Abstract. In this paper we introduce the categorical length, a homotopy version of Fox categorical sequence, and an extended version of relative L-S category which contains the classical notions of Berstein-Ganea and Fadell-Husseini. We then show that, for a space or a pair, the categorical length for categorical sequences is precisely the L-S category or the relative L-S category in the sense of Fadell-Husseini respectively. Higher Hopf invariants, cup length, module weights, and recent computations by Kono and the author are also studied within this unified L-S theory based on the categorical length of categorical sequences.

1. Introduction. Throughout this paper, we work in $\mathcal{T}$ the category of topological spaces and maps, or the category of pairs $T^A$ in which an object is a pair $(X:A)$ with an inclusion $i^X : A \hookrightarrow X$ and a morphism is a map of pairs $f : (X:A) \to (Y:A)$ with $i^Y = f \circ i^X$. A closed subset is always assumed to be a neighbourhood deformation retract, and a pair is assumed to be an NDR-pair in the sense of G. Whitehead [29]. The one-point-space is denoted by $\ast$. The (normalised) Lusternik-Schnirelmann category $\text{cat}(X)$, L-S category for short, is introduced in [22] as the least number $m$ such that there is a covering of $X$ by $m+1$ closed subsets $U_j$, $0 \leq j \leq m$, where each $U_j$ is contractible in $X$. By modifying the idea due to R. Fox [8], T. Ganea [9] gives the following definition of a strong version of L-S category for a space $X$: the strong L-S category $\text{Cat}(X)$ is the least number $m$ such that there is a space $Y \simeq X$ with a covering of $Y$ by $m+1$ closed subsets $U_j$, $0 \leq j \leq m$ where each $U_j$ is contractible in itself. By Ganea [9], it is shown that

$$\text{cat}(X) \leq \text{Cat}(X) \leq \text{cat}(X) + 1.$$
Remark 1.1. Fadell and Husseini [7] introduced a notion of relative L-S category as follows: for a pair \((K:A)\), \(\text{cat}^{FH}(K,A)\) is given as the least number \(m\) such that there is a covering of \(K\) by \(m+1\) closed subsets \(V \supset A\) and \(U_j\), \(1 \leq j \leq m\) where \(V\) is compressible relative \(A\) into \(A\) in \(K\) and each \(U_j\) is contractible in \(K\). It is also clear by definition that \(\text{cat}^{FH}(K,*) = \text{cat}(K)\).

By G. Whitehead [29], the definition of L-S category is interpreted in terms of deformation of a diagonal map as the following definition for a space \(X\).

Definition 1.2. The L-S category \(\text{cat}(X)\) of \(X\) is the least number \(m\) such that the \(m+1\) fold diagonal map \(\Delta^{m+1}_X : X \to \prod^{m+1} X\) is compressible into the fat wedge \(T^{m+1} X = \{(x_0, x_1, ..., x_m) \in \prod^{m+1} X \mid \exists i x_i = *\} \subseteq \prod^{m+1} X\).

Similarly to the above, one can give an alternative definition of a relative L-S category in the sense of Fadell and Husseini [7] for a pair \((K:A)\) to fit in with Whitehead’s definition of L-S category.

Definition 1.3. Let \(A \subseteq K\). Then the L-S category \(\text{cat}^{FH}(K,A)\) is the least number \(m \geq 0\) such that the \(m+1\) fold diagonal map \(\Delta^{m+1}_K : K \to \prod^{m+1} K\) is compressible relative \(A\) into the fat wedge \(T^{m+1}(K:A) = A \times \prod^{m} K \cup K \times T^{m} K \subseteq \prod^{m+1} K\) of a pair \((K:A)\).

Remark 1.4. For any map \(f : A \to K\), we may assume that \(f\) is an inclusion up to homotopy, and hence the definition of relative L-S category implies a definition of \(\text{cat}^{FH}(f)\) the L-S category of \(f\) in the sense of Fadell and Husseini.

In the present paper, we alter the Fox’s definition of a categorical sequence to fit in with Whitehead’s definition of L-S category:

Definition 1.5. A categorical sequence for a space \(X\) is a sequence of closed subspaces \(F_0 \subset \cdots \subset F_i \subset \cdots \subset F_m\) such that \(F_m \simeq X\), \(F_0 \simeq *\) in \(X\) and \(\Delta_i : F_i \xrightarrow{\Delta} F_i \times F_i \subset F_m \times F_m\) is compressible into \(F_{i-1} \times F_m \cup F_m \times *\) relative \(F_{i-1}\) for any \(i > 0\), where we identify \(F_{i-1}\) with its diagonal image in \(F_{i-1} \times F_{i-1} \subset F_{i-1} \times F_m \cup F_m \times *\). Let us call the least such \(m \geq 0\) the ‘categorical length’ of \(X\) and denote it by \(\text{catlen}(X)\).

Inspired by the definition of a relative L-S category due to Fadell and Husseini, we introduce a relative version of categorical sequence:

Definition 1.6. A categorical sequence for a pair \((X:A)\) is a sequence of pairs \((F_0:A) \subset \cdots \subset (F_i:A) \subset \cdots \subset (F_m:A)\) such that \((F_m:A) \simeq (X:A)\) relative \(A\), \(F_0 \simeq A\) relative \(A\) in \(X\) and \(\Delta_i : F_i \xrightarrow{\Delta} F_i \times F_i \subset F_m \times F_m\) is compressible into \(F_{i-1} \times F_m \cup F_m \times A\) relative \(F_{i-1}\), \(i > 0\). Let us call the least such \(m \geq 0\) the ‘categorical length’ of a space \(X\) relative to \(A\) and denote it by \(\text{catlen}(X:A)\).

To describe the categorical sequence in terms of a relative L-S category, we give a definition of a new extended version of relative L-S category: from now on, we work in the category \(T^A\). We remark that, if \(A = *\) the one point space, then \(T^A\) is the usual category of based connected spaces and based maps. We say that \((X,K:A)\) is a pair in \(T^A\) when \((X:A)\) and \((K:A)\) are objects in \(T^A\) and \((X,K)\) is a pair in \(T\), that \((X,K,L:A)\) is a triple in \(T^A\) when \((X:A)\), \((K:A)\), \((L:A)\) are objects in \(T^A\) and \((X,K,L)\) is a triple
in $T$, and that $(X;K,L;A)$ is a triad in $T^A$ when $(X:A)$, $(K:A)$, $(L:A)$ are objects in $T^A$ and $(X;K,L)$ is a triad in $T$.

We remark, for any pair $(X,K;A)$ in $T^A$, that the diagonal image of $A$ in $\prod_{m+1} X$ is in the subspace $T^{m+1}(X,L)$. Thus for any $(X:A) \supset (L:A) \in T^A$, we regard $(\prod_{m+1} X:A) \supset (T^{m+1}(X,L):A) \in T^A$.

**Definition 1.7.** Let $(X;K,L;A)$ be a triad in $T^A$. Then $\text{cat}(X;K,L;A)$ is the least number $m$ such that the restriction of the $m+1$ fold diagonal map of $X$ to $K$, $\Delta^{m+1}|_K : K \to \prod_{m+1} X$, is compressible relative $A$ into $T^{m+1}(X,L)$.

Using Harper's arguments on the homotopy of maps to the total space of a fibration in [12], Cornea [4] has given a proof of the following:

**Proposition 1.8.** Let $(X:A)$ be an object in $T^A$, $(Y;K;A)$ be a pair in $T^A$ with the inclusion $j : (K:A) \hookrightarrow (Y:A)$ and $f : (X:A) \to (Y:A)$ be a map in $T^A$. If $f|_X : X \to Y$ has a compression $\sigma : X \to K$ such that $j_*(\sigma) \sim f$ and $\sigma_0|_X \sim j^K$ in $T$, then there is a map $\sigma' : (X:A) \to (K:A)$ a compression relative $A$ of $f$ such that $\sigma \sim \sigma'|_X : X \to K$.

One of its direct consequences is:

**Corollary 1.9.** Let $(X;K,L;A)$ be a triple in $T^A$. Then $\text{cat}(X;K,L;A)$ is the same as the least number $m$ such that $\Delta^{m+1}|_K : K \to \prod_{m+1} X$ is compressible to a map $s : K \to T^{m+1}(X,L)$ such that $s|_A$ is homotopic to the diagonal map $\Delta_A : A \to \prod_{m+1} A \subset T^{m+1}(X,L)$.

**Remark 1.10.** (1) $\text{cat}(X;X,\ast:\ast) = \text{cat}(X)$ and $\text{cat}(X;\ast,\ast:\ast) = 0$.

(2) We denote $(X;X,L;A)$ by $(X,L:A)$, $(X;K,A;A)$ by $(X;K:A)$, $(X;X;A)$ by $(X:A)$, $(X;K,L;\ast)$ by $(X;K,L)$, $(X;K,*$ by $(X;K)$ and $(X:*$ = $(X,X,*:*) = (X;X)$ by $X$.

(3) We may replace inclusions $(L;A) \hookrightarrow (X;A)$ and $(K;A) \hookrightarrow (X;A)$ by maps $f : (L;A) \to (X;A)$ and $g : (K;A) \to (X;A)$ in $T^A$, since every such map is an inclusion map up to homotopy relative $A$ by taking the mapping cylinder of $K \cup_A L \xrightarrow{g \cup_A f} X$. Then we often denote $\text{cat}(X;K,L;A)$ by $\text{cat}(g,f)$. By applying (1), we have $\text{cat}(g,*) = \text{cat}(g)$.

