ALGEBRAIC TOPOLOGY—OLD AND NEW
BANACH CENTER PUBLICATIONS, VOLUME 85
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2009

CATEGORICAL LENGTH, RELATIVE L-S CATEGORY AND HIGHER HOPF INVARIANTS

NORIO IWASE

Faculty of Mathematics, Kyushu University
Fukuoka 810-8560, Japan
E-mail: iwase@math.kyushu-u.ac.jp

Abstract. In this paper we introduce the categorical length, a homotopy version of Fox categorical sequence, and an extended version of relative L-S category which contains the classical notions of Berstein-Ganea and Fadell-Husseini. We then show that, for a space or a pair, the categorical length for categorical sequences is precisely the L-S category or the relative L-S category in the sense of Fadell-Husseini respectively. Higher Hopf invariants, cup length, module weights, and recent computations by Kono and the author are also studied within this unified L-S theory based on the categorical length of categorical sequences.

1. Introduction. Throughout this paper, we work in $\mathcal{T}$ the category of topological spaces and maps, or the category of pairs $\mathcal{T}^A$ in which an object is a pair $(X{:}A)$ with an inclusion $i^X : A \hookrightarrow X$ and a morphism is a map of pairs $f : (X{:}A) \to (Y{:}A)$ with $i^Y = f{\circ}i^X$. A closed subset is always assumed to be a neighbourhood deformation retract, and a pair is assumed to be an NDR-pair in the sense of G. Whitehead [29]. The one-point-space is denoted by $*$. The (normalised) Lusternik-Schnirelmann category $\mathrm{cat}(X)$, L-S category for short, is introduced in [22] as the least number m such that there is a covering of X by $m{+}1$ closed subsets U_j, $0 \leq j \leq m$, where each U_j is contractible in X. By modifying the idea due to R. Fox [8], T. Ganea [9] gives the following definition of a strong version of L-S category for a space X: the strong L-S category $\mathrm{Cat}(X)$ is the least number m such that there is a space $Y \simeq X$ with a covering of Y by $m{+}1$ closed subsets U_j, $0 \leq j \leq m$ where each U_j is contractible in itself. By Ganea [9], it is shown that

$$\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1.$$

2000 *Mathematics Subject Classification*: Primary 55M30; Secondary 55Q25.
Key words and phrases: Lusternik-Schnirelmann category, categorical sequence, Hopf invariant, cone decomposition.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc85-0-15

Remark 1.1. Fadell and Husseini [7] introduced a notion of relative L-S category as follows: for a pair $(K{:}A)$, $\mathrm{cat}^{\mathrm{FH}}(K,A)$ is given as the least number m such that there is a covering of K by $m{+}1$ closed subsets $V \supset A$ and U_j, $1 \le j \le m$ where V is compressible relative A into A in K and each U_j is contractible in K. It is also clear by definition that $\mathrm{cat}^{\mathrm{FH}}(K,*) = \mathrm{cat}(K)$.

By G. Whitehead [29], the definition of L-S category is interpreted in terms of deformation of a diagonal map as the following definition for a space X.

Definition 1.2. The L-S category $\mathrm{cat}(X)$ of X is the least number m such that the $m{+}1$ fold diagonal map $\Delta^{m+1} : X \to \prod^{m+1} X$ is compressible into the fat wedge $\mathrm{T}^{m+1} X = \{(x_0, x_1, ..., x_m) \in \prod^{m+1} X \mid \exists i\, x_i = *\} \subseteq \prod^{m+1} X$.

Similarly to the above, one can give an alternative definition of a relative L-S category in the sense of Fadell and Husseini [7] for a pair $(K{:}A)$ to fit in with Whitehead's definition of L-S category.

Definition 1.3. Let $A \subseteq K$. Then the L-S category $\mathrm{cat}^{\mathrm{FH}}(K,A)$ is the least number $m \ge 0$ such that the $m{+}1$ fold diagonal map $\Delta_K^{m+1} : K \to \prod^{m+1} K$ is compressible relative A into the fat wedge $\mathrm{T}^{m+1}(K{:}A) = A \times \prod^{m} K \cup K \times \mathrm{T}^m K \subseteq \prod^{m+1} K$ of a pair $(K{:}A)$.

