THE FUKUMOTO-FURUTA AND THE OZSVÁTH-SZABÓ INVARIANTS FOR SPHERICAL 3-MANIFOLDS

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Dedicated to Professor Takao Matumoto for his 60th birthday

Abstract. We show that the Fukumoto-Furuta invariant for a rational homology 3-sphere M, which coincides with the Neumann-Siebenmann invariant for a Seifert rational homology 3-sphere, is the same as the Ozsváth-Szabó's correction term derived from the Heegaard Floer homology theory if M is a spherical 3-manifold.

1. Introduction. Let X be an oriented compact smooth 4-manifold and I_X be its intersection form. In the case where X is closed, various deep results about the constraints on I_X have been obtained by gauge theory and Seiberg-Witten theory (see [1], [3]). To obtain similar results in the case where X has a boundary, we need extra invariants of a 3-manifold that bounds X. As one of such invariants we have studied the Fukumoto-Furuta invariant $w(M, X, c_X)$ for a triple (M, X, c_X) ([5], [18], [19]), where M is a closed oriented 3-manifold, X is a compact spin 4-orbifold with $\partial X = M$, and c_X is a spin structure on X. In [19] we have shown that if M is a Seifert rational homology 3-sphere, there exists a canonical choice of X so that $w(M, X, c_X)$ is the same as the Neumann-Siebenmann invariant $\overline{\mu}(M, c_M)$, determined only by M and a spin structure c_M on M, which is induced by c_X.

Another powerful invariant giving several constraints on the intersection forms of 4-manifolds with boundary is given by Ozsváth-Szabó's correction term $d(M, t)$ defined for a rational homology 3-sphere M with spin^c structure t. These two invariants are derived from different theories (the Seiberg-Witten theory and the Heegaard Floer homology theory) and they are not the same even if M is a Seifert rational homology 3-sphere and

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t comes from a spin structure. However we can show that they are the same if M is a spherical 3-manifold. In fact in §3.1 we extend the definition of the Fukumoto-Furuta invariant $w(M, X, t_X)$ for a triple (M, X, t_X), where M and X are as before and t_X is a spin^c structure on X. If M is spherical, we can also define this invariant so that it depends only on M and the spin^c structure on M by choosing the cone over M as X. The main result of this paper is the following theorem, whose proof will be given in §3.

THEOREM 1.1. *Let (S, t) be a pair of a spherical 3-manifold and a* spin^c *structure on it. Then $4d(S, t) = -\overline{\mu}(S, t)$. Here $\overline{\mu}(S, t)$ is the Fukumoto-Furuta invariant $w(S, cS, t_{cS})$, where cS is the cone over S and t_{cS} is the unique* spin^c *structure that is an extension of t. In particular if t is a spin structure, then $-4d(S, t)$ coincides with the Neumann-Siebenmann invariant, which is a spin rational homology cobordism invariant whose value modulo 16 is the Rokhlin invariant of (S, t).*

We note that the above claim does not hold for a non-spherical Seifert rational homology 3-sphere with spin structure. In fact $d(M, c) = 0$ but $\overline{\mu}(M, c) = 8$ if M is the Brieskorn homology 3-sphere $\Sigma(2, 3, 7)$ and c is the unique spin structure on it. This implies that $-4d(M, c) \pmod{16}$ is not the Rokhlin invariant in general, in spite of the rational homology cobordism invariance of $d(M, c)$ ([11]).

In this paper we only consider the Seifert 3-manifolds over orientable base 2-orbifolds. In §1 and §2 we recall the definition and several properties of $w(M, X, c_X)$ and $d(M, t)$ respectively. In §1 we also consider a Seifert rational homology 3-sphere M with spin structure c bounding a negative definite spin 4-manifold W and give a constraint on $b_2(W)$ in terms of $\overline{\mu}(M, c)$, which is deduced from the orbifold 10/8-theorem and its refinement. In §3.4 we compare this result with the constraint given by $d(M, c)$ in [11]. In §3.4 we give the explicit formula for the above invariants for $L(p, q)$ and the correspondence between the description of the *spin^c* structures on $L(p, q)$ given in [11] and ours given in §3. In §4 we give some remarks on the case of plumbed 3-manifolds.

2. The Fukumoto-Furuta invariants. We start with a rational homology 3-sphere M with spin structure c. For such a pair (M, c), there always exists a compact spin 4-orbifold X with spin structure c_X, such that $\partial X = M$ and c_X restricted on M induces c. Let us recall the definition of the Fukumoto-Furuta invariant for a triple (M, X, c_X), which is slightly different from the original one for an integral homology 3-sphere in [5].

DEFINITION 2.1. For (M, X, c_X) given above, there exists a spin 4-manifold Y with spin structure c_Y such that $\partial(Y, c_Y) = (M, c)$, where c is the restriction of c_X on M. Then for any choice of such (Y, c_Y), we define the Fukumoto-Furuta invariant by

$$w(M, X, c_X) = -8 \operatorname{ind} \mathcal{D}(X \cup (-Y)) + \sigma(Y).$$

Here $\sigma(Y)$ is the signature of Y, $\mathcal{D}(X \cup (-Y))$ is the spin Dirac operator defined over a closed spin 4-orbifold $X \cup (-Y)$ whose spin structure is induced by c_X and c_Y, and ind denotes the index over $\mathbf{C}$.

In this paper we always assume for simplicity that
- $b_1(X) = 0$, and
- the set ΣX of singularities of X consists only of isolated points.

Then $w(M, X, c_X)$ does not depend on the choice of (Y, c_Y) by the excision principle or explicit computation of the index below. Moreover $w(M, X.c_X)$ (mod 16) is equal to the Rokhlin invariant of (M, c) since ind $\mathcal{D}$ is even for a spin 4-orbifold.

Several important properties of the Fukumoto-Furuta invariant are derived from the following theorem.

THEOREM 2.2 (The orbifold 10/8-theorem [4]). *Let Z be a closed spin 4-orbifold with $b_1(Z) = 0$. Then*

1. *Either* ind $\mathcal{D}(Z) = 0$ *or*
$$1 - b_2^-(Z) \leq \operatorname{ind} \mathcal{D}(Z) \leq b_2^+(Z) - 1.$$

2. *In particular if $b_2^\pm(Z) \leq 2$, then* ind $\mathcal{D}(Z) = 0$ *since* ind $\mathcal{D}(Z)$ *is even.*

We can compute $\mathcal{D}(Z)$ (and $w(M, X, c_X)$) by using the Kawasaki V-index theorem [7], which gives the following results in our case. Here we also assume that the set of singularities ΣZ of Z consists only of isolated ones.

$$\operatorname{ind} \mathcal{D}(Z) = -\frac{1}{24} \int_Z p_1 + \delta^{\operatorname{Dirac}}(Z),$$

$$\sigma(Z) = \frac{1}{3} \int_Z p_1 + \delta^{\operatorname{sign}}(Z).$$

Here $\delta^{\operatorname{Dirac}}(Z)$ (resp. $\delta^{\operatorname{sign}}(Z)$) denotes the sum of the contribution $\delta^{\operatorname{Dirac}}(x)$ (resp. $\delta^{\operatorname{sign}}(x)$) from the isolated singularities $x \in \Sigma Z$ to the index of the Dirac (resp. the signature) operator over Z. Thus

$$\operatorname{ind} \mathcal{D}(Z) = -\frac{1}{8}\left(\sigma(Z) + \sum_{x \in \Sigma Z} \delta(x)\right),$$

where

$$\delta(x) = -8\delta^{\operatorname{Dirac}}(x) - \delta^{\operatorname{sign}}(x).$$

Here a neighborhood of x is a cone cS over a spherical 3-manifold S. Then there is a one-to-one correspondence between the set of spin structures on S and the set of spin structures on cS [18]. Moreover $\delta(x)$ is determined uniquely by (S, c_S) and hence is denoted by $\delta(S, c_S)$, where c_S is a spin structure on S induced by c_Z. Thus $w(M, X, c)$ is represented as follows.

PROPOSITION 2.3 ([19]).
$$w(M, X, c) = \sigma(X) + \delta(X, c)$$

with

$$\delta(X, c) = \sum_S \delta(S, c_S),$$

where in the sum on the right hand side S runs over all the links of the isolated singularities of X and c_S denotes the restriction of c to S.