There is another classical notion of relative L-S category due to Berstein and Ganea [2].

**Definition 1.11.** Let $K \subset X$. Then the L-S category $\text{cat}^{BG}(X,K)$ is the least number $m \geq 0$ such that restriction to $K$ of the $m+1$ fold diagonal map $\Delta_{[X]}^{m+1} : X \to \prod_{m+1} X$ is compressible into the fat wedge $T^{m+1} X$.

**Remark 1.12.** For any map $f : K \to X$, we may assume that $f$ is an inclusion up to homotopy, and hence the above definition of the L-S category implies a definition of $\text{cat}^{BG}(f)$ the L-S category of $f$ in the sense of Berstein and Ganea.

Arkowitz and Lupton [1] have also defined their relative L-S category for a map $h : X \to Y$. Since a map is up to homotopy a fibration, we may assume that $h$ is a fibration with fibre $L = h^{-1}(\ast) \subset X$. Then the relative L-S category of $h$ in the sense of Arkowitz and Lupton depends only on the pair $(X,L)$ by its definition.
Definition 1.13. Let $L \subset X$. Then the L-S category $\text{cat}^{AL}(X, L)$ is the least number $m \geq 0$ such that the $m+1$ fold diagonal map $\Delta_{X}^{m+1}: X \to \prod_{m+1}^{X} X$ is compressible into the fat wedge $T_{m+1}^{X}(X, L)$.

Then we prove:

Theorem 1.14. The known three relative L-S categories are special cases of our new relative L-S category:

1. Let $X = K \supset L = A \supset \ast$. Then $\text{cat}(X;A) = \text{cat}(X; X, A; A) = \text{cat}^{FH}(X, A)$.
2. Let $X \supset K \supset L = A = \ast$. Then $\text{cat}(X; K) = \text{cat}(X; K, \ast ; \ast) = \text{cat}^{BG}(X, K)$. More generally, for a map $g : K \to X$ in $T_{*}$, we have $\text{cat}(g, \ast) = \text{cat}^{BG}(g)$.
3. Let $K = X \supset L \supset A = \ast$. Then $\text{cat}(X, L) = \text{cat}(X; X, L; \ast) = \text{cat}^{AL}(X, L)$.

We also introduce a new higher Hopf invariant: let $(X; K, L; A)$ be a triad in $T^{A}$, let $V$ be a co-loop co-H-space, i.e., a one-point-union of a $1$-connected co-H-space with finitely-many circles, and let $\alpha : V \to K$ be a map in $T$ such that $X \supset \tilde{K} = K \cup_{a} CV \supset K$. If $\text{cat}(X; K, L; A) \leq m$, then a relative higher Hopf invariant $H_{m}^{(X; K, L; A)}(\alpha)$ is defined as a subset of $[V, \Omega(X, L) \ast \Omega(X) \ast \cdots \ast \Omega(X)]$. If $K \supset L$ and $\text{cat}(K; K, L; A) \leq m$, then an absolute higher Hopf invariant $H_{m}^{(K, L; A)}(\alpha)$ is defined as a subset of $[V, \Omega(K, L) \ast \Omega(K) \ast \cdots \ast \Omega(K)]$ (see §4 for more details). The following result clarifies how a higher Hopf invariant determines whether a cone decomposition reduces to a categorical sequence or not.

Theorem 1.15. Let $(X; K, L; A)$ be a triad in $T^{A}$, let $V$ be a co-loop co-H-space and let $\alpha : V \to K$ be a map in $T$ such that $X \supset \tilde{K} = K \cup_{a} CV \supset K$. If $\text{cat}(X; K, L; A) \leq m$ and $H_{m}^{(X; K, L; A)}(\alpha) = 0$, then $\text{cat}(X; \tilde{K}, L; A) \leq m$.

From now on, we abbreviate $H_{m}^{(X; K, A; A)}(\alpha)$ by $H_{m}^{(X; K; A)}(\alpha)$, $H_{m}^{(X; K, A; \ast)}(\alpha)$ by $H_{m}^{(X; K)}(\alpha)$, $H_{m}^{(K, A; A)}(\alpha)$ by $H_{m}^{(K, A)}(\alpha)$ and $H_{m}^{(K, A; \ast)}(\alpha)$ by $H_{m}^{K}(\alpha)$. Note that the definition of the absolute higher Hopf invariant $H_{m}^{K}(\alpha)$ coincides with the ordinary definition of the higher Hopf invariant $H_{m}(\alpha)$ in the sense of [14].

The main goal of this paper is to prove:

Theorem 1.16. For any $X$ in $T$, we have $\text{cat}(X) = \text{catlen}(X)$. More generally, for any object $(X; A) \in T^{A}$, we have $\text{catlen}(X; A) = \text{cat}(X; A) = \text{cat}^{FH}(X, A)$.

Corollary 1.17. Let $(X; A)$ be an object in $T^{A}$. If $\text{cat}^{FH}(X, A) = m > 0$, then there exists a sequence of pairs $\{(F_{i}; A) ; 0 \leq i \leq m\}$ such that $(F_{0}; A) \simeq (A; A)$ in $(F_{m}; A)$, $(F_{m}; A) \simeq (X; A)$ relative $A$ and $\text{cat}(X; F_{i}; A) \leq i$, $i > 0$. Moreover, $\text{cat}(F_{m}; F_{i-1}; F_{i}/F_{i-1}) \leq 1$ with a partial co-action $F_{i} \to F_{m}/F_{i-1} \vee F_{m}$ along the collapsing map $F_{i} \to F_{i}/F_{i-1} \subseteq F_{m}/F_{i-1}$, $i > 0$. In particular, $F_{m}/F_{m-1}$ is a co-H-space co-acting on $F_{m}$ along the collapsing map $F_{m} \to F_{m}/F_{m-1}$.

2. $A_{\infty}$-decomposition of a map. In [9], Ganea introduced a so-called ‘fibre-cofibre’ construction for a map, which can be interpreted as the pullback construction from the view-point of Definition 1.3. We may regard this construction as an $A_{\infty}$-decomposition of a map using the pushout-pullback diagram (see [13, Lemma 2.1] and also Sakai [24] for the detailed proof in a general context):
Let us recall that, in $\mathcal{T}$, the homotopy fibre of $T^m_i(X, A_i) \hookrightarrow \prod^{m+1} X$ has the homotopy type of the join $\Omega(X, A_0) \ast \cdots \ast \Omega(X, A_m)$. Let $(X; K, L; A)$ be a triad in $\mathcal{T}^A$ and write $E^m(\Omega(X)) = \Omega(X) \ast \cdots \ast \Omega(X)$ which has the homotopy type of the homotopy fibre of $T^m(X, *) \hookrightarrow \prod^{m+1} X$. The homotopy fibre of the inclusion $T^m(X, *) \hookrightarrow \prod^{m+1} X$ has the homotopy type of $E^m(\Omega(X)) = \Omega(X) \ast \Omega(X) \ast \cdots \ast \Omega(X)$: consider the homotopy pushout-pullback diagram in $\mathcal{T}$, which is given by [13, Lemma 2.1] with $(Y, B) = (\prod^m X, T^m X)$, $Z = \ast$ and $f = g = \ast$. Thus we see that the homotopy fibre of the inclusion $T^{m+1}(X, L) \hookrightarrow \prod^{m+1} X$ has the homotopy type of $\Omega(X, L) \ast E^m(\Omega(X)) = E^{m+1}(\Omega(X, L))$ by induction.

Similarly, we define $P^m(\Omega(X, L))$ inductively from $P^0(\Omega(X, L)) = L$ as the homotopy pushout in the following homotopy pushout-pullback diagram which is given by [13, Lemma 2.1] with $(Y, B) = (\prod^m X, T^m X)$, $Z = X$ and $(f, g) = (1_X, \Delta^m_X)$:

$$E^m(\Omega(X, L)) \xrightarrow{p_{m-1}(X, L)} P^{m-1}(\Omega(X, L)) \xrightarrow{p_m(X, L)} P^m(\Omega(X, L)) \xrightarrow{\eta_m(X, L)} T^{m+1}(X, L) \xrightarrow{\Delta^{m+1}} \prod^{m+1} X,$$

where $\eta_m(X, L)$ covers the diagonal map $\Delta^{m+1} : X \to \prod^{m+1} X$. Then we define $p^{\Omega(X, L)}_{m+1} : E^{m+1}(\Omega(X, L)) \to P^m(\Omega(X, L))$ as the homotopy fibre of $\epsilon_m(X, L) : P^m(\Omega(X, L)) \to X$ given in the diagram, where $\epsilon_0(X, L) : L \hookrightarrow X$ is just the canonical inclusion. These constructions due to Ganea [9] yield the following ladder of fibrations which have the same fibre $\Omega(X)$, giving a generalisation of an $A_\infty$-structure (see Stasheff [25]):

$$\Omega(X, L) \xleftarrow{\Omega(X, L)} \cdots \xleftarrow{\epsilon_m(X, L)} E^{m+1}(\Omega(X, L)) \xleftarrow{\epsilon_m(X, L)} \cdots \xleftarrow{\epsilon_m(X, L)} E^\infty(\Omega(X, L))$$

$$\xrightarrow{p_m(X, L)} \cdots \xrightarrow{p_m(X, L)} P^m(\Omega(X, L)) \xrightarrow{p_m(X, L)} \cdots \xrightarrow{p_m(X, L)} P^\infty(\Omega(X, L))$$

