Abstract. A conjecture of [12] states that a knot is nonfibered if and only if its infinite cyclic cover has uncountably many finite covers. We prove the conjecture for a class of knots that includes all knots of genus 1, using techniques from symbolic dynamics.

1. Introduction. Let $G$ be a finitely presented group with epimorphism $\chi : G \to \mathbb{Z}$. The kernel $K$ of $\chi$ need not be finitely generated. However, $K$ is finitely presented as a $\mathbb{Z}$-operator group [10]. In [11], [12] the authors exploited this structure to show that the representations of $K$ into a fixed finite group $\Sigma$ form a shift of finite type, a simple dynamical system described by a finite directed graph. We call this dynamical system the representation shift of $K$ in $\Sigma$. When $G$ is a knot or link group, representation shifts inform us about the algebraic topology of finite covering spaces from a purely dynamical perspective.

We review basic definitions of representation shifts and give a partial solution to Conjecture 4.4 of [12]. The complete solution would characterize nonfibered knots as those knots with some representation shift having positive topological entropy.

2. Review of representation shifts. An augmented group system [7] is a triple $G = (G, \chi, x)$ consisting of a finitely presented group $G$, epimorphism $\chi : G \to \mathbb{Z}$ and distinguished element $x \in G$ such that $\chi(x) = 1$. Two such systems $G_i = (G_i, \chi_i, x_i)$, $i = 1, 2$ are equivalent (and regarded as the same) if there exists an isomorphism $f : G_1 \to G_2$ such that $f(x_1) = x_2$ and $\chi_1 = \chi_2 \circ f$.

Example 2.1. An augmented group system is associated to an oriented knot $k \subset S^3$ in a canonical manner. Let $G = \pi_1(S^3 \setminus k, p)$, where the base point $p$ is contained on the boundary $\partial N(k)$ of a tubular neighborhood $N(k) = S^1 \times D^2$ of $k$. Let $x$ be the homotopy
class of a meridian $m \subset \partial N(k)$, with orientation acquired from $k$. Finally, let $\chi : G \to \mathbb{Z}$ be the abelianization homomorphism that sends $x$ to $1$. It follows from the uniqueness of tubular neighborhoods that $\mathcal{G} = (G, \chi, x)$ is well defined.

We denote the kernel of $\chi$ by $\mathcal{K}$. Given any finite group $\Sigma$, we consider the space $\text{Hom}(K, \Sigma)$ of representations $\rho : K \to \Sigma$. The basis for its topology is given by the sets

$$\mathcal{N}_{a_1,\ldots,a_s}(\rho) = \{ \rho' \mid \rho'(a_i) = \rho(a_i), \ i = 1, \ldots, s \},$$

where $a_1, \ldots, a_s$ varies over all finite collections of elements of $K$. The topology is the compact-open topology where $K$ and $\Sigma$ are discrete spaces. Roughly speaking, representations are close in $\text{Hom}(K, \Sigma)$ if they agree on large finitely generated subgroups of $K$.

The distinguished element $x$ induces a self-map $\sigma_x$ of $\text{Hom}(K, \Sigma)$ defined by

$$\sigma_x \rho(a) = \rho(x^{-1}ax) \ \forall a \in K.$$ 

It is easily seen that $\sigma_x$ is a homeomorphism.

The representation shift associated to $\mathcal{G} = (G, \chi, x)$ and $\Sigma$ is the pair $(\text{Hom}(K, \Sigma), \sigma_x)$. We denote it by $\Phi_{\Sigma}(\mathcal{G})$. It is a dynamical system well defined up to topological conjugacy [10]. More precisely, if $\mathcal{G}_i$, $i = 1, 2$, are equivalent augmented group systems, then there exists a homeomorphism $F$ of the underlying spaces of $\Phi_{\Sigma}(\mathcal{G}_i)$ such that $F \circ \sigma_1 = \sigma_2 \circ F$.

The representation shift $\Phi_{\Sigma}(\mathcal{G})$ is an example of a shift of finite type, a special type of expansive 0-dimensional dynamical system, one that can be described by a finite directed graph. (See [4].) We use combinatorial group theory to construct such a graph for a representation shift.