REMARK 2.4. The above $\delta(S, c)$ is equal to the Fukumoto-Furuta invariant $w(S, cS, c_{cS})$, where c_{cS} is the unique spin structure on cS whose restriction to S is c.

The list of the values of $\delta(S, c)$ for (S, c) is given explicitly in [18].

(1)   The case when $S = L(p, q)$.

In the case when S is a lens space $L(p, q)$, which is the $-p/q$-surgery on the unknot in S^3, we describe a spin structure c on $L(p, q)$ by assigning a $\mathbf{Z}_2$ value $c(\mu)$ to the meridian μ of the unknot satisfying

$$pc(\mu) + pq \equiv 0 \pmod 2.$$

This equation shows that $c(\mu) \equiv q \pmod 2$ if p is odd, but $c(\mu)$ may be arbitrary if p is even. The value $c(\mu)$ determines a spin structure on the complement of the unknot ($c(\mu) = 0$ if and only if c extends to the spin structure on S^3), and the above equation is equivalent to the condition for this spin structure to be extendable to that on the solid torus attached by the $-p/q$-surgery. Then $\delta(L(p, q), c)$ is described by a σ-function as follows. (We note that either p or q may be negative according to the above convention.)

PROPOSITION 2.5 ([5]).
$$\delta(L(p, q), c) = \sigma(q, p, (-1)^{c(\mu)-1}),$$

where $\sigma(q, p, \epsilon)$ ($\gcd(p, q) = 1$, $\epsilon = \pm 1$) is uniquely determined by the following formula.

1. $\sigma(q + kp, p, \epsilon) = \sigma(q, p, (-1)^k \epsilon)$,
2. $\sigma(-q, p, \epsilon) = \sigma(q, -p, \epsilon) = -\sigma(q, p, \epsilon)$,
3. $\sigma(q, 1, \epsilon) = 0$,
4. $\sigma(p, q, -1) + \sigma(q, p, -1) = -\operatorname{sgn} pq$ if $p + q \equiv 1 \pmod 2$.

According to the V-index theorem σ is described as ([5])

$$\sigma(q, p, \epsilon) = \frac{1}{p} \sum_{k=1}^{|p|-1} \left(\cot \frac{\pi k}{p} \cot \frac{\pi k q}{p} + 2\epsilon^k \csc \frac{\pi k}{p} \csc \frac{\pi k q}{p} \right).$$

We also note that if p is odd, we only consider $\sigma(q, p, \epsilon)$ for the case when $\epsilon = (-1)^{q-1}$.

(2)   The case when $S \neq L(p, q)$.

In this case we give a complete list of $\delta(S, c)$ in [18]. In the case when S is a Seifert fibration over the 2-orbifold of type $S^2(2, 2, n)$, then $\delta(S, c)$ is described in terms of the σ-function above, and $\delta(S, c)$ is given more explicitly otherwise.

If (M, c) is a Seifert rational homology 3-sphere with spin structure, we show in [19] that (X, c_X) can be chosen so that $w(M, X, c_X)$ is equal to the Neumann-Siebenmann invariant $\overline{\mu}(M, c)$ (and hence it depends only on (M, c)), which is defined as follows.

DEFINITION 2.6. Let $P(\Gamma)$ be a plumbing associated with an integrally weighted tree Γ. Each vertex v of Γ with weight e corresponds to the D^2-bundle over S^2 with Euler class e contained in $P(\Gamma)$ whose zero section generates $H_2(P(\Gamma), \mathbf{Z})$, which we denote by $[v]$. Let $M = \partial P(\Gamma)$ and c be a spin structure on M. Then there exists a unique element of the form

$$w(P(\Gamma), c) = \sum_v \epsilon_v[v] \in H_2(P(\Gamma), \mathbf{Z}),$$

such that

1. each ϵ_v is either 0 or 1, and
2. the Poincaré dual of $w(P(\Gamma), c) \mod 2$ is the obstruction to extending c to the spin structure on $P(\Gamma)$.

Then the Neumann-Siebenmann invariant of (M, c) is defined to be

$$\overline{\mu}(M, c) = \sigma(P(\Gamma)) - w(P(\Gamma), c) \cdot w(P(\Gamma), c).$$

The value of $\overline{\mu}(M, c)$ depends only on (M, c) (does not depend on the choice of $P(\Gamma)$ with $M = \partial P(\Gamma)$).

In case of a Seifert rational homology 3-sphere M with spin structure c, we always find a star-shaped weighted graph Γ with $M = \partial P(\Gamma)$, and the Fukumoto-Furuta invariant and $\overline{\mu}$ are related as follows.

LEMMA 2.7 ([19]). *Let (M, c) be a pair of a Seifert rational homology 3-sphere and a spin structure on it. Then there exist two compact spin 4-orbifolds with only isolated singularities $X_{\pm}$ with spin structures $c_{\pm}$ such that*

 1. $\partial(X_{\pm}, c_{\pm}) = (M, c)$,
 2. $w(M, X_{+}, c_{+}) = w(M, X_{-}, c_{-}) = \overline{\mu}(M, c)$,
 3. $b_1(X_{+}) = b_1(X_{-}) = 0$, and $b_2^+(X_{+}) \leq 1$, and $b_2^-(X_{-}) \leq 1$.

Here we note that $w(M, X_{\pm}, c_{\pm}) = \sigma(X_{\pm}) + \delta(X_{\pm}, c_{\pm})$.

Combining this lemma with the orbifold 10/8-theorem we proved the following theorem in [19], which extends the theorem by Saveliev for Seifert integral homology 3-spheres [15].

THEOREM 2.8 ([19]). *Let (M_i, c_i) $(i = 1, 2)$ be Seifert rational homology 3-spheres with spin structure. If there exists a spin cobordism (W, c) from (M_1, c_1) to (M_2, c_2) with $b_2(W) = 0$, then $\overline{\mu}(M_1, c_1) = \overline{\mu}(M_2, c_2)$.*

We also obtain the following constraint on the intersection form of a spin 4-manifold bounded by a Seifert rational homology 3-sphere.

THEOREM 2.9. *Suppose that a Seifert rational homology 3-sphere M with spin structure c bounds a negative definite spin 4-manifold Y with spin structure c_Y. Then*

$$b_2(Y) \equiv -\overline{\mu}(M, c) \pmod{16}, \tag{1}$$

$$-\frac{\overline{\mu}(M, c)}{9} \leq b_2(Y) \leq -\overline{\mu}(M, c). \tag{2}$$

Proof. We choose two spin 4-orbifolds $(X_{\pm}, c_{\pm})$ bounded by (M, c) as in Lemma 2.7 so that

$$\overline{\mu}(M, c) = w(M, X_{\pm}, c_{\pm}).$$

Furthermore by applying the formula stated above to a closed spin 4-orbifold $(Z_{\pm}, c_{Z_{\pm}}) = (X_{\pm}, c_{X_{\pm}}) \cup (-Y, c_Y)$ we have

$$w(M, X_{\pm}, c_{\pm}) = -8 \operatorname{ind} \mathcal{D}(Z_{\pm}) + \sigma(Y) = -8 \operatorname{ind} \mathcal{D}(Z_{\pm}) - b_2(Y).$$

Thus since

$$\operatorname{ind} \mathcal{D}(Z_{\pm}) = -\frac{1}{8}(\overline{\mu}(M, c) + b_2(Y))$$

is even, we have the first formula above. Next applying the orbifold 10/8-theorem to $Z_{\pm}$ we have either

$$\operatorname{ind} \mathcal{D}(Z_{\pm}) = 0,$$

in which case $b_2(Y) = -\overline{\mu}(M, c)$, or

$$1 - b_2^-(Z_\pm) \leq \mathrm{ind}(Z_\pm) \leq b_2^+(Z_\pm) - 1.$$

Here we note that

$$b_2^+(Z_+) = b_2^+(X_+) + b_2^-(Y) \leq 1 + b_2(Y),$$

and

$$b_2^-(Z_-) = b_2^-(X_-) + b_2^+(Y) \leq 1.$$

It follows that

$$0 \leq -\overline{\mu}(M, c) - b_2(Y) \leq 8b_2(Y),$$

which gives the second formula ∎

We can slightly strengthen the above result by applying the following refined inequality for the index of the Dirac operator, which is the orbifold version of the theorem in [6].