(2.3)
together with \( e_\infty^{(X,L)} : P^\infty(\Omega(X,L)) = \bigcup_m P^m(\Omega(X,L)) \rightarrow X \) given by \( e_\infty^{(X,L)}|_{P^m(\Omega(X,L))} = e_m^{(X,L)} \) with fibre \( E^\infty(\Omega(X,L)) \), where the upper horizontal arrows are null-homotopic. Since \( E^\infty(\Omega(X,L)) = \bigcup_m E^m(\Omega(X,L)) \) is weakly contractible, \( e_\infty^{(X,L)} : P^\infty(\Omega(X,L)) = \bigcup_m P^m(\Omega(X,L)) \rightarrow X \) is a weak equivalence. If further \( X \) is a CW complex, then there is a right homotopy inverse \( h^{(X,L)} : X \rightarrow P^\infty(\Omega(X,L)) \) of \( e_\infty^{(X,L)} \), where \( h^{(X,L)} \) is also a weak equivalence.

The ladder (2.3) is natural with respect to a map of triads in \( T^A \):

**Lemma 2.1.** For any map \( f : (X;K,L:A) \rightarrow (X';K',L':A) \) of triads in \( T^A \), there is the following commutative diagram with \( f|_{(X,L)} : (X,L) \rightarrow (X',L') \) and \( f|_L : L \rightarrow L' \) the restrictions of \( f \).

\[
\begin{array}{ccc}
E^m(\Omega(X,L)) & \xrightarrow{p_1^{\Omega(X,L)}} & E^m(\Omega(f|_{(X,L)})) \\
\downarrow{E^m(\Omega(f|_{(X,L)}))} & & \downarrow{E^m(\Omega(f|_{(X,L)}))} \\
P^m(\Omega(X,L)) & \xleftarrow{P^m(\Omega(X,L))} & P^m(\Omega(X,L)) \\
\downarrow{p^{\Omega(X',L')}} & & \downarrow{p^{\Omega(X',L')}} \\
P^{m-1}(\Omega(X',L')) & \xleftarrow{P^{m-1}(\Omega(f|_{(X,L)}))} & P^{m-1}(\Omega(X',L')).
\end{array}
\]

We give here another kind of naturality of the ladder (2.3) in \( T^A \) induced from the structure map \( \sigma : K \rightarrow P^m(\Omega(X,L)) \) of \( \text{cat}(X;K,L>A) \leq m \).

**Lemma 2.2.** For any triad \( (X;K,L:A) \) in \( T^A \) with a compression \( \sigma : K \rightarrow P^m(\Omega(X,L)) \) relative \( A \) of the inclusion \( K \hookrightarrow X \), there is a sequence of maps \( \sigma_n : P^n(\Omega(X,K)) \rightarrow P^{m+n}(\Omega(X,L)) \) \((n \geq 0)\) with \( \sigma_0 = \sigma \), which makes the following diagram commutative up to homotopy relative \( A \).

\[
\begin{array}{ccc}
P^{n-1}(\Omega(X,K)) & \xleftarrow{\sigma_{n-1}} & P^n(\Omega(X,K)) \\
\downarrow{\sigma_n} & & \downarrow{id_X} \\
P^{m+n-1}(\Omega(X,L)) & \xleftarrow{\sigma_n} & P^{m+n}(\Omega(X,L)) \\
\downarrow{\epsilon^{(X,L)}} & & \downarrow{\epsilon^{(X,L)}} \\
X & & X.
\end{array}
\]

**Proof.** We construct \( \sigma_n \) inductively on \( n \geq 1 \): the homotopy commutativity relative \( A \) of (2.5) without the dotted arrow induces a map of fibres in \( T \), \( \hat{\sigma}_n : E^n(\Omega(X,K)) \rightarrow E^{m+n}(\Omega(X,L)) \).
Proposition 3.3 to cat

A standard argument shows that the homotopy commutativity of the left square implies the existence of \( \sigma_n : P^n(\Omega(X, L)) \to P^{m+n}(\Omega(X, L)) \) which makes (2.4) commutative up to homotopy relative \( A \).

3. Properties of a new relative \( \text{L-S} \) category. Here we prove Theorem 1.14 and some consequences. For that we need

Lemma 3.1. cat\((X; K, L; A) \leq m\) if and only if the inclusion \( g : K \hookrightarrow X \) is compressible into \( P^m(\Omega(X, L)) \subset P^\infty(\Omega(X, L)) \cong X \) relative \( A \) as \( \sigma : K \to P^m(\Omega(X, L)) \) the structure map for cat\((X; K, L; A) \leq m\).

Proof. Let us assume that cat\((X; K, L; A) \leq m\). Then by the definition of the relative category, the diagonal map \( \Delta^{m+1}|_K : K \hookrightarrow X \to \prod^{m+1} X \) is compressible relative \( A \) into \( T^{m+1}(X, L) \). This implies that there exists a map \( \sigma \) from \( K \) to \( P^m(\Omega(X, L)) \), which is a compression relative \( A \) of the inclusion \( g : K \hookrightarrow X \). Conversely, we assume that there is a compression relative \( A \) of the inclusion \( g : K \hookrightarrow X \) into \( P^m(\Omega(X, L)) \). Composing with \( q_m : P^m(\Omega(X, L)) \to T^{m+1}(X, L) \), we obtain a compression relative \( A \) of the diagonal map \( \Delta^{m+1}|_K : K \hookrightarrow X \to \prod^{m+1} X \) into \( T^{m+1}(X, L) \). The following propositions complete the proof of Theorem 1.14.

Proposition 3.2. Assume \( X = K \supset L = A \supset \ast \). Then cat\((X; A) = \text{cat}(X; X, A; A) = \text{cat}^{\text{FH}}(X, A)\).

Proof. By Lemma 3.1 with \( X = K \) and \( L = A \), cat\((X; X, A; A) \leq m\) if and only if there is a right homotopy inverse of \( e_m^{(X; X; A)} : P^m(\Omega(X; A)) \to X \) relative \( A \), which is equivalent to cat\(^{\text{FH}}(X, K) \leq m\).

Proposition 3.3. Assume \( X \supset K \supset L = A = \ast \). Then cat\((X; K) = \text{cat}(X; K, \ast; \ast) = \text{cat}^{\text{BG}}(X, K)\).

Proof. By Lemma 3.1 with \( A = \ast \), cat\((X; K) \leq m\) if and only if the inclusion \( K \hookrightarrow X \) is compressible into \( P^m(\Omega(X)) \), which is equivalent to cat\(^{\text{BG}}(X, K) \leq m\).

Proposition 3.4. Assume \( X = K \supset L \supset A = \ast \). Then cat\((X, L) = \text{cat}(X; X, L; \ast) = \text{cat}^{\text{AL}}(X, L)\).

Proof. By Lemma 3.1 with \( X = K \) and \( A = \ast \), cat\((X, L) = \text{cat}(X; X, L; \ast) \leq m\) if and only if there is a right homotopy inverse of \( e_m^{(X; X, L)} : P^m(\Omega(X, L)) \to X \), which is equivalent to cat\(^{\text{AL}}(X, L) \leq m\).

For relative L-S categories, one has:
**Theorem 3.5.** (1) Let \((X; K, L; A)\) be a triad in \(\mathcal{T}^A\). Then
\[
\text{cat}(X; K, L; A) \leq \text{cat}(X; K; A) \leq \text{cat}(X; L; A) + \text{cat}(X; K, L; A),
\]
\[
\text{cat}(X; K, L; A) \leq \text{cat}(X, L; A) \leq \text{cat}(X, K; A) + \text{cat}(X; K, L; A).
\]

More generally, for any maps \(f : (L; A) \to (X; A)\) and \(g : (K; A) \to (X; A)\),
\[
\text{cat}(g, f) \leq \text{cat}(g, *_A) \leq \text{cat}(f, *_A) + \text{cat}(g, f),
\]
\[
\text{cat}(g, f) \leq \text{cat}(1_{(X; A)}, f) \leq \text{cat}(1_X, g) + \text{cat}(g, f),
\]
where \(1_X : (X; A) = (X; A)\) denotes the identity and \(*_A : (A; A) \to (X; A)\) denotes
the trivial inclusion.