Remark 1.4. For any map $f : A \to K$, we may assume that f is an inclusion up to homotopy, and hence the definition of relative L-S category implies a definition of $\mathrm{cat}^{\mathrm{FH}}(f)$ the L-S category of f in the sense of Fadell and Husseini.

In the present paper, we alter the Fox's definition of a categorical sequence to fit in with Whitehead's definition of L-S category:

Definition 1.5. A categorical sequence for a space X is a sequence of closed subspaces $F_0 \subset \cdots \subset F_i \subset \cdots \subset F_m$ such that $F_m \simeq X$, $F_0 \simeq *$ in X and $\Delta_i : F_i \xrightarrow{\Delta} F_i \times F_i \subset F_m \times F_m$ is compressible into $F_{i-1} \times F_m \cup F_m \times *$ relative F_{i-1} for any $i > 0$, where we identify F_{i-1} with its diagonal image in $F_{i-1} \times F_{i-1} \subset F_{i-1} \times F_m \cup F_m \times *$. Let us call the least such $m \ge 0$ the 'categorical length' of X and denote it by $\mathrm{catlen}(X)$.

Inspired by the definition of a relative L-S category due to Fadell and Husseini, we introduce a relative version of categorical sequence:

Definition 1.6. A categorical sequence for a pair $(X{:}A)$ is a sequence of pairs $(F_0{:}A) \subset \cdots \subset (F_i{:}A) \subset \cdots \subset (F_m{:}A)$ such that $(F_m{:}A) \simeq (X{:}A)$ relative A, $F_0 \simeq A$ relative A in X and $\Delta_i : F_i \xrightarrow{\Delta} F_i \times F_i \subset F_m \times F_m$ is compressible into $F_{i-1} \times F_m \cup F_m \times A$ relative F_{i-1}, $i > 0$. Let us call the least such $m \ge 0$ the 'categorical length' of a space X relative to A and denote it by $\mathrm{catlen}(X{:}A)$.

To describe the categorical sequence in terms of a relative L-S category, we give a definition of a new extended version of relative L-S category: from now on, we work in the category $\mathcal{T}^A$. We remark that, if $A = *$ the one point space, then $\mathcal{T}^A$ is the usual category of based connected spaces and based maps. We say that $(X, K{:}A)$ is a pair in $\mathcal{T}^A$ when $(X{:}A)$ and $(K{:}A)$ are objects in $\mathcal{T}^A$ and (X,K) is a pair in $\mathcal{T}$, that $(X,K,L{:}A)$ is a triple in $\mathcal{T}^A$ when $(X{:}A)$, $(K{:}A)$, $(L{:}A)$ are objects in $\mathcal{T}^A$ and (X,K,L) is a triple in $\mathcal{T}$, and that $(X;K,L{:}A)$ is a triad in $\mathcal{T}^A$ when $(X{:}A)$, $(K{:}A)$, $(L{:}A)$ are objects in $\mathcal{T}^A$ and $(X;K,L)$ is a triad in $\mathcal{T}$.

We remark, for any pair $(X,K{:}A)$ in $\mathcal{T}^A$, that the diagonal image of A in $\prod^{m+1} X$ is in the subspace $\mathrm{T}^{m+1}(X,L)$. Thus for any $(X{:}A) \supset (L{:}A) \in \mathcal{T}^A$, we regard $(\prod^{m+1} X{:}A) \supset (\mathrm{T}^{m+1}(X,L){:}A) \in \mathcal{T}^A$.

Definition 1.7. Let $(X;K,L{:}A)$ be a triad in $\mathcal{T}^A$. Then $\mathrm{cat}(X;K,L{:}A)$ is the least number m such that the restriction of the $m{+}1$ fold diagonal map of X to K, $\Delta^{m+1}|_K : K \to \prod^{m+1} X$, is compressible relative A into $\mathrm{T}^{m+1}(X,L)$.