THEOREM 2.10 ([6]). *Let Z be a closed spin 4-orbifold with $b_1(Z) = 0$. Then either $\mathrm{ind}\,\mathcal{D}(Z) = 0$ or*

$$\epsilon\left(-\frac{1}{2}\,\mathrm{ind}\,\mathcal{D}(Z), b_2^-(Z)\right) - b_2^-(Z) \leq \mathrm{ind}\,\mathcal{D}(Z) \leq b_2^+(Z) - \epsilon\left(\frac{1}{2}\,\mathrm{ind}\,\mathcal{D}(Z), b_2^+(Z)\right)$$

where $\epsilon(d, \ell)$ is determined by the following rule.

$$\epsilon(d, \ell) = \begin{cases} 3 & (d \equiv 0 \mod 4, \ell \geq 4), \\ 1 & (d \equiv 0 \mod 4, \ell < 4), \\ 1 & (d \equiv 1 \mod 4), \\ 2 & (d \equiv 2 \mod 4), \\ 3 & (d \equiv 3 \mod 4). \end{cases}$$

The following result is obtained by modifying the proof of Theorem 2.9, so we omit the proof.

THEOREM 2.11. *Suppose that a Seifert rational homology 3-sphere M with spin structure c bounds a negative definite spin 4-manifold Y. We put $d = -(\overline{\mu}(M, c) + b_2(Y))/16$, which is an integer by the first claim of Theorem 2.9. Then we have either $b_2(Y) = -\overline{\mu}(M, c)$ or*

$$\frac{1}{9}(-\overline{\mu}(M, c) + 8(\epsilon - 1)) \leq b_2(Y) \leq -\overline{\mu}(M, c) - 8(\epsilon' - 1),$$

where

$$(\epsilon, \epsilon') = \begin{cases} (1, 1) & \text{if } d \equiv 0 \mod 4, \\ (1, 3) & \text{if } d \equiv 1 \mod 4, \\ (2, 2) & \text{if } d \equiv 2 \mod 4, \\ (3, 1) & \text{if } d \equiv 3 \mod 4. \end{cases}$$

If M is a spherical 3-manifold we can replace the above inequality by

$$\frac{1}{9}(-\overline{\mu}(M, c) + 8\epsilon) \leq b_2(Y) \leq -\overline{\mu}(M, c) - 8\epsilon'$$

for the same (ϵ, ϵ').

3. The Ozsváth-Szabó invariant. In this section we recall the definition and some properties of the Ozsváth-Szabó's correction term. Let M be a rational homology 3-sphere with spin^c structure t. In [10] Ozsváth and Szabó defined the Heegaard Floer homologies HF^+, HF^∞, and HF_{red} for (M, t). Here $HF^+(M, t)$ has an absolute grading with value in $\mathbf{Q}$, and $HF_{\mathrm{red}}(M, t)$, which is the cokernel of the natural map from $HF^\infty(M, t)$ to $HF^+(M, t)$, is a finitely generated abelian group with an absolute $\mathbf{Z}_2$ grading and hence its Euler chracteristic $\chi(HF_{\mathrm{red}}(M, t))$ is well-defined.

DEFINITION 3.1 ([11]). The correction term $d(M, t)$ is defined to be the minimum of the absolute grading of the elements in Im : $HF^\infty(M, t) \to HF^+(M, t)$.

In [11] Ozsváth and Szabó proved several properties of $d(M, t)$. For example, d changes the sign if the orientation of M is reversed, $d(M, t) = d(M, \bar{t})$ where $\bar{t}$ is the conjugation of t, and d is additive under a connected sum operation. Furthermore they proved the following results.

THEOREM 3.2 ([11]).

1. *Let (M_i, t_i) $(i = 1, 2)$ be a rational homology 3-sphere with* spin^c *structure. Suppose that there exists a* spin^c *rational homology cobordism (W, t_W) from (M_1, t_1) to (M_2, t_2). Then $d(M_1, t_1) = d(M_2, t_2)$.*
2. *If (M, t) bounds a 4-manifold (W, t_W) with* spin^c *structure t_W, then*

$$d(M, t) \equiv \frac{c_1(t_W)^2 - \sigma(W)}{4} \pmod{2\mathbf{Z}}.$$

3. *If (M, t) bounds a negative definite 4-manifold (W, t_W) with* spin^c *structure t_W, then*

$$c_1(t_W)^2 + b_2(W) \leq 4d(M, t).$$

Rustamov [14] (together with the result by Nicolaescu [13]) gives the relations between the correction term $d(M, t)$ and the modified Seiberg-Witten invariant for (M, t) via the modified Reidemeister torsion as follows. To describe his statement we introduce several notations.

DEFINITION 3.3. For a rational homology 3-sphere with spin^c structure (M, t), we put

$$\xi^0(M, t) = \frac{1}{2} d(M, t) - \chi(HF_{\mathrm{red}}(M, t)).$$

The sign convention is opposite to that in [14]. In addition we choose a Riemann metric g of M and let $sw(M, t)$ be the 3-dimensional Seiberg-Witten invariant for (M, t) with respect to g (the number of the irreducible solutions to the Seiberg-Witten equation counted with sign). We also denote by $\eta^{\mathrm{Dirac}}(M, t)$ (resp. $\eta^{\mathrm{sign}}(M)$) the eta invariant of the Dirac (resp. the signature) operator with respect to g. Then we define the modified Seiberg-Witten invariant to be

$$sw^0(M, t) = sw(M, t) + \frac{1}{8}(4\eta^{\mathrm{Dirac}}(M, t) + \eta^{\mathrm{sign}}(M)).$$

The value of $sw^0(M, t)$ does not depend on the choice of g.

Theorem 3.4 ([14]). $\xi^0(M,t) = sw^0(M,t)$. In fact the both invariants are equal to the modified Reidemeister torsion

$$\tau^0(M,t) = \tau(M,t) + \frac{1}{2}|H_1(M,\mathbf{Z})|\lambda(M)$$

where $\tau(M,t)$ is the Reidemeister-Turaev torsion of (M,t) and $\lambda(M)$ is the Casson-Walker invariant of M. (The equality $\tau^0 = sw^0$ is proved in [13].)

4. Proof of Theorem 1.1. In this section we focus on the Fukumoto-Furuta invariant for a spherical 3-manifold.

4.1. *The Fukumoto-Furuta invariant for* spin^c *structures.* We discuss the Fukumoto-Furuta invariant for a triple (M,X,t_X), where M, X are as before and t_X is a spin^c structure on X. We can define $w(M,X,t_X)$ for such a triple as in Definition 2.1 by choosing a 4-manifold Y with spin^c structure t_Y such that $\partial(Y,t_Y) = (M,t_X|_M)$. We note that M always bounds a 1-connected spin 4-manifold Y, in which case the inclusion $M \to Y$ induces the surjection $H^2(Y,\mathbf{Z}) \to H^2(M,\mathbf{Z})$. It follows that any given spin^c structure on M extends to that on Y. However we can also define $w(M,X,t_X)$ without any auxiliary manifold as follows. First we start with a spherical 3-manifold S with spin^c structure t. We write $S = S^3/G$, where G is a finite subgroup of $SO(4)$ acting freely on S^3. Up to conjugacy we can assume that G is a subgroup of $U(2) = S^3 \times_{\mathbf{Z}_2} S^1$. Then cS is represented as D^4/G. Next we fix a spin structure c_S on S, which extends uniquely to the spin structure c_{cS} on cS [19]. We note that c_{cS} is determined by the choice of the double covering $\sqrt{K}$ of the canonical bundle K over cS. We denote by $P_{\mathrm{spin}(4)}$ the corresponding orbifold principal $\mathrm{spin}(4)$-bundle over cS that covers the frame bundle over cS. The difference between t and c_S is represented by a complex line bundle $L = S^3 \times_\rho \mathbf{C}$ over S, which is a flat bundle associated with some representation $\rho : G \to U(1)$. Note that ρ corresponds to $c_1(L) \in H^2(S,\mathbf{Z})$ via the isomorphism $H^2(S,\mathbf{Z}) \cong \mathrm{Ext}(H_1(S,\mathbf{Z}),\mathbf{Z}) \cong \mathrm{Hom}(G,U(1))$. Then L extends to the orbifold line bundle $\widetilde{L} = D^4 \times_\rho \mathbf{C}$ over cS, and using $P_{\mathrm{spin}(4)}$ and the orbifold principal $U(1)$-bundle associated with $\widetilde{L}$, we obtain a spin^c structure $\widetilde{t}$ on cS that is an extension of t. We define $\delta(S,t) = w(S,cS,t_{cS})$ by