Given an augmented group system $\mathcal{G} = (G, \chi, x)$, we can describe $G$ as an HNN extension $\langle x, B \mid x^{-1}ax = \phi(a), \forall a \in U \rangle$, where $B$ is a finitely generated subgroup of $K$, and $U, V$ are isomorphic finitely generated subgroups of $B$ with isomorphism $\phi : U \to V$ (see [6]). The subgroup $B$ is an HNN base. One can choose $B$ so that it contains any prescribed finite subset of $K$ (see [8]).

**Example 2.2.** Let $\mathcal{G} = (G, \chi, x)$ be an augmented group system associated to a knot, as in Example 2.1. An HNN decomposition for $G$ can be obtained in a natural way. Begin with a $\pi_1$-incompressible Seifert surface for $k$ meeting the exterior $E(k) = S^3 \setminus \text{int} \ N(k)$ in a connected surface $S$. Split $E(k)$ along $S$, and denote by $(W; S_0, S_1)$ the resulting cobordism, with boundary comprising two copies $S_0, S_1$ of $S$ joined by an annulus $\partial S \times I$. Let $B = \pi_1(W, p)$, where the basepoint $p$ lies on the boundary of $S_0$. Let $U = \pi_1(S_0, p)$. The meridian $m$ appears as a path from $p \in S_0$ to a point $p_1 \in S_1$. Use the path to regard $\pi_1(S_1, p_1)$ as a subgroup $V$ of $B$. Clearly $G$ is described as $(B; U, V, \phi)$, where $\phi$ is induced by the gluing of $S_0$ to $S_1$ when recovering the exterior $E(k)$.

Conjugation by $x$ induces an automorphism of $K$. Let $B_j = x^{-j}Bx^j$, $U_j = x^{-j}Ux^j$ and $V_j = x^{-j}Vx^j$, for $j \in \mathbb{Z}$. Then $K$ is described as an infinite amalgamated free product

$$K = \langle B_j \mid V_j = U_{j+1}, \forall j \in \mathbb{Z} \rangle.$$ 

A graph $\Gamma$ describing the representation shift $\Phi_{\Sigma}(\mathcal{G})$ is constructed as follows. The vertex set consists of all representations $\rho_0 : U \to \Sigma$, a finite set since $U$ is finitely generated. If $\overline{\rho}_0$ is a representation from $B$ to $\Sigma$, then we draw a directed edge labeled
\( \tilde{\rho}_0 \) from the vertex \( \rho_0 = \tilde{\rho}_0|_U \) to the vertex \( \rho'_0 = \tilde{\rho}_0|_V \circ \phi \). (\( \Gamma \) may have parallel edges.) Consider a bi-infinite path in \( \Gamma \) given by an edge sequence

\[ \cdots \tilde{\rho}_{-2} \tilde{\rho}_{-1} \tilde{\rho}_0 \tilde{\rho}_1 \tilde{\rho}_2 \cdots \]

The representations \( B_j : \to \Sigma \) given by \( a \mapsto \tilde{\rho}_j(x^j a x^{-j}) \) have a unique common extension \( \rho : K \to \Sigma \). Conversely, any representation \( \rho : K \to \Sigma \) arises from such a path, and uniquely. Thus bi-infinite paths of the graph \( \Gamma \) correspond bijectively to elements of \( \text{Hom}(K, \Sigma) \). The map \( \sigma_x \) acts as the left coordinate shift on the sequence of edges.

We may “prune” \( \Gamma \) by removing any vertex or edge that is not contained in a bi-infinite path. The resulting graph has finitely many bi-infinite paths iff it consists of a collection of disjoint cycles. It contains uncountably many bi-infinite paths iff it contains two cycles with at least one common vertex.

A representation \( \rho \in \Phi_\Sigma(\mathcal{G}) \) has period \( r \) if \( \sigma_x^r(\rho) = \rho \). Such representations correspond to closed paths in \( \Gamma \) with length dividing \( r \). The set of representations with period \( r \) is denoted by \( \text{Fix}(\sigma_x^r) \). If \( M_r \) is the \( r \)-fold cyclic cover of \( S^3 \) branched over a knot \( k \), then \( \text{Fix}(\sigma_x^r) \) is in natural bijective correspondence with \( \text{Hom}(\pi_1 M_r, \Sigma) \) [12]. This correspondence connects dynamical properties of the representation shift with topological properties of \( k \).