Using Harper's arguments on the homotopy of maps to the total space of a fibration in [12], Cornea [4] has given a proof of the following:

Proposition 1.8. *Let $(X{:}A)$ be an object in $\mathcal{T}^A$, $(Y,K{:}A)$ be a pair in $\mathcal{T}^A$ with the inclusion $j : (K{:}A) \hookrightarrow (Y{:}A)$ and $f : (X{:}A) \to (Y{:}A)$ be a map in $\mathcal{T}^A$. If $f|_X : X \to Y$ has a compression $\sigma : X \to K$ such that $j\circ\sigma \sim f$ and $\sigma\circ i^X \sim i^K$ in $\mathcal{T}$, then there is a map $\sigma' : (X{:}A) \to (K{:}A)$ a compression relative A of f such that $\sigma \sim \sigma'|_X : X \to K$.*

One of its direct consequences is:

Corollary 1.9. *Let $(X;K.L{:}A)$ be a triple in $\mathcal{T}^A$. Then $\mathrm{cat}(X;K,L{:}A)$ is the same as the least number m such that $\Delta^{m+1}|_K : K \to \prod^{m+1} X$ is compressible to a map $s : K \to \mathrm{T}^{m+1}(X,L)$ such that $s|_A$ is homotopic to the diagonal map $\Delta_A : A \to \prod^{m+1} A \subset \mathrm{T}^{m+1}(X,L)$.*

Remark 1.10. (1) $\mathrm{cat}(X;X,*{:}*) = \mathrm{cat}(X)$ and $\mathrm{cat}(X;*,*{:}*) = 0$.

(2) We denote $(X;X,L{:}A)$ by $(X,L{:}A)$, $(X;K,A{:}A)$ by $(X;K{:}A)$, $(X;X{:}A)$ by $(X{:}A)$, $(X;K,L{:}*)$ by $(X;K,L)$, $(X;K,*)$ by $(X;K)$ and $(X{:}*) = (X;X,*{:}*) = (X;X)$ by X.

(3) We may replace inclusions $(L{:}A) \hookrightarrow (X{:}A)$ and $(K{:}A) \hookrightarrow (X{:}A)$ by maps $f : (L{:}A) \to (X{:}A)$ and $g : (K{:}A) \to (X{:}A)$ in $\mathcal{T}^A$, since every such map is an inclusion map up to homotopy relative A by taking the mapping cylinder of $K \cup_A L \xrightarrow{g\cup_A f} X$. Then we often denote $\mathrm{cat}(X;K,L{:}A)$ by $\mathrm{cat}(g,f)$. By applying (1), we have $\mathrm{cat}(g,*) = \mathrm{cat}(g)$.

There is another classical notion of relative L-S category due to Berstein and Ganea [2].

Definition 1.11. Let $K \subset X$. Then the L-S category $\mathrm{cat}^{\mathrm{BG}}(X,K)$ is the least number $m \geq 0$ such that restriction to K of the $m{+}1$ fold diagonal map $\Delta_X^{m+1} : X \to \prod^{m+1} X$ is compressible into the fat wedge $\mathrm{T}^{m+1} X$.

Remark 1.12. For any map $f : K \to X$, we may assume that f is an inclusion up to homotopy, and hence the above definition of the L-S category implies a definition of $\mathrm{cat}^{\mathrm{BG}}(f)$ the L-S category of f in the sense of Berstein and Ganea.

Arkowitz and Lupton [1] have also defined their relative L-S category for a map $h : X \to Y$. Since a map is up to homotopy a fibration, we may assume that h is a fibration with fibre $L = h^{-1}(*) \subset X$. Then the relative L-S category of h in the sense of Arkowitz and Lupton depends only on the pair (X,L) by its definition.

Definition 1.13. Let $L \subset X$. Then the L-S category $\mathrm{cat}^{\mathrm{AL}}(X, L)$ is the least number $m \geq 0$ such that the $m{+}1$ fold diagonal map $\Delta_X^{m+1} : X \to \prod^{m+1} X$ is compressible into the fat wedge $\mathrm{T}^{m+1}(X, L)$.