$$\delta(S,t) = -8\delta^{\mathrm{Dirac}}(S,t) - \delta^{\mathrm{sign}}(S),$$

where $\delta^{\mathrm{sign}}(S)$ (resp. $\delta^{\mathrm{Dirac}}(S,t)$) is the contribution from the singularity of cS to the index of the signature operator over cS (resp. the spin^c Dirac operator over cS associated with $\widetilde{t}$). Thus if a pair (M,t) of a rational homology 3-sphere and a spin^c structure on it bounds (X,t_X), where X is a compact 4-orbifold with spin^c structure t_X and all the singular points of X are isolated, we formally define the Fukumoto-Furuta invariant as

$$w(M,X,t_X) = \sigma(X) + \sum_{(S,t_X|_S)} \delta(S,t_X|_S),$$

where in the sum on the right hand side S runs over all the links of the singularities of X and $t_X|_S$ is the spin^c structure of S induced by t_X.

4.2. *The equivariant eta invariants.* The proof of the main theorem is deduced from the following proposition.

PROPOSITION 4.1. *Let S be a spherical 3-manifold with spin^c structure t, and g be the metric of positive constant curvature on S. Then*

$$\eta^{\mathrm{Dirac}}(S,t) = 2\delta^{\mathrm{Dirac}}(S,t), \quad \eta^{\mathrm{sign}}(S) = \delta^{\mathrm{sign}}(S),$$

where the eta invariants above are defined with respect to g.

The computation of the eta invariants goes in the same way as in [9]. So let us recall the theorem by Donnelly about the equivariant eta invariant. Let $\widetilde{Y}$ be a closed oriented smooth n-manifold (n odd) and G be a finite group acting freely on $\widetilde{Y}$. We assume that every $h \in G$ acts on $\widetilde{Y}$ as an orientation-preserving diffeomorphism of $\widetilde{Y}$. Let $\widetilde{Y} \to Y = \widetilde{Y}/G$ be the associated unbranched G covering, and $E_\rho = \widetilde{Y} \times_\rho \mathbf{C}^k$ be a flat $\mathbf{C}^k$-bundle over Y associated with some representation $\rho : G \to U(k)$. Now we consider the Dirac or the signature operator $D : \Gamma(W) \to \Gamma(W)$ over Y (where W is the corresponding spinor bundle), which is covered by a G-invariant operator $\widetilde{D} : \Gamma(\widetilde{W}) \to \Gamma(\widetilde{W})$ over $\widetilde{Y}$, where $\widetilde{W}$ is the spinor bundle covering W. We also consider the Dirac or the signature operator coupled with a flat connection A_ρ on E_ρ,

$$D_\rho : \Gamma(W \otimes E_\rho) \to \Gamma(W \otimes E_\rho).$$

Then we can define the usual eta function of D_ρ of the form

$$\eta_\rho(s, Y) = \sum_{\lambda \neq 0} (\mathrm{sgn}\,\lambda)|\lambda|^{-s},$$

where the sum is taken over all the nonzero eigenvalues of D_ρ, and also the equivariant eta function with respect to $h \in G$ of the form

$$\eta_h(s, \widetilde{Y}) = \sum_{\lambda \neq 0} (\mathrm{sgn}\,\lambda)\,\mathrm{Tr}(h_\lambda^* : E_\lambda \to E_\lambda)|\lambda|^{-s}$$

where E_λ is the λ-eigenspace of $\widetilde{D}$ and h_λ^* is induced by h. Both of these functions can be extended analytically over 0 and let $\eta_\rho(Y) = \eta_\rho(0, Y)$ and $\eta_h(\widetilde{Y}) = \eta_h(0, \widetilde{Y})$. Then we have

THEOREM 4.2 ([2]).

$$\eta_\rho(Y) = \frac{1}{|G|} \sum_{h \in G} \eta_h(\widetilde{Y})\chi_\rho(h),$$

where χ_ρ is the character of ρ.

4.3. *Proof of Proposition 4.1.* We apply Theorem 4.2 to $\widetilde{Y} = S^3$ and $Y = S = S^3/G$. Here for every $h \in G$, the fixed point set $(D^4)^h$ only consists of the origin 0. If we denote by p the order of h, then the quotient space $S^3/\langle h \rangle$ of S^3 by the cyclic group $\langle h \rangle$ generated by h is a lens space of the form $L(p,q)$ for some q. It follows that for some appropriate choice of the coordinates (z, w) of $D^4 \subset \mathbf{C}^2$, we can assume that the action of h on D^4 is given by

$$h(z,w) = (\zeta^k z, \zeta^{kq} w) \quad (\zeta = \exp(2\pi i/p))$$

for some k. We extend the standard metric g of S^3 to the metric $\widetilde{g}$ of D^4 of non-negative scalar curvature, and first we consider the case when D is the Dirac operator. According to 4.1, we fix one spin structure c_0 on S and its unique extension $\widetilde{c}_0$ on $cS = D^4/G$,

which is determined by the choice of the double covering $\sqrt{K}$ of the canonical bundle K over cS. Here K is the quotient space of some G-action on $\widetilde{K} = D^4 \times \mathbf{C}$, which induces the covering $D^4 \to cS = D^4/G$ of the base spaces. Hence $\sqrt{K}$ is determined by choosing a lift of the G-action on $\widetilde{K}$ to the double covering $\sqrt{\widetilde{K}} = D^4 \times \mathbf{C} \to \widetilde{K} = D^4 \times \mathbf{C}$ defined by $(z, w, \widetilde{w}) \to (z, w, \widetilde{w}^2)$ ($\widetilde{w} \in \mathbf{C}$). For each $h \in G$, a lift of the action of h on $\widetilde{K}$ to $\sqrt{\widetilde{K}}$, which we denote by $\widetilde{h}$, is represented as $\widetilde{h}(z, w, \widetilde{w}) = (\zeta^k z, \zeta^{kq} w, \epsilon(h) \zeta^{k(-1-q)/2} \widetilde{w})$ for some choice of $\epsilon(h) = \pm 1$ (depending on h). Then $\widetilde{h}$ determines the lifts of the h action to that on the $\pm$ spinor bundles $\widetilde{W}^{\pm}$ over D^4 according to the representations $\widetilde{W}^{+} = (\wedge^{0,0} \oplus \wedge^{0,2})(D^4) \otimes \sqrt{\widetilde{K}}$ and $\widetilde{W}^{-} = \wedge^{0,1}(D^4) \otimes \sqrt{\widetilde{K}}$. We also denote these actions by the same symbol $\widetilde{h}$. Then by the theorem of Donnelly [2], the $\widetilde{h}$-index $\mathrm{ind}(\widetilde{\mathcal{D}}, \widetilde{h})$ of the Dirac operator $\widetilde{\mathcal{D}} : \Gamma(\widetilde{W}^{+}) \to \Gamma(\widetilde{W}^{-})$ over D^4, which is given by

$$\mathrm{Tr}\, \widetilde{h}|_{\ker \widetilde{\mathcal{D}}} - \mathrm{Tr}\, \widetilde{h}|_{\ker \widetilde{\mathcal{D}}^*}$$

with the Atiyah-Patodi-Singer condition, is represented as follows.

$$\mathrm{ind}(\widetilde{\mathcal{D}}, \widetilde{h}) = \frac{\mathrm{Tr}\, \widetilde{h}|_{(\widetilde{W}^{+})_0} - \mathrm{Tr}\, \widetilde{h}|_{(\widetilde{W}^{-})_0}}{\det(1 - (d\widetilde{h})_0)(N_0 \otimes \mathbf{C})} - \frac{1}{2}(\eta_{\widetilde{h}}^{\mathrm{Dirac}}(S^3) + k_{\widetilde{h}}).$$