(2) If \((X', L'; A) \supset (X, L; A)\) and \((K'; A') \subset (K; A)\), then
\[
\text{cat}(X'; K', L'; A') \leq \text{Min}\{\text{cat}(X'; K, L'; A), \text{cat}(X; K', L'; A')\}
\]
\[
\leq \text{Max}\{\text{cat}(X'; K, L'; A), \text{cat}(X; K', L'; A')\} \leq \text{cat}(X; K, L; A).
\]

More generally, for any maps \(f' : (L'; A) \to (X'; A)\), \(f : (L; A) \to (X; A)\), \(g : (K; A) \to (X; A)\), \(h : (X; A) \to (X'; A)\), \(k : (K'; A') \to (K; A)\) and \(\ell : (L; A) \to (L'; A)\)
which satisfy the relation \(f' \circ \ell = h \circ f\), we have
\[
\text{cat}(h \circ g \circ k, f') \leq \text{Min}\{\text{cat}(h \circ g, f'), \text{cat}(g \circ k, f)\}
\]
\[
\leq \text{Max}\{\text{cat}(h \circ g, f'), \text{cat}(g \circ k, f)\} \leq \text{cat}(g, f).
\]

The following corollaries are immediate consequences of Theorem 3.5:

**Corollary 3.6.** (1) For a triad \((X; K, L; * )\) in \(\mathcal{T}_*\), we have
\[
\text{cat}(X; K, L) \leq \text{cat}(X; K) = \text{cat}^\text{BG}(X; K)
\]
\[
\leq \text{cat}(X; L) + \text{cat}(X; K, L) = \text{cat}^\text{BG}(X, L) + \text{cat}(X; K, L),
\]
\[
\text{cat}(X; K, L) \leq \text{cat}(X, L) = \text{cat}^\text{AL}(X, L)
\]
\[
\leq \text{cat}(X, K) + \text{cat}(X; K, L) = \text{cat}^\text{AL}(X, K) + \text{cat}(X; K, L).
\]

(2) For a pair \((X, L; A)\) in \(\mathcal{T}^A\), we have
\[
\text{cat}(X, L; A) \leq \text{cat}(X; A) = \text{cat}^\text{FH}(X, A)
\]
\[
\leq \text{cat}(X; L; A) + \text{cat}(X, L; A) \leq \text{cat}(X; L; A) + \text{cat}^\text{FH}(X, L).
\]

If we further assume that \(A = *\), then
\[
\text{cat}(X; L) \leq \text{cat}(X) \leq \text{cat}(X; L) + \text{cat}(X, L).
\]

(3) For maps \(f : L \subset X, f' : * \subset Y, g = 1_X : X \to X, h : X \to Y, k = 1_X : X \to X\)
and \(\ell : L \to *\) in \(\mathcal{T}_*\) with \(h|_L = \ell\), we have
\[
\text{cat}^\text{BG}(h) = \text{cat}(h, *) = \text{cat}(h \circ g, f') \leq \text{cat}(g, f) = \text{cat}^\text{AL}(X, L).
\]

In Definition 1.6, we have \(\text{cat}(X; F_i, F_{i-1}; A) \leq 1\) for the filtration \(\{F_i\}\). Hence we have:

**Corollary 3.7.** \(\text{cat}(X; F_i, A; A) \leq i\) for every \(i\).
Proof of Theorem 3.5. The case of maps is left to the reader, and we concentrate on the case of spaces.

Firstly, we show (1) for a triad \((X; K, L:A)\) in \(T^A:\)

To show \(\text{cat}(X; K, L:A) \leq \text{cat}(X; K:A)\), we assume that \(\text{cat}(X; K:A) = m\). By Lemma 3.1 for the triad \((X; K, A:A)\), \(\text{cat}(X; K:A) = \text{cat}(X; K, A:A) \leq m\) if and only if there is a compression \(\sigma : X \rightarrow P^m(\Omega(X:A))\) relative \(A\) of the inclusion \(K \hookrightarrow X\). By Lemma 2.1 for the inclusion \((X; K, A:A) \hookrightarrow (X; K, L:A)\), the composition \(P^m(\Omega(f|_{X:A})) \circ \sigma : K \rightarrow P^m(\Omega(X, L))\) gives a compression of the inclusion \(K \hookrightarrow X\), which implies \(\text{cat}(X; K, L:A) \leq m = \text{cat}(X; K:A)\).

To show \(\text{cat}(X; K, L:A) \leq \text{cat}(X, L:A)\), we assume that \(\text{cat}(X, L:A) = m\). By Lemma 3.1 for the triad \((X, L:A)\), \(\text{cat}(X, L:A) \leq m\) if and only if there is a compression \(\sigma : X \rightarrow P^m(\Omega(X, L))\) relative \(A\) of the identity \(1_X\). By restricting \(\sigma\) to \(K\), we obtain a compression \(\sigma|_K : K \rightarrow P^m(\Omega(X, L))\) relative \(A\) of the inclusion \(K \hookrightarrow X\), which implies \(\text{cat}(X; K, L:A) \leq m = \text{cat}(X, L:A)\).

To show the inequality \(\text{cat}(X; K, L:A) \leq \text{cat}(X; L:A) + \text{cat}(X; K, L:A)\), we assume that \(\text{cat}(X; L:A) = m\) and \(\text{cat}(X; K, L:A) = n\). By Lemma 3.1 for the triad \((X; L, A:A)\), \(\text{cat}(X; L:A) \leq m\) if and only if there is a compression \(\sigma : L \rightarrow P^m(\Omega(X:A))\) relative \(A\) of the inclusion \(L \hookrightarrow X\). Then by Lemma 2.2 for the triad \((X; L, A:A)\), we have the following commutative ladder with \(\sigma_0 = \sigma\) up to homotopy relative \(A\):

\[
P^{n-1}(\Omega(X, L)) \xrightarrow{\sigma_{n-1}} P^n(\Omega(X, L)) \xrightarrow{e^{(X, L)}_n} X
\]

Again by Lemma 3.1 for the triad \((X; K, L:L)\), \(\text{cat}(X; K, L:L) \leq n\) if and only if there is a compression \(\tau : K \rightarrow P^n(\Omega(X, L))\) relative \(A\) of the inclusion \(K \hookrightarrow X\). Then the composition \(\sigma_n \circ \tau : K \rightarrow P^{m+n}(\Omega(X:A))\) gives a compression relative \(A\) of the inclusion \(K \hookrightarrow X\), which implies that \(\text{cat}(X; K, L:A) \leq m + n = \text{cat}(X; L:A) + \text{cat}(X; K, L:A)\).

To show the inequality \(\text{cat}(X; L:A) \leq \text{cat}(X; K, A:A) + \text{cat}(X; K, L:A)\), we assume that \(\text{cat}(X; K, A:A) = m\) and \(\text{cat}(X; K, L:A) = n\). By Lemma 3.1 for the triad \((X; K, L:A)\), \(\text{cat}(X; L:A) \leq m\) if and only if there is a compression \(\tau : X \rightarrow P^m(\Omega(X, L))\) relative \(A\) of the inclusion \(K \hookrightarrow X\). Then by Lemma 2.2 for the triad \((X; K, L:A)\), we have the following commutative ladder with \(\tau_0 = \tau\) up to homotopy relative \(A\):

\[
P^{n-1}(\Omega(X, K)) \xrightarrow{\tau_{n-1}} P^n(\Omega(X, K)) \xrightarrow{e^{(X, K)}_n} X
\]

Again by Lemma 3.1 for the triad \((X; K, A:A)\), \(\text{cat}(X; K, A:A) \leq n\) if and only if there is a compression \(\rho : X \rightarrow P^n(\Omega(X, K))\) relative \(A\) of the identity \(1_X : X \rightarrow X\). Then the composition \(\tau_{n} \circ \rho : X \rightarrow P^{m+n}(\Omega(X, L))\) gives a compression relative \(A\) of the identity \(1_X : X \rightarrow X\), which implies that \(\text{cat}(X, L:A) \leq m + n = \text{cat}(X; K, A:A) + \text{cat}(X; K, L:A)\).
Secondly, we show (2) for a triad \((X; K, L; A)\) with spaces \(X' \supset X\), \((K'; A') \subset (K; A)\) and \((L'; A') \subset (L; A)\), which is sufficient to show that \(\text{cat}(X'; K', L'; A') \leq \text{cat}(X; K, L; A)\) and \(\text{cat}(X; K', L'; A') \leq \text{cat}(X; K, L; A)\): 

To show \(\text{cat}(X'; K', L'; A') \leq \text{cat}(X; K, L; A)\), we assume that \(\text{cat}(X; K, L; A) = m\). By Lemma 3.1 for the triad \((X; K, L; A)\), \(\text{cat}(X; K, L; A) = m\) if and only if there is a compression \(\sigma : K \to P^m(\Omega(X, L))\) relative \(A\) of the inclusion \(K \hookrightarrow X\). Since \(X' \supset X\), we have the inclusion of triads \((X; K, L; A) \hookrightarrow (X'; K', L'; A')\). Then by Lemma 2.1 for the map of triads \(j : (X; K, L; A) \hookrightarrow (X'; K', L'; A')\), we have the following commutative diagram:

\[
\begin{array}{ccc}
P^{m-1}(\Omega(X, L)) & \xrightarrow{\epsilon_{(X,L)}^{(m)}} & P^m(\Omega(X, L)) \xrightarrow{j_m} X \\
\downarrow j_{m-1} & & \downarrow j_m \\
P^{m-1}(\Omega(X', L')) & \xrightarrow{\epsilon_{(X',L')}^{(m)}} & P^m(\Omega(X', L')) \xrightarrow{j_{m}|_X} X'
\end{array}
\]

with \(j_0 = \text{id}_L\) and \(j_k = P^k(\Omega(j|_{X,L}))\), \(1 \leq k \leq m\). Thus the map \(j_m \circ \sigma\) gives a compression relative \(A\) of the inclusion \(K \hookrightarrow X \subset X'\), and hence \(\text{cat}(X'; K', L'; A') \leq m = \text{cat}(X; K, L; A)\).

