the corresponding subset of $\mathbb{Z}_J(\mathbb{Q}^{2n})$ in (2.12). One has again (2.8), (2.9), appropriately adapted. We indicate now $n \in \mathbb{N}$ in (1.1) and use temporarily $\|x \,|\, \ell^n_q\|$ in place of $|x|_q$. Let $Q^{2n}_{J,M}$ be one of the cubes in (2.14) and let $x \in \mathbb{Q}^n$, $y \in \mathbb{Q}^n$ with $(x,y) \in Q^{2n}_{J,M}$. Let $(z,z) \in \mathbb{D}^n$ be a point in $\mathbb{D}^n$ nearest to $Q^{2n}_{J,M}$.

Then one has by (2.9), (2.6),

$$\begin{aligned}\|x-y\,|\,\ell_q^n\| &\le 2^{\max(1/q,1)-1}\big(\|x-z\,|\,\ell_q^n\|+\|y-z\,|\,\ell_q^n\|\big)\\ &\le 2^{\max(1/q,1)}\|(x,y)-(z,z)\,|\,\ell_q^{2n}\|\\ &\le 2^{2+2\max(1/q,1)}\,2^{-J}\,(2n)^{1/q}.\end{aligned}$$

In particular,

$$2^J \le 2^{2+1/q+2\max(1/q,1)}\,n^{1/q}\,\|x-y\,|\,\ell_q^n\|^{-1} = c_q\,n^{1/q}\|x-y\,|\,\ell_q^n\|^{-1}. \tag{2.15}$$

*Step 3.* We wish to estimate the integrals on the right-hand side of (2.13) and deal first with the prototype $j=0$. From (2.14) and (2.15) (switching back to $|x|_q$ as in (1.1)) and $Q^n_{0,0}=\mathbb{Q}^n$ follows

$$\begin{aligned}&\int_{Q^n_{0,0}\times Q^n_{0,0}} |f(x)-f(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad\le \sum_{J\in\mathbb{N},M\in\mathbb{Z}'_J(\mathbb{Q}^{2n})} \big(2^{-J}c_q\,n^{1/q}\big)^{sp+n}\int_{Q^{2n}_{J,M}}\frac{|f(x)-f(y)|^p}{|x-y|_q^{sp+n}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{2.16}$$

Let $j\in\mathbb{N}$ and let $\mathbb{Z}_j(\mathbb{Q}^n)$ be as in (2.12), (2.13). By construction one has in $\mathbb{Q}^{2n}$ for the related tubular neighbourhood of $\mathbb{D}^n$,

$$\Big(\bigcup_{m\in\mathbb{Z}_j(\mathbb{Q}^n)} Q^n_{j,m}\times Q^n_{j,m}\Big)\setminus\mathbb{D}^n = \bigcup_{J>j,M\in\mathbb{Z}'_J(\mathbb{Q}^{2n})}\overline{Q^{2n}_{J,M}},$$

hence a sub-decomposition of (2.14). Then

$$\begin{aligned}&2^{j(sp+n)}\sum_{m\in\mathbb{Z}_j(\mathbb{Q}^n)}\int_{Q^n_{j,m}\times Q^n_{j,m}}|f(x)-f(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad\le\sum_{J>j,M\in\mathbb{Z}'_J(\mathbb{Q}^{2n})}2^{-(J-j)(sp+n)}\big(c_q n^{1/q}\big)^{sp+n}\int_{Q^{2n}_{J,M}}\frac{|f(x)-f(y)|^p}{|x-y|_q^{sp+n}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

is the counterpart of (2.16). Summing over $j$ one obtains by (2.13)

$$\int_0^1 |\log t|^{sp}\,f^*(t)^p\,\mathrm{d}t \le 2^{cn}\,n^{n/q}\int_{\mathbb{Q}^n\times\mathbb{Q}^n}\frac{|f(x)-f(y)|^p}{|x-y|_q^{sp+n}}\,\mathrm{d}x\,\mathrm{d}y, \tag{2.17}$$

where $c>0$ is independent of $n$ (but may depend on $s,p,q$). Then (1.4) with $\int_{\mathbb{Q}^n} f(y)\,\mathrm{d}y=0$ follows from (2.17) and (2.5), (2.1). ∎

Remark 2.6. We formulated in [13, Theorem 3.3, p. 369] an assertion of type (1.4) for $q=2$, but without the factor $|\mathbb{B}^n_2|\sim n^{-n/2}$ according to (2.5). But there was a weak point in the proof which we now corrected. However we do not know whether the original version is valid.

Remark 2.7. In [13, Theorem 3.7] we dealt with corresponding assertions for all spaces $B^s_{p,p}(\mathbb{Q}^n)$ with $1<p<\infty$ and $s>0$. This was based on expansions in terms of Daubechies wavelets and some sophisticated polynomial approximations. There is little doubt that this technique can be combined with the above arguments resulting in counterparts of (1.4) for all $s>0$.

## References


[1] C. Bennett, R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston 1988.

[2] D. E. Edmunds, W. D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer Monogr. Math., Springer, Berlin 2004.

[3] D. E. Edmunds, H. Triebel, *Function Spaces, Entropy Numbers, Differential Operators*, Cambridge Tracts in Math. 120, Cambridge Univ. Press, Cambridge, 1996.

[4] A. Fiorenza, M. Krbec, H.-J. Schmeisser, *An improvement of dimension-free Sobolev imbeddings in r.i. spaces*, J. Funct. Anal. 267 (2014), 243–261.

[5] M. Krbec, H.-J. Schmeisser, *A limiting case of the uncertainty principle*, in: Proceedings of Equadiff 11, Proceedings of Minisymposia and Contributed Talks (Bratislava, July 2005), Bratislava 2007, 181–187.

[6] M. Krbec, H.-J. Schmeisser, *Dimension-invariant Sobolev imbeddings*, in: Function Spaces IX, Banach Center Publ. 92, Polish Acad. Sci. Inst. Math., Warsaw 2011, 205–217.

[7] M. Krbec, H.-J. Schmeisser, *On dimension-free Sobolev imbeddings.* I, J. Math. Anal. Appl. 387 (2012), 114–125.

[8] M. Krbec, H.-J. Schmeisser, *On dimension-free Sobolev imbeddings.* II, Rev. Mat. Complut. 25 (2012), 247–265.

[9] J. Martín, M. Milman, *Isoperimetry and symmetrization for logarithmic Sobolev inequalities*, J. Funct. Anal. 256 (2009), 149–178.

[10] J. Martín, M. Milman, *Pointwise symmetrization inequalities for Sobolev functions and applications*, Adv. Math. 225 (2010), 121–199.

[11] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton 1970.

[12] H. Triebel, *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*, EMS Tracts Math. 11, European Math. Soc., Zürich 2010.

[13] H. Triebel, *Tractable embeddings of Besov spaces into Zygmund spaces*, in: Function Spaces IX, Banach Center Publ. 92, Polish Acad. Sci. Inst. Math., Warsaw 2011, 361–377.

[14] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge 1952.