Then we prove:

Theorem 1.14. *The known three relative L-S categories are special cases of our new relative L-S category:*

(1) Let $X = K \supset L = A \supset *$. Then $\mathrm{cat}(X{:}A) = \mathrm{cat}(X; X, A{:}A) = \mathrm{cat}^{\mathrm{FH}}(X, A)$.
(2) Let $X \supset K \supset L = A = *$. Then $\mathrm{cat}(X; K) = \mathrm{cat}(X; K, *{:}*) = \mathrm{cat}^{\mathrm{BG}}(X, K)$. *More generally for a map $g : K \to X$ in $\mathcal{T}_*$, we have* $\mathrm{cat}(g, *) = \mathrm{cat}^{\mathrm{BG}}(g)$.
(3) Let $K = X \supset L \supset A = *$. Then $\mathrm{cat}(X, L) = \mathrm{cat}(X; X, L{:}*) = \mathrm{cat}^{\mathrm{AL}}(X, L)$.

We also introduce a new higher Hopf invariant: let $(X; K, L{:}A)$ be a triad in $\mathcal{T}^A$, let V be a co-loop co-H-space, i.e., a one-point-union of a 1-connected co-H-space with finitely-many circles, and let $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. If $\mathrm{cat}(X; K, L{:}A) \leq m$, then a relative higher Hopf invariant $H_m^{(X;K,L:A)}(\alpha)$ is defined as a subset of $[V, \Omega(X, L) * \Omega(X) * \overset{m}{\cdots} * \Omega(X)]$. If $K \supset L$ and $\mathrm{cat}(K; K, L{:}A) \leq m$, then an absolute higher Hopf invariant $H_m^{(K,L:A)}(\alpha)$ is defined as a subset of $[V, \Omega(K, L) * \Omega(K) * \overset{m}{\cdots} * \Omega(K)]$ (see §4 for more details). The following result clarifies how a higher Hopf invariant determines whether a cone decomposition reduces to a categorical sequence or not.

Theorem 1.15. *Let $(X; K, L{:}A)$ be a triad in $\mathcal{T}^A$, let V be a co-loop co-H-space and let $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. If* $\mathrm{cat}(X; K, L{:}A) \leq m$ *and $H_m^{(X;K,L:A)}(\alpha) = 0$, then* $\mathrm{cat}(X; \hat{K}, L{:}A) \leq m$.

From now on, we abbreviate $H_m^{(X;K,A:A)}(\alpha)$ by $H_m^{(X;K:A)}(\alpha)$, $H_m^{(X;K,*)}(\alpha)$ by $H_m^{(X;K)}(\alpha)$, $H_m^{(K,A:A)}(\alpha)$ by $H_m^{(K:A)}(\alpha)$ and $H_m^{(K,*)}(\alpha)$ by $H_m^{K}(\alpha)$. Note that the definition of the absolute higher Hopf invariant $H_m^{K}(\alpha)$ coincides with the ordinary definition of the higher Hopf invariant $H_m(\alpha)$ in the sense of [14].

The main goal of this paper is to proof:

Theorem 1.16. *For any X in $\mathcal{T}$, we have* $\mathrm{cat}(X) = \mathrm{catlen}(X)$. *More generally, for any object $(X{:}A) \in \mathcal{T}^A$, we have* $\mathrm{catlen}(X{:}A) = \mathrm{cat}(X{:}A) = \mathrm{cat}^{\mathrm{FH}}(X, A)$.

Corollary 1.17. *Let $(X{:}A)$ be an object in $\mathcal{T}^A$. If* $\mathrm{cat}^{\mathrm{FH}}(X, A) = m > 0$, *then there exists a sequence of pairs $\{(F_i{:}A)\,;\, 0{\leq}i{\leq}m\}$ such that $(F_0{:}A) \simeq (A{:}A)$ in $(F_m{:}A)$, $(F_m{:}A) \simeq (X{:}A)$ relative A and* $\mathrm{cat}(X; F_i{:}A) \leq i$, $i > 0$. *Moreover,* $\mathrm{cat}(F_m/F_{i-1}; F_i/F_{i-1}) \leq 1$ *with a partial co-action $F_i \to F_m/F_{i-1} \vee F_m$ along the collapsing map $F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$, $i > 0$. In particular, F_m/F_{m-1} is a co-H-space co-acting on F_m along the collapsing map $F_m \to F_m/F_{m-1}$.*