To show \(\text{cat}(X; K', L; A') \leq \text{cat}(X; K, L; A)\), we may assume that \(A = A'\), since it is clear by definition that \(\text{cat}(X; K, L; A') \leq \text{cat}(X; K, L; A)\) if \(A' \subset A\): let us assume that \(\text{cat}(X; K, L; A) = m\) if and only if there is a compression \(\sigma : K \to P^m(\Omega(X, L))\) relative \(A\) of the inclusion \(K \hookrightarrow X\). Hence the restriction \(\sigma|_{K'}\) of the map \(\sigma\) to \(K'\) gives a compression relative \(A\) of the inclusion \(K' \hookrightarrow X\), and hence \(\text{cat}(X; K', L; A') \leq m = \text{cat}(X; K, L; A)\).

4. A higher Hopf invariant for a triad. Let us consider the following exact sequences of abelian groups and algebraic loops:

\[
0 \to [\Sigma V, E^{m+1}(\Omega(X, L))] \xrightarrow{p^{(X,L)}_{m+1}} [\Sigma V, P^m(\Omega(X, L))] \xrightarrow{\epsilon_{m}^{(X,L)}} [\Sigma V, X] \to 0, \quad (4.1)
\]

\[
1 \to [V, E^{m+1}(\Omega(X, L))] \xrightarrow{p^{(X,L)}_{m+1}*} [V, P^m(\Omega(X, L))] \xrightarrow{\epsilon_{m}^{(X,L)}} [V, X]. \quad (4.2)
\]

Since the fibre \(\Omega(X)\) of a fibration \(p^{(X,L)}_{m+1}\) is contractible in the total space \(E^{m+1}(\Omega(X, L))\) of \(p^{(X,L)}_{m+1}\), we know \(\epsilon_{m}^{(X,L)} : [\Sigma V, P^m(\Omega(X, L))] \to [\Sigma V, X]\) is an epimorphism of abelian groups and \(p^{(X,L)}_{m+1*} : [\Sigma V, E^{m+1}(\Omega(X, L))] \to [\Sigma V, P^m(\Omega(X, L))]\) is a monomorphism of abelian groups. Similarly, \(p^{(X,L)}_{m+1*} : [V, E^{m+1}(\Omega(X, L))] \to [V, P^m(\Omega(X, L))]\) is a monomorphism of algebraic loops. Thus we obtain the following proposition:

**Proposition 4.1.** (1) \(\epsilon_{m}^{(X,L)} : [\Sigma V, P^m(\Omega(X, L))] \to [\Sigma V, X]\) is an epimorphism of abelian groups.

(2) \(p_{m+1}^{(X,L)} : [\Sigma V, E^{m+1}(\Omega(X, L))] \to [\Sigma V, P^m(\Omega(X, L))]\) is a monomorphism of abelian groups.

(3) \(p_{m+1}^{(X,L)} : [V, E^{m+1}(\Omega(X, L))] \to [V, P^m(\Omega(X, L))]\) is a monomorphism of algebraic loops.
We give here a definition of higher Hopf invariants in a slightly different form: let $(X; K, L; A)$ be a triad in $T^A$, let $V$ be a co-loop co-H-space, and let $\alpha : V \to K$ be a map in $T$ such that $X \supset \hat{K} = K \cup_{\alpha} CV \supset K$. We assume that $\text{cat}(X; K, L; A) \leq m$. Then by Lemma 3.1 for the triad $(X; K, L; A)$, $\text{cat}(X; K, L; A) \leq m$ implies that the inclusion $i : K \hookrightarrow X$ is compressible into $P^m(\Omega(X, L))$ relative $A$ as a map $\sigma : K \to P^m(\Omega(X, L))$. Since $e_m^{(X, L)} \circ \sigma \alpha \sim i \circ \alpha$ is trivial in $\hat{K} \subset X$, we obtain a unique lift $H_m^\sigma(\alpha) : V \to E^{m+1}(\Omega(X, L)) \simeq \Omega(X, L) \ast \Omega(X) \ast \cdots \ast \Omega(X)$ of $\sigma \circ \alpha$.

**Definition 4.2.** We define $H_m^{(X; K, L; A)}(\alpha)$ as follows:

$$H_m^{(X; K, L; A)}(\alpha) = \left\{ [H_m^\sigma(\alpha)] \mid \sigma : K \to P^m(\Omega(X, L)) \text{ is a compression relative } A \text{ of the inclusion } K \hookrightarrow X \right\}$$

$$\subset \{ V, \Omega(X, L) \ast \Omega(X) \ast \cdots \ast \Omega(X) \}.$$

Now let $(K, L; A)$ be a pair in $T^A$ and let $\alpha : V \to K$ a map in $T$. We assume that $\text{cat}(K, L; A) \leq m$. By Lemma 3.1 for the triad $(K; K, L; A)$, $\text{cat}(K, L; A) \leq m$ implies that the identity $1_K : K \to K$ is compressible into $P^m(\Omega(K, L))$ relative $A$ as a map $\sigma : K \to P^m(\Omega(K, L))$. By Lemma 2.1 for the inclusion $j : (K; K, *:* \hookrightarrow (K; K, L; *)$, the following ladder is commutative up to homotopy:

$$\xymatrix{ \ast \ar[r] \ar[d] & \Sigma \Omega(K) \ar[r] \ar[d] & P^m(\Omega(K)) \ar[r] \ar[d] & \Sigma \Omega(K) \ar[d] & K \ar[l] \ar[d] \\ \Sigma \Omega(K) \ar[r] \ar[d] & L \ar[r] \ar[d] & \Sigma \Omega(K) \ar[r] \ar[d] & \Sigma \Omega(K) \ar[r] \ar[d] & \Sigma \Omega(K) \ar[r] \ar[d] & K, }
$$

where $e_1^K = e_m^K|_{\Sigma \Omega(K)} : \Sigma \Omega(K) \to K$ is given by the evaluation map (see Ganea [9] or [14]). Since $V$ is a co-loop co-H-space, the evaluation map $e_1^V : \Sigma \Omega(V) \to V$ admits a right homotopy inverse, say the co-H-structure map $P^V : V \to \Sigma \Omega(V)$ for $V$, by Ganea [10]. Then we have $e_1^K \circ \Sigma \Omega(\alpha) \circ P^V \sim \alpha \circ e_1^V \circ P^V \sim \alpha$, and hence $e_1^{(K, L)} \circ j_1 \circ \Sigma \Omega(\alpha) \circ P^V \sim \text{id}_K \circ e_1^K \circ \Sigma \Omega(\alpha) \circ P^V \sim \alpha$. Since both maps $e_1^{(K, L)} \circ \Sigma \Omega(\alpha) \circ P^V$ are homotopic to $\alpha$, the difference $d(\alpha) = e_1^K \circ \Sigma \Omega(\alpha) - j_1 \circ \Sigma \Omega(\alpha) \circ P^V$ is trivial in $K$. Thus we obtain a unique lift $H_m^\sigma(\alpha) : V \to E^{m+1}(\Omega(K, L)) \simeq \Omega(X, L) \ast \Omega(X) \ast \cdots \ast \Omega(X)$ of $d(\alpha)$.

**Definition 4.3.** We define $H_m^{(K, L; A)}(\alpha)$ as follows:

$$H_m^{(K, L; A)}(\alpha) = \left\{ [H_m^\sigma(\alpha)] \mid \sigma \text{ is a compression relative } A \text{ of } 1_K \right\}$$

$$\subset \{ V, \Omega(K, L) \ast \Omega(K) \ast \cdots \ast \Omega(K) \}.$$

Proof of Theorem 1.15. Let $(X; K, L; A)$ be a triad in $T^A$, $V$ be a co-loop co-H-space and $\alpha : V \to K$ a map in $T$ such that $X \supset \hat{K} = K \cup_{\alpha} CV \supset K$. Assuming $\text{cat}(X; K, L; A) \leq m$ and $H_m^{(X; K, L; A)}(\alpha) = 0$, we show cat$(X; \hat{K}, L; A) \leq m$ by the assumption, there is a compression $\sigma : K \to P^m(\Omega(X, L))$ relative $A$ of the inclusion $K \hookrightarrow X$ such that $\sigma \circ \alpha \sim p_{m+1}^{(X, L)} \circ H_m(\alpha) \sim \ast$, and hence there is a map $\hat{\sigma} : \hat{K} \to P^m(\Omega(X, L))$ whose restriction to $K$ is $\sigma$. Since $e_m^{(X, L)} \circ \sigma$ and the inclusion $K \hookrightarrow X$ are homotopic relative $A$, the difference between $e_m^{(X, L)} \circ \hat{\sigma}$ and the inclusion $\hat{K} \hookrightarrow X$ is given by an element $[\delta] \in [TV, X]$. By Proposition 4.1 (1), we have a map $\hat{\delta} : SV \to P^m(\Omega(X, L))$ such that $e_m^{(X, L)} \circ \hat{\delta} \sim \delta$. By subtracting $\hat{\delta}$ from $\hat{\sigma}$, we obtain a genuine compression
\[ \sigma' = \sigma - \delta : \Sigma V \to P^m(\Omega(X, L)) \] of the inclusion \( \hat{K} \to P^m(\Omega(X, L)) \) relative \( A \), where the subtraction is given by the co-action of \( \Sigma V \) under \( K \cup_{\alpha} \mathcal{C}^2V = \hat{K} \) the mapping cone of \( \alpha \). This implies that \( \text{cat}(X; \hat{K}, L; A) \leq m \).