Here N_0 is the normal bundle of $\{0\} \subset D^4$ and the first term on the right hand side is the "Lefschetz number", i.e., the contribution from the unique fixed point $\{0\}$ to the $\widetilde{h}$-index, which we denote by $L(\widetilde{h}, \widetilde{\mathcal{D}})$, and $k_{\widetilde{h}} = \mathrm{Tr}\, \widetilde{h}|_{\ker \widetilde{D}}$ for $\widetilde{D} = \widetilde{\mathcal{D}}|_{S^3}$. Since the metrics g and $\widetilde{g}$ on S^3 and D^4 have non-negative (not identically zero) scalar curvature, we have

$$\mathrm{ind}(\mathcal{D}, \widetilde{h}) = 0, \quad k_{\widetilde{h}} = 0$$

by the Weitzenböck formula, and hence

$$\eta_{\widetilde{h}}^{\mathrm{Dirac}}(S^3) = 2L(\mathcal{D}, \widetilde{h}).$$

Next we consider a spin^c structure t on S such that the difference between t and c_0 is given by a flat complex line bundle L_ρ over S with flat connection A_ρ corresponding to some representation $\rho : G \to U(1)$, which extends to the flat line bundle $\widetilde{L}_\rho$ over cS. Then the spin^c Dirac operator corresponding to t is equal to $D_{A_\rho} : \Gamma(W \otimes L_\rho) \to \Gamma(W \otimes L_\rho)$, covered by the G-invariant Dirac operator $\widetilde{D} : \Gamma(\widetilde{W}) \to \Gamma(\widetilde{W})$ over S^3, which also extends to the Dirac operator $\mathcal{D} : \Gamma(\widetilde{W}^{+}) \to \Gamma(\widetilde{W}^{-})$ over D^4. Here W and $\widetilde{W}$ are the spinor bundles over S and S^3 respectively, and $\widetilde{W}^{\pm}$ is the $\pm$ spinor bundle over D^4 defined in the spin case, while for each $h \in G$ the lift of the h-action on $\widetilde{W}^{\pm}$ or $\widetilde{W}$, which we denote by $\widetilde{h}'$, is defined as follows. If the order of h is p as before, $\rho(h) = \zeta^a$ for some $a \in \mathbf{Z}$. Hence $\widetilde{h}'$ acts on $\sqrt{\widetilde{K}}$ by $\widetilde{h}'(z, w, \widetilde{w}) = (\zeta^k z, \zeta^{qk} w, \epsilon(h) \zeta^{k(-1-q)/2+a} \widetilde{w})$ so that the quotient of $\sqrt{\widetilde{K}}$ by $\widetilde{h}'$ is $\sqrt{K} \otimes \widetilde{L}_\rho$. Thus the Lefschetz number $L(\widetilde{h}', \widetilde{\mathcal{D}})$ is given by

$$L(\widetilde{h}', \widetilde{\mathcal{D}}) = L(\widetilde{h}, \widetilde{\mathcal{D}})\chi_\rho(h).$$

On the other hand by the theorem of Donnelly, the eta invariant $\eta^{\mathrm{Dirac}}(S, t)$, which is equal to $\eta_\rho(S)$, satisfies

$$\eta^{\mathrm{Dirac}}(S, t) = \frac{1}{|G|} \sum_{h \in G} \eta_{\widetilde{h}}^{\mathrm{Dirac}}(S^3)\chi_\rho(h) = \frac{1}{|G|} \sum_{h \in G, h \neq 1} \eta_{\widetilde{h}}^{\mathrm{Dirac}}(S^3)\chi_\rho(h).$$

We note that $\eta^{\mathrm{Dirac}}(S^3)$ (which is equal to $\eta_{\widetilde{h}}^{\mathrm{Dirac}}(S^3)$ for $h = 1$) is 0 since there is an orientation-reversing isometry of S^3. Here by the previous results we have

$$\eta_{\widetilde{h}}^{\mathrm{Dirac}}(S^3)\chi_\rho(h) = 2L(\widetilde{h}, \widetilde{\mathcal{D}})\chi_\rho(h) = 2L(\widetilde{h}', \widetilde{\mathcal{D}}).$$

On the other hand the contribution $\delta^{\mathrm{Dirac}}(S, t)$ from cS to the index of the Dirac operator over cS with respect to t_{cS} (which is an extension of t as in 4.1) is given by

$$\delta^{\mathrm{Dirac}}(S, t) = \frac{1}{|G|} \sum_{h \in G, h \neq 1} L(\widetilde{h}', \widetilde{\mathcal{D}})$$

by the V-index theorem. Thus we have the first claim of the proposition.

To prove the second claim of Proposition 4.1, we note that in [2] the h-signature $\mathrm{sign}(h, D^4)$ of D^4 for each $h \in G$ is represented as

$$\mathrm{sign}(h, D^4) = \frac{\mathrm{Tr}\,\widetilde{h}|_{(\wedge^+)_0} - \mathrm{Tr}\,\widetilde{h}|_{(\wedge^-)_0}}{\det(1 - (d\widetilde{h})_0)(N_0 \otimes \mathbf{C})} - \eta_h^{\mathrm{sign}}(S^3).$$

Here the actions of h on $\wedge^\pm$ over D^4 and $\wedge^{ev}$ over S^3 induced by the original action on D^4 are also denoted by h. Since $\mathrm{sign}(h, D^4) = 0$, the first term on the right hand side of the above equation, which comes from the h-index of the signature operator $\widetilde{\mathcal{D}}^{\mathrm{sign}}$ over D^4 and we denote by $L(h, \widetilde{\mathcal{D}}^{\mathrm{sign}})$, is equal to $\eta_h^{\mathrm{sign}}(S^3)$. For the above representation of the action of h, it is known that

$$\eta_h^{\mathrm{sign}}(S^3) = -\cot\frac{\pi k}{p}\cot\frac{\pi kq}{p}.$$

Moreover we can see ([2]) that

$$\eta^{\mathrm{sign}}(S) = \frac{1}{|G|} \sum_{h \in G} \eta_h^{\mathrm{sign}}(S^3) = \frac{1}{|G|} \sum_{h \in G, h \neq 1} \eta_h^{\mathrm{sign}}(S^3) = \frac{1}{|G|} \sum_{h \in G, h \neq 1} L(h, \widetilde{\mathcal{D}}^{\mathrm{sign}}).$$

Here by the same reason as before we have $\eta^{\mathrm{sign}}(S^3) = 0$, which yields the second equality above. Finally by the V-index theorem the last term of the above equation is also equal to the contribution $\delta^{\mathrm{sign}}(S)$ from cS to the V-signature. This proves the second claim of the proposition.

4.4. *Proof of Theorem 1.1.* Let (S, t) be a pair of a spherical 3-manifold and a spin^c structure on it. We fix a metric g of S of positive constant curvature. Then by Rustamov's theorem we have

$$\frac{1}{2}d(S, t) - \chi(HF_{\mathrm{red}}(S, t)) = sw(S, t) + \frac{1}{8}(4\eta^{\mathrm{Dirac}}(S, t) + \eta^{\mathrm{sign}}(S)).$$

Here the 3-dimensional Seiberg-Witten invariant $sw(S, t)$ and the eta inavariants are defined with respect to g. Hence $sw(S, t)$ is zero since g has positive scalar curvature, and by Proposition 4.1, we have

$$\delta(S, t) := -(8\delta^{\mathrm{Dirac}}(S, t) + \delta^{\mathrm{sign}}(S)) = -(4\eta^{\mathrm{Dirac}}(S, t) + \eta^{\mathrm{sign}}(S, t)).$$

For the estimate of $\chi(HF_{\mathrm{red}}(S, t))$, we recall Némethi's result [12], where he observed a Heegaard Floer homology of a rational homology 3-sphere that bounds a negative definite plumbing $P(\Gamma)$ such that Γ is an almost rational (AR) graph. Every link of a rational

surface singularity and every Seifert rational homology 3-sphere (up to orientation) satisfy
these conditions. In general we have

$$\chi(HF_{\mathrm{red}}(M,t)) = \operatorname{rank} HF_{\mathrm{even}}(M,t) - \operatorname{rank} HF_{\mathrm{odd}}(M,t)$$

where HF_{odd} and HF_{even} denote the sets of elements of HF_{red} with odd and even absolute
$\mathbf{Z}_2$ grading respectively. We also note that $\chi(HF(-M,t)) = -\chi(HF(M,t))$. If M is a
plumbed homology 3-sphere satisfying the above conditions, then Theorem 8.3 in [12]
shows that $HF_{\mathrm{odd}}(-M,t) = 0$ for any spin^c structure t of M, and moreover we have