Topological entropy is a measure of complexity of a dynamical system. For a shift of finite type, it can be computed as the log of the spectral radius of the adjacency matrix \( A \) of any directed graph that describes the shift. (Here \( A_{i,j} \) is the number of edges from the \( i \)th vertex to the \( j \)th vertex.) Consequently, the topological entropy of \( \Phi_\Sigma(\mathcal{G}) \), which we denote by \( h_\Sigma(\mathcal{G}) \), is the exponential growth rate of \( \text{tr} A^r = |\text{Fix}(\sigma_x^r)| = |\text{Hom}(\pi_1 M_r, \Sigma)| \) (see [12]). This is positive if and only if \( \Phi_\Sigma(\mathcal{G}) \) is uncountable. Notice that if \( K \) is finitely generated, then \( \Phi_\Sigma(\mathcal{G}) \) is finite for all \( \Sigma \), and so in this case \( h_\Sigma(\mathcal{G})=0 \).

Let \( S_N \) denote the symmetric group on \( \{1, \ldots, N\} \). It is well known that elements \( \rho \in \text{Hom}(K, S_N) \) correspond in a finite-to-one manner with subgroups \( H \leq K \) with index no greater than \( N \). The correspondence is

\[ \rho \mapsto \{ g \in K \mid \rho(g)(1) = 1 \}. \]

The preimage of a subgroup of index \( N \) consists of \((N - 1)! \) transitive representations. (A representation \( \rho \) is transitive if \( \rho(K) \) operates transitively on \( \{1, \ldots, N\} \).) Note that if \( \Phi_{S_N}(\mathcal{G}) \) is uncountable, then \( K \) contains uncountably many subgroups of some index no greater than \( N \). Hence the infinite cyclic cover of \( k \) has uncountably many finite covers.

We summarize the results of this section. Recall that any finite group embeds in a sufficiently large symmetric group.

**Proposition 2.3.** Let \( k \subset S^3 \) be a knot with associated augmented group system \( \mathcal{G} \). Then the following statements are equivalent.

(i) The infinite cyclic cover of \( k \) has uncountably many finite covers.

(ii) The representation shift \( \Phi_\Sigma(\mathcal{G}) \) is uncountable for some finite group \( \Sigma \).

(iii) The topological entropy \( h_\Sigma(\mathcal{G}) \) is positive for some finite group \( \Sigma \).

(iv) \( \lim_{r \to \infty} \frac{1}{r} \log |\text{Hom}(\pi_1 M_r, \Sigma)| \) is positive for some finite group \( \Sigma \).
3. Nonfibered knots. We recall that a knot $k \subset S^3$ is fibered if its exterior $E(k) = S^3 \setminus \text{int } N(k)$ fibers over the circle. It is no loss of generality to assume that the fibration restricts to the standard projection $\partial N(k) \simeq k \times S^1 \to S^1$. Hence $E(k)$ is seen to be homeomorphic to a mapping torus $S \times I/F$, where $F : S \to S$ is a homeomorphism of a minimal-genus Seifert surface $S$ of $k$.

If $k$ is fibered, then the commutator subgroup $G'$ of its group is finitely generated and free, isomorphic to $\pi_1 S$. Conversely, a theorem of J. Stallings [13] implies that if $k$ is a knot such that $G'$ is finitely generated, then in fact $G'$ is free and $k$ is fibered.

If $k$ is fibered and $\mathcal{G}$ is its associated augmented group system, then for any finite group $\Sigma$, the representation shift $\Phi_\Sigma(\mathcal{G})$ is finite. Its order is $|\Sigma|^{2g}$, where $g$ is the genus of $k$ (equal to the genus of its fiber). The trefoil and figure-eight knots are the only fibered knots of genus 1.

Conjecture 4.4 of [12] proposes a characterization of nonfibered knots. It states that $k$ is nonfibered iff the entropy $h_\Sigma(\mathcal{G})$ is positive for some finite group $\Sigma$.