We describe here the relationship among higher Hopf invariants. The following definition is essentially due to Berstein and Hilton [3]:

**Definition 4.4.** Let \((X; K, L; A)\) and \((X'; K', L'; A)\) be triads in \( T^A \), \( V \) be a co-loop co-\( H \)-space, and \( s : K \to T^{m+1}(X, L) \) and \( s' : K' \to T^{m+1}(X', L') \) be compressions of \( \Delta^{m+1} \circ i : K \hookrightarrow \prod^{m+1} X \) and \( \Delta^{m+1} \circ i' : K' \hookrightarrow \prod^{m+1} X' \) relative \( A \), respectively, so that \( \text{cat}(X; K, L; A) \leq m \) and \( \text{cat}(X'; K', L'; A) \leq m \). A map \( f : (X; K, L; A) \to (X'; K', L'; A) \) of triads in \( T^A \) is called \( m \)-primitive (with respect to \( s \) and \( s' \)), if \( s' \circ f|_K \sim T^{m+1}(f|_{(X, L)}) \circ s \).

Let \((X; K, L; A)\) and \((X'; K', L'; A)\) be triads in \( T^A \), and let \( \text{cat}(X; K, L; A) \leq m \) and \( \text{cat}(X'; K', L'; A) \leq m \) with compressions \( s : K \to T^{m+1}(X, L) \) and \( s' : K' \to T^{m+1}(X', L') \) of \( \Delta^{m+1} \circ i : K \hookrightarrow \prod^{m+1} X \) and \( \Delta^{m+1} \circ i' : K' \hookrightarrow \prod^{m+1} X' \) relative \( A \), respectively. By using the lower right square of the diagram (2.2), we obtain structure maps \( \sigma, \sigma' \) for \( \text{cat}(X; K, L; A) \leq m \) and \( \text{cat}(X'; K', L'; A) \leq m \) corresponding to \( s \) and \( s' \), respectively by \( s \sim q_m(X, L) \circ \sigma \) and \( s' \sim q_m(X', L') \circ \sigma' \) relative \( A \).

**Lemma 4.5.** Let \( f : (X; K, L; A) \to (X'; K', L'; A) \) be a map of triads in \( T^A \). Then \( f \) is \( m \)-primitive with respect to \( s \) and \( s' \), if and only if \( \sigma' \circ f|_K \sim P^m(\Omega(f|_{(X, L)})) \circ \sigma \) relative \( A \) for the corresponding structure maps \( \sigma \) and \( \sigma' \).

**Proof.** Assume that \( f \) satisfies that \( \sigma' \circ f|_K \sim P^m(\Omega(f|_{(X, L)})) \circ \sigma \). By composing \( q_m(X', L') : P^m(\Omega(X', L')) \to T^{m+1}(X', L') \) with both sides, we obtain

\[
\begin{align*}
s' \circ f|_K \sim q_m(X', L') \circ \sigma' \circ f|_K & \sim q_m(X', L') \circ P^m(\Omega(f|_{(X, L)})) \circ \sigma \\
& \sim T^{m+1}(f|_{(X, L)}) \circ q_m(X, L) \circ \sigma \sim T^{m+1}(f|_{(X, L)}) \circ s
\end{align*}
\]

relative \( A \), and hence \( f \) is \( m \)-primitive with respect to \( s \) and \( s' \). Conversely assume that \( f \) is \( m \)-primitive with respect to \( s \) and \( s' \). Then the naturality of the lower right square of the diagram (2.2) immediately induces the homotopy \( \sigma' \circ f|_K \sim P^m(\Omega(f|_{(X, L)})) \circ \sigma \) relative \( A \).

**Theorem 4.6.** Let \((X; K, L; A)\) and \((X'; K', L'; A)\) be triads in \( T^A \), \( V \) be a co-loop co-\( H \)-space, and \( s : K \to T^{m+1}(X, L) \) and \( s' : K' \to T^{m+1}(X', L') \) be compressions of the inclusions \( i : K \hookrightarrow X \) and \( i' : K' \hookrightarrow X' \) relative \( A \), respectively, so that \( \text{cat}(X; K, L; A) \leq m \) and \( \text{cat}(X'; K', L'; A) \leq m \), respectively. Let \( f : (X; K, L; A) \to (X'; K', L'; A) \) be a map of triads in \( T^A \) and let \( \alpha : V \to K \) be a map in \( T \) such that \( X \supset \hat{K} = K \cup_{\alpha} \mathcal{C}V \supset K \) and \( X' \supset \hat{K}' = K' \cup_{f|_K \circ \alpha} \mathcal{C}V \supset K \). If \( f \) is \( m \)-primitive with respect to \( s \) and \( s' \), then

\[ E^{m+1}(\Omega(f|_{(X, L)})) \# H_m(X; K, L; A)(\alpha) \subset H_m(X'; K', L'; A)(f|_{K \circ \alpha}). \]

**Proof.** By Lemma 2.1 for \( f : (X; K, L; A) \to (X'; K', L'; A) \) a map of triads in \( T^A \), the following diagram is commutative up to homotopy relative \( A \):
Proposition 4.1 (3). Thus we have

Hence we obtain

relation:

From the definition of a higher Hopf invariant, we obtain

Since $f$ is $m$-primitive with respect to $s$ and $s'$, we have the homotopy relation relative $A$ $P_m^m(\Omega(f|_{(X,L)})) \circ \sigma \sim \sigma' \circ f|_K$ for the corresponding compressions $\sigma$ and $\sigma'$ relative $A$ of the inclusions $i : K \hookrightarrow X$ and $i' : K' \hookrightarrow X'$, resp. Thus we have the following homotopy relation:

$$p_m^m(X',L') \circ E^{m+1}(\Omega(f|_{(X,L)})) \circ H^\sigma_m(\alpha)$$

$$\sim P^m_m(\Omega(f|_{(X,L)})) \circ p_m^m(X,L) \circ H^{(X;K:L:A)}_m(\alpha)$$

$$\sim P^m_m(\Omega(f|_{(X,L)})) \circ \sigma \circ \alpha \sim \sigma' \circ f|_{K \cup \alpha} \sim p_m^m(X',L') \circ H^\sigma_m(f|_K \circ \alpha).$$

Hence we obtain $E^{m+1}(\Omega(f|_{(X,L)})) \circ H^\sigma_m(\alpha) \sim H^\sigma_m(f|_K \circ \alpha)$, since $p_m^m(X',L')$ is monic by Proposition 4.1 (3). Thus we have

$$E^{m+1}(\Omega(f|_{(X,L)})) \# H^{(X;K,L:A)}_m(\alpha) \subset H^{(X';K',L':A)}_m(f|_K \circ \alpha).$$

This completes the proof of Theorem 4.6.

**Theorem 4.7.** Let $(X,K,L:A)$ be a triple in $T^A$, $V$ be a co-loop co-H-space, and $\alpha : V \to K$ be a map in $T$ such that $X \supset K = K \cup \alpha CV \supset K$. If $\text{cat}(K,L:A) \leq m$, then

$$E^{m+1}(\Omega(j|_{(K,L)})) \# H^{(K,L:A)}_m(\alpha) \subset H^{(X;K,L:A)}_m(\alpha),$$

where $j : (K;K,L:A) \to (X;K,L:A)$ is the inclusion.

**Corollary 4.8.** For the filtration $\{F_i\}$ in Definition 1.6, we have

$$E^{m+1}(\Omega(j_i|_{(F_i,F_{i-1})})) \# H^{(F_i,F_{i-1};A)}_i(\alpha) \subset H^{(X;F_i,F_{i-1};A)}_i(\alpha)$$

for every $i$, where $j_i : (F_i;F_i,F_{i-1};A) \hookrightarrow (X;F_i,F_{i-1};A)$ denote the inclusion.