THEOREM 4.3 ([12]). *Let M be a link of a rational surface singularity. Then $HF_{\mathrm{red}}(M,t)$
$= 0$ for every* spin^c*structure of M.*

It is well known that every spherical manifold S is a link of a quotient singularity
(up to orientation), and every such singularity is a rational surface singularity. Hence we
have $HF_{\mathrm{red}}(S,t) = 0$ for every spin^c structure t on S. It follows that

$$4d(S,t) = -\delta(S,t).$$

This proves Theorem 1.1. ∎

Now we compare Theorem 2.9 with the Ozsváth-Szabó's theorem for a pair (M,c)
of a Seifert rational homology 3-sphere and a spin structure on it. Suppose that (M,c)
bounds a negative definite spin 4-manifold (W,c_W). Then Theorem 2.9 shows that

$$b_2(W) \equiv -\overline{\mu}(M,c) \pmod{16}, \quad -\frac{\overline{\mu}(M,t)}{9} \le b_2(W) \le -\overline{\mu}(M,c),$$

while Theorem 3.2 shows that

$$b_2(W) \equiv 4d(M,c) \pmod 8, \quad b_2(W) \le 4d(M,c).$$

If M is non-spherical, the above two estimates are different in general, but Theorem 1.1
suggests that they derive similar results if M is spherical. If a spherical 3-manifold S is
neither a lens space nor a Seifert fibration over $S^2(2,2,n)$ for some $n \ge 2$ (i.e., a prism
manifold), we have $|\overline{\mu}(S,c)| \le 8$ according to the list of $\overline{\mu}(S,c) = \delta(S,c)$ in [18]. It follows
that the above two estimates lead to the same conclusion: $b_2(W) = -\overline{\mu}(S,c) = 4d(S,c)$.
In fact we have the following result (which is the extension of Theorem 2 in [18]).

COROLLARY 4.4. *Suppose that a pair (M,c) of a Seifert rational homology 3-sphere and a
spin structure on it bounds a negative definite spin 4-manifold (W,c_W). If $|\overline{\mu}(M,c)| \le 18$,
then $b_2(W) = -\overline{\mu}(M,c)$.*

REMARK 4.5. Strictly speaking, for the case when S is a spherical manifold with $\overline{\mu}(S,c) =
-4d(S,c) = -8$, Theorem 3.2 shows that either $b_2(W) = 8$ or $b_2(W) = 0$, while the
estimate by Theorem 2.9 eliminates the second possibility. This case occurs if S is a
Seifert fibration over $S^2(2,3,5)$ with Seifert invariants $\{(2,1),(3,1),(5,10k-4)\}$ for $k \ge 0$
([18]), and c is the unique spin structure. In such a case the Rokhlin invariant of (S,c) is
nonzero, and hence the case $b_2(W) = 0$ is eliminated.

On the other hand, if S is either a lens space or a prism manifold, the absolute value
of $\overline{\mu}(S,c) = -4d(S,c)$ may be arbitrary large. In fact in such a case, $\overline{\mu}(S,c) = \delta(S,c)$ is
represented by a σ function in §2. As in [5], if p/q has a continued fraction expansion of

the form $[a_1, \ldots, a_s]$ for a_i even and $a_i \neq 0$, (in such a case p and q have opposite parity), then $\sigma(q, p, -1) = -\sum \mathrm{sgn}(a_i)$. Since $\overline{\mu}(L(p, q), c) = \sigma(q, p, (-1)^{c-1})$, where $c = c(\mu)$ is chosen as in §1, there exist p, q, c so that $-\overline{\mu}(L(p, q), c) \geq N$ for any given $N > 0$ by some choice of a_i's above. The same statement is also true for the prism manifolds according to the list in [18]. In such cases the estimate given by Theorem 2.9 might be slightly stronger than that given by Theorem 3.2.

4.5. *Explicit computation for the lens spaces.* In this subsection we focus on the lens space $L(p, q)$ with $\gcd(p, q) = 1$ and $p > q > 0$ and give more explicit relations between the two invariants in Theorem 1.1. First we compare the description of the spin^c structures on the lens spaces given by [11] and the one described in §2. Let (E, α, γ) be the standard Heegaard diagram corresponding to the decomposition of $L(p, q)$ into two solid tori U_0 and U_1, where α (resp. γ) is the attaching circle of U_0 (resp. U_1). Here the Heegaard torus $E = \partial U_0 = -\partial U_1$ is identified with $S^1 \times S^1$ so that α corresponds to $S^1 \times \{1 \text{ point}\}$ and, if we put $\beta = \{1 \text{point}\} \times S^1$, the homology class of γ in E is represented by $p\beta - q\alpha$. Moreover if $L(p, q)$ is represented by the $-p/q$-surgery on the unknot of S^3, then U_0 is identified with the exterior of the unknot so that the meridian μ (resp. the preferred longitude λ) of the unknot corresponds to β (resp. α). Note that the orientation convention for the lens space in [11] is opposite to ours ($L(p, q)$ is described as $-L(p, q)$ in [11]). If we choose a base point z on E, we have a map defined in [10]

$$s_z : \alpha \cap \gamma \to \mathrm{spin}^c(L(p, q))$$

from $\alpha \cap \gamma$ to the set of the spin^c structures $\mathrm{spin}^c(L(p, q))$ on $L(p, q)$, which is a bijection in the case of lens spaces. Furthermore in [11] Ozsváth-Szabó defined the circular ordering of $\mathrm{spin}^c(L(p, q))$, which is represented by a natural number in $\{0, 1, \ldots, p - 1\}$ as follows. First we consider the pointed Heegaard triple $(E, \alpha, \beta, \gamma, z)$, where z is placed so that all the coefficients of the triply periodic domain connecting α, β, and γ are negative. Let $x_0, \ldots, x_{p-1}$ be the p intersection points of $\alpha \cap \gamma$ which are placed in order along α so that x_{p-1} is adjacent to z. Then the spin^c structure $s_z(x_i)$ is represented by $i \pmod{p}$. They also gave the following recursive formula.

PROPOSITION 4.6 ([11], Proposition 4.8). *For any integer i with $0 \leq i < p + q$,*

$$d(L(p, q), i) = \left(\frac{pq - (2i + 1 - p - q)^2}{4pq} \right) - d(L(q, r), j).$$

Here r and j are the reductions modulo q of p and i respectively.

In fact $(E, \alpha, \beta, \gamma)$ represents a compact 4-manifold $X_{\alpha, \beta, \gamma}$ with

$$\partial X_{\alpha, \beta, \gamma} = S^3 - L(q, r) + L(p, q),$$

where S^3 and $L(q, r)$ are represented by the Heegaard diagrams (E, α, β) and (E, β, γ) respectively, and the intersection points $\beta \cap \gamma$ are placed in order along β so that the corresponding spin^c structures (via s_z as before) have the circular ordering as in the case of $L(p, q)$. Moreover it is shown in [11] that there are $p + q$ triangles ϕ_i ($0 \leq i \leq p + q - 1$) connecting the intersection points in $\alpha \cap \gamma$ and $\beta \cap \gamma$ corresponding to the spin^c structures with the same order i (which is considered modulo p and q respectively), and the unique intersection point $\alpha \cap \beta$ (which corresponds to the unique spin^c structure

on S^3). Furthermore as in [11], there is a correspondence (which we denote also by s_z) between the triangles ϕ_i and the spin^c structures on $X_{\alpha,\beta,\gamma}$ so that $s_z(\phi_i)$ induces the i-th spin^c structures $s_z(x_i)$ on $L(p,q)$ and $L(q,r)$. It is also shown in [11] that

$$\langle c_1(s_z(\phi_i)), H \rangle = 2i + 1 - p - q$$

where H is the generator of $H_2(X_{\alpha,\beta,\gamma})$. Hence the map $q : i \to p + q - 1 - i$ leaves the above value invariant, and in fact the correspondence $i \to q(i)$ represents the conjugation $t \to \bar{t}$ of the spin^c structures on $L(p,q)$ and $L(q,r)$ if the numbers are counted modulo p and q respectively (see also [8]). It follows that the spin^c structure t on $L(p,q)$ comes from a spin structure (in this case $c_1(\det t) = t - \bar{t} = 0$) if and only if t corresponds to one of the integers in $\{(q-1)/2, (p+q-1)/2\}$ (just one of them is an integer if p is odd and both of them are integers if p is even).