**Remark 3.1.** (1) In terms of the HNN base $B$ described above, the condition that $k$ is not fibered is equivalent to the condition that $U$ is a proper subgroup of $B$. Lemma 2.3 (Substitution Lemma) of [11] provides a strategy for showing that some $\Phi_\Sigma(\mathcal{G})$ is uncountable: Find a periodic element of $\Phi_{S_N}(\mathcal{G})$ such that some symbol, say $N$, is fixed by every permutation in the image of $U$ but moved by some element of $\rho(K)$. Recall that periodic representations correspond to cycles in the graph $\Gamma$. Enlarging $S_N$ to $S_{N+1}$, we can construct another periodic representation $\rho'$ by replacing $N$ by $N+1$ in each permutation in the image of $\rho$. In the graph of $\Phi_{S_{N+1}}(\mathcal{G})$, $\rho$ and $\rho'$ correspond to cycles with a common vertex, and hence $\Phi_{S_{N+1}}(\mathcal{G})$ is uncountable.

(2) For our strategy, it suffices to find any representation $\tilde{\rho} : G \to \Sigma$ such that $\rho(U)$ is a proper subgroup of $\rho(K)$. For given such a representation, and letting $\rho : K \to \Sigma$ be the restriction, we enumerate the cosets of $\rho(U)$ in $\rho(K)$, say $1, \ldots, N$ ($N > 1$). In a natural way, $\rho$ determines an element of $\Phi_{S_N}(\mathcal{G})$: $a \in K$ is sent to the transitive permutation of cosets given by right multiplication by $\rho(a)$. Note that if $a \in U$, then such a permutation fixes the symbol corresponding to $\rho(U)$. Finally, we note that if $r$ is the order of $\tilde{\rho}(x)$ in $\Sigma$, then $\sigma_x^r \rho = \rho$, since $(\sigma_x^r \rho)(a) = \rho(x^{-r}ax^r) = \tilde{\rho}(x^{-1})^r \rho(a) \tilde{\rho}(x)^r = \rho(a)$, for all $a \in K$.

The representation $\tilde{\rho}$ in the Remark 3.1 (2) “separates” the subgroup $U$ from some element $a \in K$.

In general, a subgroup $U$ of a group $G$ is *separable* if for any element $a \in G \setminus U$, there exists a finite-index subgroup of $G$ that contains $U$ but not $a$. Equivalently, there exists a finite representation $\tilde{\rho} : G \to \Sigma$ such that $\tilde{\rho}(a) \notin \tilde{\rho}(U)$. The strategy outlined in Remark 3.1(2) requires only that $U$ can be separated from some element of $K \setminus U$.

**Definition 3.2.** An element $a \in G \setminus U$ is *separable from $U$* if there exists a subgroup $H$ of finite index in $G$ containing $U$ but not $a$.

Question 15 of [14] asks if any finitely generated subgroup of a finitely-generated Kleinian group is separable. An affirmative answer would establish Theorem 3.4 for all hyperbolic knots. Although Thurston’s question remains open, a result of D. Long and
G. Niblo [5] enables us to apply our strategy in the case of genus-1 knots (see also remarks that follow).

The theorem of Long and Niblo has been used by S. Friedl and S. Vidussi in [1] to show that twisted Alexander polynomials corresponding to finite representations decide if a genus-1 knot is fibered.

**Theorem 3.3** (D. Long and G. Niblo [5]). Let $M$ be an orientable Haken 3-manifold. If $i: T \hookrightarrow M$ is an incompressibly embedded torus, then $i_*(\pi_1 T)$ is separable in $\pi_1 M$.

**Theorem 3.4**. Let $k$ be a knot of genus 1. Then $k$ is nonfibered iff the conclusions of Proposition 2.3 hold.

**Proof.** One implication of the theorem is clear: if the conclusion of Proposition 2.3 holds, then $k$ is nonfibered.

Assume that $k$ is nonfibered. Consider the 3-manifold $M$ obtained by 0-framed surgery on $k$; that is, by removing and replacing a tubular neighborhood $N(k) \equiv k \times \mathbb{D}^2$ in such a way that each disk $* \times \mathbb{D}^2$ bounds a longitude of $k$. By results of [3], $M$ is irreducible. We denote the fundamental group of $M$ by $\hat{G}$.

The addition of a meridianal disk converts a genus-1 Seifert surface $S$ for $k$ to a torus $\hat{S}$ in $M$. Since $\hat{S}$ is dual to a nontrivial cohomology class and $M$ is irreducible, we see that $\hat{S}$ is incompressible. Note in particular that $M$ is Haken.