**Proof.** Proof of Theorem 4.7 Let $(X,K,L:A)$ be a triple in $T^A$, $V$ be a co-loop co-H-space and $\alpha : V \to K$ be a map in $T$ such that $X \supset K = K \cup \alpha CV \supset K$. Assuming $\text{cat}(K,L:A) \leq m$, we show $E^{m+1}(\Omega(j|_{(K,L)})) \# H^{(K,L:A)}_m(\alpha) \subset H^{(X;K,L:A)}_m(\alpha)$, where $j : (K;K,L:A) \to (X;K,L:A)$ denotes the inclusion: By Lemma 2.1 for $j : (K;K,L:A) \to (X;K,L:A)$ an inclusion map of triads in $T^A$, the following diagram is commutative up to homotopy relative $A$:

$$\begin{array}{ccc}
E^{m+1}(\Omega(K,L)) & \overset{p_m^{\Omega(K,L)}}{\rightarrow} & P^m_m(\Omega(X,L)) \overset{e_m^{(K,L)}}{\rightarrow} K \\
E^{m+1}(\Omega(j|_{(K,L)})) & \downarrow & P^m_m(\Omega(j|_{(K,L)})) \downarrow j|_K \\
E^{m+1}(\Omega(X,L)) & \overset{p_m^{\Omega(X,L)}}{\rightarrow} & P^m_m(\Omega(X,L)) \overset{e_m^{(X,L)}}{\rightarrow} X.
\end{array}$$

From the definition of a higher Hopf invariant, we obtain $p_m^{\Omega(K,L)} \circ H^\sigma_m(\alpha) \sim \sigma \circ \alpha -$
The identity of $F$ where $F$ is the homotopy pullback of $\sigma$ and $\iota$. Let $(X:A) \in T^A$. Assume $\text{caten}(X:A) = m$ with a categorical sequence $(F_i^X:A)$, $0 \leq i \leq m$ for $(X:A)$. Then by Corollary 3.7, we have $\text{cat}(X:A) = \text{cat}(X; F_m^X:A) \leq m = \text{caten}(X:A)$. Hence we have $\text{cat}(X:A) \leq \text{caten}(X:A)$. Conversely assume $\text{cat}(X:A) = m$. Then the pair $(X:A)$ is dominated by $(P^m(\Omega(X:A)):A)$ which has the cone decomposition $(P^i(\Omega(X:A))I:A)$, $0 \leq i \leq m$ as the canonical categorical sequence. Thus by Lemma 5.1, we have that $(X:A)$ has also a categorical sequence of length $m$, and hence that $\text{caten}(X:A) \leq m = \text{cat}(X:A)$. This completes the proof of Theorem 1.16.

Proof of Lemma 5.1 Let $(F_i^Y:A)$, $0 \leq i \leq m$, be a categorical sequence for $(Y:A) \in T^A$ and $\sigma : X \to Y$ and $\rho : Y \to X$ be maps such that $\rho \sigma \approx 1_X$. Then we define $F_i$ as the homotopy pullback of $\sigma$ and the inclusion $\iota_i : F_i^Y \hookrightarrow F_m^Y$. Since the image of $\sigma|_A$ is the same as the inclusion $A \subseteq F_0^Y \hookrightarrow F_m^Y$, the space $A$ is canonically embedded in $F_0$ and hence in $F_i \supseteq F_0$ for any $i \geq 0$.

\[ F \xrightarrow{\sigma} F_0 \xrightarrow{\iota_0} F_i \xrightarrow{\iota_i} F_m \xrightarrow{\text{id}} X \]

\[ \text{PB } \sigma_0 \quad \text{PB } \sigma_i \quad \text{PB } \sigma_m \quad \text{HPB } \sigma \]

\[ * \xrightarrow{\iota_0} A \xrightarrow{\iota_i} F_i^Y \xrightarrow{\text{id}} F_m^Y \]

where $F$ denotes the homotopy fibre of $\sigma$ and $F_m$ is the homotopy pullback of $\sigma$ and the identity of $F_m^Y$. Since $\rho \sigma \approx 1_X$, $\rho|_{F^Y}$ can be compressed into $F_i$ and we have the
By composing $\rho$ and $F$, \[X\] sequence for $\text{Proof}$. By the definition of a categorical sequence, the diagonal map $\Delta : F \to F \times *$ $\cup F \times F_m$ of the diagonal map $\Delta F : F \to F \times F \subseteq F \times F_m$ relative to $F_{i-1}$:

$$
\begin{array}{c}
F_i \\
\downarrow \sigma_i \\
F_i^Y \\
\downarrow \Delta F_i \\
F_i \times F_i^Y \\
\downarrow \nu_i^Y \\
F_i \times * \cup F_{i-1} \times F_i^Y
\end{array}
\quad
\begin{array}{c}
F_i \\
\downarrow \Delta F_i \\
F_i \times F_i^Y \\
\downarrow \nu_i^Y \\
F_i^Y \times * \cup F_{i-1} \times F_i^Y
\end{array}
\quad
\begin{array}{c}
F_i \\
\downarrow \Delta F_i \\
F_i \times F_i^Y \\
\downarrow \nu_i^Y \\
F_i^Y \times * \cup F_{i-1} \times F_i^Y
\end{array}
\quad
\begin{array}{c}
F_i \\
\downarrow \Delta F_i \\
F_i \times F_i^Y \\
\downarrow \nu_i^Y \\
F_i^Y \times * \cup F_{i-1} \times F_i^Y
\end{array}
\quad
\begin{array}{c}
F_i \\
\downarrow \Delta F_i \\
F_i \times F_i^Y \\
\downarrow \nu_i^Y \\
F_i^Y \times * \cup F_{i-1} \times F_i^Y
\end{array}
$$

By composing $\rho_i$ and $\sigma_i$, we obtain a compression of the diagonal map $\Delta F_i : F \to F \times F_i \subseteq F \times F_m$ as follows:

This implies $\text{cat}(X'; X', F_{m-1}; A) \leq 1$, and hence $X' = F^X_m \supset F_{m-1} \supset \cdots \supset F_0 = A$ gives a categorical sequence for $X$.

The following lemma is our version of the result of Arkowitz and Lupton [1]:

**Lemma 5.2.** Let $X$ be a space in $T$ with $\text{cat}(X) = m$ and $\{F_i ; 0 \leq i \leq m\}$ be a categorical sequence for $X$. Then there is a map $\mu : F \to F_m/F_{i-1} \vee F_m$ in $T$ with axes $F \to F_m/F_{i-1}$ and the inclusion $F_i \hookrightarrow F_m$.

**Proof.** By the definition of a categorical sequence, the diagonal map $\Delta : F \to F \times F_i \subseteq F \times F_m$ is compressible into $F_{i-1} \times F_m \cup F \times *$ as $F_i \xrightarrow{\hat{\mu}} F_{i-1} \times F_m \cup F \times * \subseteq F \times F_m$. Since $F_m/F_{i-1} \vee F_m$ can be regarded as the pushout of the second projection $\text{pr}_2 : F_{i-1} \times F_m \to F_m$ and the canonical inclusion $i : F_{i-1} \times F_m \hookrightarrow F_{i-1} \times F_m \cup F \times *$, we have
the following diagram:

\[
\begin{array}{ccc}
F_{i-1} \times F_m & \xrightarrow{\mu} & F_{i-1} \times F_m \cup F_m \\
\downarrow & & \downarrow \\
F_m & \xrightarrow{\text{in}_2} & F_m/F_{i-1} \cup F_m \\
\downarrow & & \downarrow \\
F_m & \xrightarrow{j} & F_m/F_{i-1} \times F_m
\end{array}
\]

where \( q_i^{F_i} : F_i \rightarrow F_i/F_{i-1} \subseteq F_m/F_{i-1} \) denotes the canonical collapsing map in \( \mathcal{T} \). Let \( \mu \) be the composition \( q_i^{F_i} \circ \hat{\mu} : F_i \rightarrow F_i/F_{i-1} \vee F_m \) so that \( j \circ \mu \) is homotopic to \( (q_i^{F_i} \times \text{id}_{F_i}) \circ \Delta \). Thus \( \mu \) has axes \( q_i^{F_i} : F_i \rightarrow F_i/F_{i-1} \subseteq F_m/F_{i-1} \) and the inclusion \( F_i \hookrightarrow F_m \).

From this one immediately deduces corollary 1.17 of the introduction.

6. Cup length and module weight for the relative L-S category. A computable lower estimate in L-S category theory is given by the classical cup-length. Here we give the definition for our new relative L-S category.

**Definition 6.1.** For any two maps \( f : (L;A) \subset (X;A) \) and \( g : (K;A) \rightarrow (X;A) \) in \( \mathcal{T}^A \), we define cup length for \( (g, f) = (X; K, L; A) \):

1. Let \( h \) be a multiplicative generalized cohomology theory.
   \[
   \text{cup}(g, f; h) = \min \left\{ m \geq 0 \mid \forall \{v_0 \in h^*(X, L); v_1, \ldots , v_m \in h^*(X, A)\} \left| g^* (v_0 \cdot v_1 \cdots v_m) = 0 \right. \text{in } h^*(K, A) \right\}
   \]

2. \( \text{cup}(g, f) = \max \left\{ \text{cup}(g, f; h) \mid h \text{ is a multiplicative generalized cohomology theory} \right\} \).

Then we have \( \text{cup}(g, f; h) \leq \text{cup}(g, f) \leq \text{cat}(g, f) \) for any multiplicative generalized cohomology \( h \). When \( h \) is the ordinary cohomology with a coefficient ring \( R \), we denote \( \text{cup}(g, f; h) \) by \( \text{cup}(g, f; R) \). This definition immediately implies the following.

**Remark 6.2.** For \( (g, f) = (X; K, L; A) \), using the arguments in [16], we have \( \text{cup}(g, f) = \min \{ m \geq 0 \mid \hat{\Delta}_K^{m+1} : K/A \rightarrow X/L \wedge \wedge^m X/A \text{ is stably trivial} \} \).

Let us recall that Rudyak [23] and Strom [26] introduced a homotopy theoretical version of Fadell-Husseini’s category weight (see [6]). But unfortunately, we have not been able to give a version of category weight for our new relative L-S category. In this paper, we give instead a version of module weight which is a better computable lower estimate for our relative L-S category than cup length: let \( f : (L; A) \subset (X; A) \) and \( g : (K; A) \rightarrow (X; A) \) be maps in \( \mathcal{T}^A \) and let \( h \) be a generalized cohomology theory.