Now we compare the representation of spin$^c(L(p,q))$ above with that given in §4.1, where we describe the spin^c structure on $L(p,q)$ as the restriction to $L(p,q)$ of some spin^c structure on $cL(p,q)$. Note that the associated $+$ spinor bundle is of the form $(1 \oplus K^{-1}) \otimes L$, where K is the canonical line bundle and L is some line bundle over $cL(p,q)$. We denote the restriction of the line bundle over $cL(p,q)$ to $L(p,q)$ by the same symbol. Note that the covering projection $\pi : S^3 \to L(p,q)$ maps the curve (ξ^t, ξ^{qt}) $(0 \le t < 1)$ in S^3 to the curve in $L(p,q)$ (which is the $-p/q$ surgery on the unknot) corresponding to the meridian μ of the unknot, which generates $\pi_1(L(p,q))$. Let L_ρ be the flat line bundle over $L(p,q)$ corresponding to the representation $\rho : \pi_1(L(p,q)) \to U(1)$ defined by $\rho(\mu) = \zeta := \exp(2\pi i/p)$, i.e.,

$$L_\rho = S^3 \times \mathbf{C}/\widetilde{\rho}$$

where $\widetilde{\rho}(z_1, z_2, w) = (\zeta z_1, \zeta^q z_2, \zeta w)$. Then every line bundle over $L(p,q)$ is of the form $L_\rho^{\otimes k}$ for some k, and $K = L_\rho^{\otimes(-q-1)}$. Moreover the spin^c structure on $L(p,q)$ corresponding to $(1 \oplus K^{-1}) \otimes L_\rho^{\otimes k}$ comes from a spin structure if and only if L_ρ^k is the double covering $\sqrt{K}$ of K, in which case k is either $-(q+1)/2$ if q is odd, or $-(p+q+1)/2$ if $p+q$ is odd. Once we fix a spin structure s_0 corresponding to $\sqrt{K} \oplus \sqrt{K^{-1}}$ and denote by t the spin^c structure on $L(p,q)$ corresponding to $(\sqrt{K} \oplus \sqrt{K^{-1}}) \otimes L_\rho^{\otimes \ell}$, then the conjugation $\bar{t}$ of t corresponds to $(\sqrt{K} \oplus \sqrt{K^{-1}}) \otimes L_\rho^{\otimes(-\ell)}$.

In the meantime we have

PROPOSITION 4.7. *Let $\{x_0, \ldots, x_{p-1}\}$ be the intersection points $\alpha \cap \gamma$ for the pointed Heegaard diagram (E, α, γ, z) of $L(p,q)$ described and ordered as above. Then the difference $s_z(x_{i+k}) - s_z(x_i)$ is represented by $c_1(L_\rho^{\otimes k}) \in H^2(L(p,q), \mathbf{Z})$, where the subscripts are counted modulo p.*

Proof. It is sufficient to prove the case where $k = 1$. As is shown in [10], $s_z(x_{i+1}) - s_z(x_i)$ is the Poincaré dual of the homology class in $L(p,q)$ of a curve $\epsilon(x_i, x_{i+1})$, which is the union of the curve in α from x_i to x_{i+1} and the curve in γ from x_{i+1} to x_i. Since the oriented curve γ is represented by $(\exp(-2\pi iqt), \exp(2\pi ipt))$ $(0 \le t \le 1)$ in $E = S^1 \times S^1$ (where $\alpha = S^1 \times \{*\}$ and $\beta = \{*\} \times S^1$), we see that $\epsilon(x_i, x_{i+1})$ is $q^*\mu = q^*\beta$, where q^* is the integer with $qq^* \equiv 1 \pmod{p}$. On the other hand by the Poincaré duality and the universal coefficient theorem, $q^*\mu \in H_1(L(p,q), \mathbf{Z})$ is identified via $\mathrm{Ext}(H_1(L(p,q)), \mathbf{Z})$

with the element $\widetilde{\rho}$ of $\mathrm{Hom}(H_1(L(p,q),\mathbf{Z}),\mathbf{Q}/\mathbf{Z})$ defined by

$$\widetilde{\rho}(c) = \mathrm{lk}_{\mathbf{Q}/\mathbf{Z}}(q^*\mu, c)$$

for $c \in H_1(L(p,q),\mathbf{Z})$, where $\mathrm{lk}_{\mathbf{Q}/\mathbf{Z}}$ denotes the linking form over $H_1(L(p,q),\mathbf{Z})$. Since

$$\mathrm{lk}_{\mathbf{Q}/\mathbf{Z}}(\mu,\mu) = q/p \pmod 1$$

for the meridian μ, we see that $c_1(L_\rho)$ is the Poincaré dual of $q^*\mu$, which gives the desired result. ∎

Thus if the spin structure on $L(p,q)$ given by some choice of $\sqrt{K}$ is labelled by s, then the spin^c structure corresponding to $(\sqrt{K} \oplus \sqrt{K^{-1}}) \otimes L_\rho^k$ is labelled by $s + k$, and its conjugate is labelled by $s - k$ (modulo p). The value of the correction term for $L(p,q)$ with spin structure is related to the σ-function as follows.

PROPOSITION 4.8. *Let p, q be the integers with $\gcd(p,q) = 1$ and $p > q > 0$. If q is odd, then*

$$4d\left(L(p,q), \frac{q-1}{2}\right) = -\sigma(q,p,1),$$

while if p and q have opposite parity, then

$$4d\left(L(p,q), \frac{p+q-1}{2}\right) = -\sigma(q,p,-1).$$

Proof. The case where p is odd comes from Theorem 1.1 since the spin structure on $L(p,q)$ is unique. Note that in the description of $\overline{\mu}(L(p,q),c)$ in terms of $\sigma(q,p,\epsilon)$, the last term ϵ is determined so that $\epsilon = 1$ if $\sqrt{K}$ is $L_\rho^{\otimes(-(1+q)/2)}$ and $\epsilon = -1$ if $\sqrt{K}$ is $L_\rho^{\otimes(-(p+q+1)/2)}$. Suppose that p is even. Then we already have

$$4d\left(L(q,p), \frac{p+q-1}{2}\right) = -\sigma(p,q,-1)$$

since q is odd and hence $(p+q-1)/2 \pmod q$ and -1 are the unique choices for the spin structure on $L(q,p)$ on the both sides. Strictly speaking, the number p in the above equation should be replaced by its residue r modulo q. Suppose that $p = nq + r$ for some $n > 0$. If n is even, then the above equation is the same as that obtained by replacing p by r on both sides. If n is odd, then the above equation is equivalent to

$$4d\left(L(q,r), \frac{r-1}{2}\right) = -\sigma(r,q,1),$$

which appeared in the first claim since $\sigma(nq+r,q,\epsilon) = \sigma(r,q,(-1)^n\epsilon)$. Then the claim for p follows by comparing the formula in Proposition 4.6 with that given in §2, i.e.,

$$\sigma(p,q,-1) + \sigma(q,p,-1) = -\operatorname{sgn} pq = -1.$$

In fact, this reciprocity is the same as the Ozsváth-Szabó's recursive formula for this particular case. ∎

The explicit correspondence between the correction term d and $\overline{\mu}$ for lens spaces with general spin^c structure is given as follows.