Obtain an HNN decomposition $(\hat{B}; \hat{U}, \hat{V})$ for $\hat{G}$ much as we did for $G$, by splitting $M$ along $\hat{S}$. Here $\hat{U} = \pi_1 \hat{S}$. Since $k$ is not fibered, neither is $M$ [2]. Hence $\hat{U}$ must be a proper subgroup of $\hat{B}$. Select an element $\hat{a} \in \hat{B} \setminus \hat{U}$. By Theorem 3.3 there exists a finite group $\Sigma$ and homomorphism $\hat{\rho} : \hat{G} \to \Sigma$ such that $\hat{\rho}(\hat{a}) \notin \hat{U}$.

The group $\hat{G}$ is a quotient of $G$. Let $p$ be the natural projection. Note that $p(U) = \hat{U}$.

Choose $a \in K$ such that $p(a) = \hat{a}$. Define $\rho = \hat{\rho} \circ p : G \to \Sigma$.

Remark 3.1(2) completes the proof. 

Genus-1 knots are plentiful, the simplest examples being the twist knots (e.g. the knots $5_2, 6_1$) and doubled knots (obtained from a knot and any push-off by joining with a clasp). We extend the collection of nonfibered knots with uncountable representation shifts by considering also any knot $k$ with group $G$ that maps homomorphically onto the group $\bar{G}$ of a nonfibered genus-1 knot $\bar{k}$. Examples of such knots $k$ include satellite knots with genus-1 pattern knot [9].

**Corollary 3.5.** Let $k$ be a knot. Assume that the group of $k$ maps onto the group of a nonfibered knot $\bar{k}$ of genus 1. Then $k$ is nonfibered and the conclusions of Proposition 2.3 hold.

**Proof.** Assume that $h : G \to \bar{G}$ is an epimorphism, where $G, \bar{G}$ are the groups of $k, \bar{k}$, respectively. Let $K, \bar{K}$ denote the respective commutator subgroups, and $x, \bar{x}$ the meridianal generators of $k, \bar{k}$. Since $h(K) = \bar{K}$ and $\bar{K}$ is not finitely generated, we see at once that $\bar{K}$ is not finitely generated. Hence $k$ is nonfibered.

For any group $\Sigma$, the restricted epimorphism $h : K \to \bar{K}$ induces an injection $\text{Hom}(\bar{K}, \Sigma) \hookrightarrow \text{Hom}(K, \Sigma)$. By Theorem 3.4, there exists a finite group $\Sigma$ such that
Hom(\bar{K}, \Sigma) is uncountable. Hence Hom(K, \Sigma), the underlying space of the representation shift \Phi_\Sigma(\bar{G}) is uncountable. 

**Remark 3.6.** For any finite group \Sigma, the topological entropy \( h_\Sigma(G) \) is at least as great as \( h_\Sigma(\bar{G}) \), where \( \bar{G} \) is the augmented group system of \( \bar{k} \). The reason is the following.

If \( h(x) = \bar{x} \), then for any finite group \( \Sigma \), the representation shift \( \Phi_\Sigma(\bar{G}) \) corresponding to \( \bar{k} \) is a subshift of the representation shift \( \Phi_\Sigma(G) \) corresponding to \( k \); that is, Hom(\bar{K}, \Sigma) is a subspace of Hom(K, \Sigma) with the shift map \( \sigma_x \) restricting to \( \sigma_{\bar{x}} \). The epimorphism \( h \) induces an embedding: \( h^* \rho = \rho \circ h \). It follows that the topological entropy \( h_\Sigma(\bar{G}) \) is at least \( h_\Sigma(\bar{G}) \).

If \( h(x) \neq \bar{x} \), then there exists \( a \in K \) such that \( h(ax) = \bar{x}^\epsilon \), where \( \epsilon = \pm 1 \). We may assume without loss of generality that \( \epsilon = 1 \). In this case, we replace \( x \) by \( ax \). Of course the augmented group system \( G \) and associated representation shifts \( \Phi_\Sigma(G) \) change. However, by a result of [10], the topological entropy of the representation shift remains unchanged. Again \( h_\Sigma(G) \geq h_\Sigma(\bar{G}) \).

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**References**