**Definition 6.3 ([16]).** A homomorphism \( \phi : h^*(Y, L) \rightarrow h^*(K, A) \) of \( h_* \)-modules is called an (unstable) \( h \)-morphism if it preserves the action of any (unstable) cohomology operation on \( h^* \).
**Definition 6.4.** An (unstable) module weight $Mwgt(g, f; h)$ of $(g, f)$ with respect to $h$ is defined as follows.

$$
Mwgt(g, f; h) = \text{Min} \left\{ m \geq 0 \left| \begin{array}{l}
\text{There is a (unstable) } h\text{-morphism } \phi : \\
h^*(P^m(\Omega(X, L)), L) \to h^*(K, A) \text{ such that }
\phi(e_n^X) = g^*: h^*(X, L) \to h^*(K, A).
\end{array} \right. \right\}.
$$

When $h$ is the ordinary cohomology theory with coefficients in a ring $R$, we denote $Mwgt(g, f; h)$ by $Mwgt(g, f; R)$.

**Remark 6.5.** The invariants introduced in this paper satisfy the following inequality for any generalised cohomology theory $h^*$:

$$
cup(g, f; h) \leq Mwgt(g, f; h) \leq \text{cat}(g, f) = \text{catlen}(g, f),
$$

and hence for any ring $R$, we have

$$
cup(g, f; R) \leq Mwgt(g, f; R) \leq \text{cat}(g, f) = \text{catlen}(g, f).
$$

Similar to the above definition of $\cup(g, f)$, we define the following invariants.

**Definition 6.6.** For any $(g, f) = (X; K, L; A)$, we define

$$
Mwgt(g, f) = \text{Max} \left\{ m \geq 0 \left| \begin{array}{l}
Mwgt(g, f; h) = m \text{ for some generalized }
\text{cohomology theory } h
\end{array} \right. \right\}.
$$

**Remark 6.7.** $\cup(g, f) \leq Mwgt(g, f) \leq \text{cat}(g, f) = \text{catlen}(g, f)$.

### 7. Examples of Categorical Sequences.

In [3], Berstein and Hilton showed that the L-S category of the cell complex $Q(\alpha) = S^r \cup_{\alpha} e^{q+1}$, $\alpha \in \pi_q(S^r)$, is determined by the Hopf invariant $H_1(\alpha) \in \pi_{q+1}(S^r \times S^r, S^r \vee S^r) \cong \pi_q(\Omega(S^r) \ast \Omega(S^r))$ by Ganea. We can easily observe that $F_0 = *, F_1 = S^r$ and $F_2 = Q(\alpha)$ give a cone decomposition of $Q(\alpha)$ of length 2. If $H_1(\alpha) = 0$, then by Theorem 1.15, we obtain that $F'_0 = F_0 = *, F'_1 = F_1 \cup_{\alpha} e^{q+1} = F_2 = Q(\alpha)$ give a categorical sequence of length 1.

In [15], the author showed that the L-S category of total space $E(\beta) = Q(\beta) \cup_{\psi(\beta)} e^{q+r+1}$, $\beta \in \pi_q(S^r)$, $\psi(\beta) \in \pi_{q+r}(Q(\beta))$ is determined by $\Sigma^r H_1(\alpha) \in \pi_{q+r}(\Omega(S^r) \ast \Omega(Q(\beta)) \ast \Omega(Q(\beta)))$, if $H_1(\beta) \neq 0$. We can easily observe that $F_0 = *, F_1 = S^r, F_2 = Q(\beta)$ and $F_3 = E(\beta)$ give a cone decomposition of $E(\beta)$ of length 3. If $\Sigma^r H_1(\alpha) = 0$, then by Theorem 1.15, we obtain that $F'_0 = F_0 = *, F'_1 = F_1 = S^r, F'_2 = F_2 \cup_{\psi(\beta)} e^{q+r+1} = F_3 = E(\beta)$ give a categorical sequence of length 2.

Let us denote by $Z^{(k)}$ the $k$-skeleton of a CW complex $Z$. To give an upper-bound for L-S category of the total space of a fibre bundle $F \hookrightarrow E \to B$, we need a refinement of results of Varadarajan [28] and Hardie [11], and the corresponding result for strong category of Ganea [9]:

**Theorem 7.1.** ([28, 11, 9]).

1. $\text{cat}(E)+1 \leq (\text{cat}(F)+1) \cdot (\text{cat}(B)+1)$.
2. $\text{Cat}(E)+1 \leq (\text{Cat}(F)+1) \cdot (\text{Cat}(B)+1)$.

In [18], Iwase-Mimura-Nishimoto gave a refinement in the case when the base space $B$ is non-simply connected. On the other hand in the case when $B$ is simply connected, Iwase-Kono [17] gave another refinement if the higher Hopf invariant of the characteristic map is 0.
By assuming the fibre $F$ is of categorical length $m$, we obtain a further refinement using categorical sequence in place of cone decomposition:

**Theorem 7.2.** Let $B$ be a $(d-1)$-connected finite dimensional CW complex $(d \geq 1)$, whose cells are concentrated in dimensions $0, 1, \ldots, s \mod d$ for some $s$, $0 \leq s \leq d-1$. Let $F \hookrightarrow X \to B$ be a fibre bundle with fibre $F$ whose structure group is a compact Lie group $G$. Then we have $\text{cat}(X) \leq m + \left\lceil \frac{\dim B}{d} \right\rceil$, if $F$ has a categorical sequence of length $m$ with the following compatibility assumption for some $d \geq 1$:

1. $\mu|_{F_{i-1+1} \times Q} = \mu_{i-1}$.

In [17], Kono and the author showed that there is a cone decomposition $E_i$, $0 \leq i \leq 8$ and $E'_8$ of $\text{Spin}(9)$ of length 9, while the L-S category of $\text{Spin}(9)$ is 8 by a combination of a higher Hopf invariant and the cone decomposition: we can easily see that the construction in §1 in [17] gives the following proposition:

**Proposition 7.3.** Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with a characteristic map $\alpha : \Sigma V \to Q$ a subspace of $G$. Then $\text{catlen}(E) \leq m+n+1$ if $G$ has a categorical sequence $\ast = F_0 \subset \cdots \subset F_m \simeq G$ with the following compatibility assumption for a positive integer $n$:

1. the restriction of the multiplication $\mu : G \times G \to G$ to the subspace $F_j \times Q \subseteq F_m \times F_m$ $\simeq G \times G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j \geq 0$ as $\mu_j : F_j \times Q \to F_{j+n}$ such that $\mu_j|_{F_{j-1} \times Q} = \mu_{j-1}$.

**Remark 7.4.** If we choose $n = m$, then the assumption (1) above is automatically satisfied and we always have $\text{cat}(E) \leq 2 \text{cat}(G)+1$ which is a special case of a theorem of Hardie and Varadarajan [11, 28] (see Theorem 7.1 (1)).

Moreover, Lemma 1.1 in [17] implies that the higher Hopf invariant of the attaching map of the top cell of $\text{Spin}(9)$ must vanish, since the structure map of $\text{cat}(E'_8) = 8$ can be chosen to be compatible to the structure map of $\text{cat}(E_8) = 8$ by the argument given in the proof of Lemma 1.1 in [17]. Hence by Theorem 1.15, we obtain that $E_i$, $0 \leq i \leq 7$ and $E'_8$ give the categorical sequence of length 8: we can easily see that the proof of Lemma 1.1 in [17] gives the following theorem:

**Proposition 7.5.** Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with a characteristic map $\alpha : \Sigma V \to Q$ a subspace of $G$. Then $\text{cat}(E) \leq \text{Max}\{m+n, m+2\}$ if $G$ has a categorical sequence $\ast = F_0 \subset \cdots \subset F_m \simeq G$ with the following compatibility assumptions for a positive integer $n$:

1. the restriction of the multiplication $\mu : G \times G \to G$ to the subspace $F_j \times Q \subseteq F_m \times F_m$ $\simeq G \times G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j \geq 0$ as $\mu_j : F_j \times Q \to F_{j+n}$ such that $\mu_j|_{F_{j-1} \times Q} = \mu_{j-1}$ and
2. $H_n^{(E; \{Q \cup \alpha C \Sigma V, Q\}; \ast)}(\alpha) = 0$.

These propositions imply the following result.

**Theorem 7.6.** Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with a characteristic map $\alpha : \Sigma V \to Q$, a subspace of $G$. Then $\text{cat}(E) \leq \text{Max}\{m+n+\text{cat}(E; Q \cup \alpha C \Sigma V, Q; \ast), m+2\}$
if $G$ has a categorical sequence $\ast = F_0 \subset \cdots \subset F_m \simeq G$ with the following compatibility assumption for a positive integer $n \geq 1$:

(1) the restriction of the multiplication $\mu : G \times G \to G$ to the subspace $F_j \times Q \subseteq F_m \times F_m \simeq G \times G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j \geq 0$ as $\mu_j : F_j \times Q \to F_{j+n}$ such that $\mu_j|_{F_{j-1} \times Q} = \mu_{j-1}$.

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References