2. A_∞-decomposition of a map. In [9], Ganea introduced a so-called 'fibre-cofibre' construction for a map, which can be interpreted as the pullback construction from the view-point of Definition 1.3. We may regard this construction as an A_∞-decomposition of a map using the pushout-pullback diagram (see [13, Lemma 2.1] and also Sakai [24] for the detailed proof in a general context):

$$
\begin{array}{ccccc}
\Omega(X,L)\times E^m(\Omega(X)) & \xrightarrow{\mathrm{pr}_2} & E^m(\Omega(X)) & & \\
\downarrow{\scriptstyle \mathrm{pr}_1} & \text{HPO} & \downarrow & & \\
\Omega(X,L) & \longrightarrow & E^{m+1}(\Omega(X,L)) & \longrightarrow & \mathrm{T}^{m+1}(X,L) \\
 & & \downarrow & \text{HPB} & \downarrow \\
 & & * & \longrightarrow & \prod^{m+1} X.
\end{array}
\tag{2.1}
$$

Let us recall that, in $\mathcal{T}$, the homotopy fibre of $\mathrm{T}^m_{i=0}(X,A_i) \hookrightarrow \prod^{m+1} X$ has the homotopy type of the join $\Omega(X,A_0) * \overset{m+1}{\cdots} * \Omega(X,A_m)$. Let $(X;K,L{:}A)$ be a triad in $\mathcal{T}^A$ and write $E^m(\Omega(X)) = \Omega(X) * \overset{m}{\cdots} * \Omega(X)$ which has the homotopy type of the homotopy fibre of $\mathrm{T}^m(X,*) \hookrightarrow \prod^m X$. The homotopy fibre of the inclusion $\mathrm{T}^{m+1}(X,L) \hookrightarrow \prod^{m+1} X$ has the homotopy type of $E^{m+1}(\Omega(X,L)) = \Omega(X,L) * \Omega(X) * \overset{m}{\cdots} * \Omega(X)$: consider the homotopy pushout-pullback diagram in $\mathcal{T}$, which is given by [13, Lemma 2.1] with $(Y,B) = (\prod^m X, \mathrm{T}^m X)$, $Z = *$ and $f = g = *$. Thus we see that the homotopy fibre of the inclusion $\mathrm{T}^{m+1}(X,L) \hookrightarrow \prod^{m+1} X$ has the homotopy type of $\Omega(X,L)*E^m(\Omega(X)) = E^{m+1}(\Omega(X,L))$ by induction.

Similarly, we define $P^m(\Omega(X,L))$ inductively from $P^0(\Omega(X,L)) = L$ as the homotopy pushout in the following homotopy pushout-pullback diagram which is given by [13, Lemma 2.1] with $(Y,B) = (\prod^m X, \mathrm{T}^m X)$, $Z = X$ and $(f,g) = (1_X, \Delta^m_X)$:

$$
\begin{array}{ccccc}
E^m(\Omega(X,L)) & \xrightarrow{p^{\Omega(X,L)}_{m-1}} & P^{m-1}(\Omega(X,L)) & & \\
\downarrow & \text{HPO} & \downarrow & & \\
* & \longrightarrow & P^m(\Omega(X,L)) & \xrightarrow{q^{(X,L)}_m} & \mathrm{T}^{m+1}(X,L) \\
 & & \downarrow{\scriptstyle e^{(X,L)}_m} & \text{HPB} & \downarrow \\
 & & X & \xrightarrow{\Delta^{m+1}} & \prod^{m+1} X,
\end{array}
\tag{2.2}
$$

where $q^{(X,L)}_m$ covers the diagonal map $\Delta^{m+1} : X \to \prod^{m+1} X$. Then we define $p^{\Omega(X,L)}_{m+1} : E^{m+1}(\Omega(X,L)) \to P^m(\Omega(X,L))$ as the homotopy fibre of $e^{(X,L)}_m : P^m(\Omega(X,L)) \to X$ given in the diagram, where $e^{(X,L)}_0 : L \hookrightarrow X$ is just the canonical inclusion. These constructions due to Ganea [9] yield the following ladder of fibrations which have the same fibre $\Omega(X)$, giving a generalisation of an A_∞-structure (see Stasheff [25]):