REMARK 4.9. Consider a spin structure on $L(p,q)$ ($p > q > 0$) such that its unique extension on $cL(p,q)$ has the $+$ spinor bundle of the form $\sqrt{K} \oplus \sqrt{K^{-1}}$. Here we put $\epsilon = 1$

if $\sqrt{K} = L_\rho^{\otimes(-(q+1)/2)}$ and $\epsilon = -1$ if $\sqrt{K} = L_\rho^{\otimes(-(p+q+1)/2)}$. We note that in Proposition 4.8 the first and the second cases correspond to $(q-1)/2$ and $(p+q-1)/2 \pmod{p}$ respectively in Ozsváth-Szabó's notation. Once we choose such a spin structure (and $\sqrt{K}$), any spin^c structure t on $L(p,q)$ is a restriction of that on $cL(p,q)$ whose + spinor bundle is given by $(\sqrt{K} \oplus \sqrt{K^{-1}}) \otimes L_\rho^{\otimes a}$ for some a. Then we can compute $4d(L(p,q),t) = -\overline{\mu}(L(p,q),t)$ explicitly by using the formula in §4.3 (see also the computation in [5]) as follows. First let us recall that

$$4d(L(p,q),t) = \delta^{\text{sign}}(L(p,q)) + 8\delta^{\text{Dirac}}(S,t).$$

Here the first term is equal to

$$\frac{1}{p} \sum_{k=1}^{p-1} \left(-\cot \frac{\pi k}{p} \cot \frac{\pi k q}{p} \right)$$

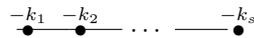(which is the Dedekind sum up to multiplicative constant), while the second term is equal to

$$\frac{1}{p} \sum_{k=1}^{p-1} \left(-2\epsilon^k \csc \frac{\pi k}{p} \csc \frac{\pi k q}{p} \exp\left(\frac{2\pi i k a}{p} \right) \right) = \frac{1}{p} \sum_{k=1}^{p-1} \left(-2\epsilon^k \csc \frac{\pi k}{p} \csc \frac{\pi k q}{p} \cos \frac{2\pi k a}{p} \right).$$

By Proposition 4.7 we see that $d(L(p,q),t) = d(L(p,q), a + \frac{q-1}{2})$ if $\sqrt{K} = L_\rho^{\otimes(-(q+1)/2)}$ ($\epsilon = 1$), and $d(L(p,q),t) = d(L(p,q), a + \frac{p+q-1}{2})$ if $\sqrt{K} = L_\rho^{\otimes(-(p+q+1)/2)}$ ($\epsilon = -1$). Since $\cos(2\pi(a + \frac{p}{2})/p) = -\cos(2\pi a/p)$, in either case we have

$$4d(L(p,q),i) = -\frac{1}{p} \left(\sum_{k=1}^{p-1} \cot \frac{\pi k}{p} \cot \frac{\pi k q}{p} + 2\csc \frac{\pi k}{p} \csc \frac{\pi k q}{p} \cos \frac{2\pi k (i - \frac{q-1}{2})}{p} \right).$$

The above explicit formula is also obtained by Tange [17] by using the formulae for the Dedekind sums and for the correction term deduced from Proposition 4.6 (without using the index theorem).

5. Concluding remarks. A Seifert rational homology 3-sphere is a boundary of a plumbing $P(\Gamma)$ for some weighted tree Γ. In particular $L(p,q)$ is a boundary of a negative definite plumbing $P(\Gamma)$, where Γ is of the form

and

$$\frac{p}{q} = [k_1, \ldots, k_s] := k_1 - \cfrac{1}{\cdots - \cfrac{1}{k_s}}$$

with $k_i \geq 2$ for all i. Let b_j be the generator of $L := H_2(P(\Gamma), \mathbf{Z})$ corresponding to the jth vertex of Γ, g_j be the dual generator of $L' := H_2(P(\Gamma).\partial P(\Gamma), \mathbf{Z})$, and $[g_j] = \partial g_j \in H_1(L(p,q), \mathbf{Z})$. Then if we write

$$\frac{n_{i,j}}{n_{i+1,j}} = [k_i, \ldots, k_j]$$

(we have $p = n_{1,s}$ and $q = n_{2,s}$), as in [12] we have

$$[g_j] = n_{j+1,s}[g_s].$$

(We put $n_{s+1,s} = 1$ and $n_{s+2,s} = 0$.) As is shown in [12], for any spin^c structure s on $L(p,q)$ there exists a unique distinguished representative in L' for s, which is defined as follows. First note that s is a restriction of a spin^c structure $\widetilde{s}$ on $P(\Gamma)$, which is uniquely determined by $c_1(\det \widetilde{s})$ since $P(\Gamma)$ is 1-connected. Here $c_1(\det \widetilde{s})$ is represented as $K_\Gamma + 2\ell_r \in L'$ via Poincaré duality, where K_Γ is the canonical class of $P(\Gamma)$ defined by $K_\Gamma = \sum_j (k_j - 2)g_j$ and $\ell_r = -\sum_j a_j g_j$ for some a_j. Then the images of them in $H_1(L(p,q), \mathbf{Z})$ are given by

$$[K_\Gamma] = \sum_j (k_j - 2) n_{j+1,s}[g_s] = \sum_{j=1}^s (n_{j,s} - 2n_{j+1,s} + n_{j+2,s})[g_s] = (p - q - 1)[g_s]$$

and

$$[\ell_r] = -\sum_{j=1}^s a_j n_{j+1,s}[g_s] =: -a[g_s].$$

Then according to [12] $K_\Gamma + 2\ell_r$ is distinguished if and only if the coefficients of ℓ_r satisfy

$$(SI) \quad a_j \geq 0, \quad j = 1, \dots, s$$

$$n_{js} > \sum_{i=j}^s n_{i+1,s} a_i \quad j = 1, \dots, s.$$

We note that if $L(p,q)$ is represented as the $-p/q$ surgery on the unknot, $[g_1]$ corresponds to the generator of $H_1(L(p,q), \mathbf{Z}) \cong \mathbf{Z}_p$ represented by the meridian of the unknot. Furthermore since $[g_1] = n_{2,s}[g_s] = q[g_s]$ we have $[g_s] = q'[g_1]$, where $q' := n_{1,s-1}$ satisfies $qq' \equiv 1 \pmod{p}$.

Now we compare the distinguished representative for s with the representation of s given in §4. We denote by L_ρ the line bundle over $L(p,q)$ defined by $L_\rho = S^3 \times \mathbf{C}/\mathbf{Z}_p$ where the generator g of $\mathbf{Z}_p$ acts on $D^4 \times \mathbf{C}$ by

$$g(z_1, z_2, w) = (\eta z_1, \eta^q z_2, \eta w) \quad (\eta = \exp(2\pi i/p)).$$

Then if we idetify $H_1(L(p,q), \mathbf{Z})$ with $H^2(L(p,q), \mathbf{Z})$ via the Poincaré duality we see that

$$c_1(L_\rho) = q'[g_1] = [g_s].$$

Hence if we denote by $\widetilde{s}_{\mathrm{can}}$ the spin^c structure on $cL(p,q)$ whose determinant line bundle is the canonical line bundle K over $cL(p,q)$, we see that

$$c_1(\det \widetilde{s}_{\mathrm{can}})|_{L(p,q)} = c_1(L_\rho^{\otimes(-q-1)}) = -(q+1)[g_s]$$

It follows from the above formula that the spin^c structure s_{can} on $L(p,q)$ induced by $\widetilde{s}_{\mathrm{can}}$ coincides with that corresponding to K_Γ. Moreover using Proposition 4.8 we can see the following proposition.

PROPOSITION 5.1. *For s_{can} on $L(p,q)$, $d(L(p,q), s_{\mathrm{can}}) = d(L(p,q), -1)$ in Oszváth-Szabó's notation. For the* spin^c *structure s on $L(p,q)$ represented by $K + 2\ell_r$ defined above, which coincides with $s_{\mathrm{can}} \otimes L_\rho^{\otimes(-a)}$, we have $d(L(p,q), s) = d(L(p,q), -1-a)$.*

Finally we point out that the claim (for spin structures) in Theorem 1.1 also holds for certain plumbed 3-manifolds as follows. The proof will be discussed elsewhere.

THEOREM 5.2 ([20]). *Let M be a boundary of a negative definite plumbing $P(\Gamma)$. Suppose that $m_v \leq -2$ for any vertex v of Γ and $m_v \leq -\nu_v$ for any v with $\nu_v \geq 3$. Here m_v is the weight of v and ν_v is the number of edges in Γ connected with v. Then for any spin structure c on M, $-4d(M,c)$ coincides with the Neumann-Siebenmann invariant of (M,c).*

We note that any plumbed 3-manifold satisfying the conditions in Theorem 5.2 is a link of a rational surface singularity ([12]) as in the cases of spherical 3-manifolds. But not all spherical 3-manifolds satisfy the above condition.

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