$$
\begin{array}{ccccccccc}
\Omega(X,L) & \hookrightarrow & \cdots & \hookrightarrow & E^{m+1}(\Omega(X,L)) & \hookrightarrow & \cdots & \hookrightarrow & E^\infty(\Omega(X,L)) \\
\downarrow{\scriptstyle p^{\Omega(X,L)}_1} & & & & \downarrow & & & & \downarrow{\scriptstyle p^{\Omega(X,L)}_\infty} \\
L & \hookrightarrow & \cdots & \hookrightarrow & P^m(\Omega(X,L)) & \hookrightarrow & \cdots & \hookrightarrow & P^\infty(\Omega(X,L))
\end{array}
\tag{2.3}
$$

together with $e_\infty^{(X,L)} : P^\infty(\Omega(X,L)) = \bigcup_m P^m(\Omega(X,L)) \to X$ given by $e_\infty^{(X,L)}|_{P^m(\Omega(X,L))} = e_m^{(X,L)}$ with fibre $E^\infty(\Omega(X,L))$, where the upper horizontal arrows are null-homotopic. Since $E^\infty(\Omega(X,L)) = \bigcup_m E^m(\Omega(X,L))$ is weakly contractible, $e_\infty^{(X,L)} : P^\infty(\Omega(X,L)) = \bigcup_m P^m(\Omega(X,L)) \to X$ is a week equivalence. If further X is a CW complex, then there is a right homotopy inverse $h^{(X,L)} : X \to P^\infty(\Omega(X,L))$ of $e_\infty^{(X,L)}$, where $h^{(X,L)}$ is also a weak equivalence.

The ladder (2.3) is natural with respect to a map of triads in $\mathcal{T}^A$:

LEMMA 2.1. *For any map $f : (X; K, L{:}A) \to (X'; K', L'{:}A)$ of triads in $\mathcal{T}^A$, there is the following commutative diagram with $f|_{(X,L)} : (X,L) \to (X',L')$ and $f|_L : L \to L'$ the restrictions of f.*

$$\begin{array}{ccccccc}
E^m(\Omega(X,L)) & & \hookrightarrow & & E^{m+1}(\Omega(X,L)) & & \\
\downarrow{\scriptstyle p_1^{\Omega(X,L)}} & \searrow^{E^m(\Omega(f|_{(X,L)}))} & & & \downarrow{\scriptstyle p_{m+1}^{\Omega(X,L)}} & \searrow^{E^{m+1}(\Omega(f|_{(X,L)}))} & \\
 & & E^m(\Omega(X',L')) & \hookrightarrow & & & E^{m+1}(\Omega(X',L')) \\
P^{m-1}(\Omega(X,L)) & & \hookrightarrow & & P^m(\Omega(X,L)) & & \\
 & \searrow_{P^{m-1}(\Omega(f|_{(X,L)}))} & \downarrow{\scriptstyle p_m^{\Omega(X',L')}} & & & \searrow_{P^m(\Omega(f|_{(X,L)}))} & \downarrow{\scriptstyle p_{m+1}^{\Omega(X',L')}} \\
 & & P^{m-1}(\Omega(X',L')) & \hookrightarrow & & & P^m(\Omega(X',L')).
\end{array}$$

We give here another kind of naturality of the ladder (2.3) in $\mathcal{T}^A$ induced from the structure map $\sigma : K \to P^m(\Omega(X,L))$ of $\mathrm{cat}(X; K, L{:}A) \leq m$.

LEMMA 2.2. *For any triad $(X; K, L{:}A)$ in $\mathcal{T}^A$ with a compression $\sigma : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$, there is a sequence of maps $\sigma_n : P^n(\Omega(X,K)) \to P^{m+n}(\Omega(X,L))$ $(n \geq 0)$ with $\sigma_0 = \sigma$, which makes the following diagram commutative up to homotopy relative A.*

$$\begin{array}{ccccc}
P^{n-1}(\Omega(X,K)) & \hookrightarrow & P^n(\Omega(X,K)) & \xrightarrow{e_n^{(X,K)}} & X \\
\downarrow{\scriptstyle \sigma_{n-1}} & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \mathrm{id}_X} \\
P^{m+n-1}(\Omega(X,L)) & \hookrightarrow & P^{m+n}(\Omega(X,L)) & \xrightarrow{e_{m+n}^{(X,L)}} & X.
\end{array} \tag{2.4}$$

Proof. We construct σ_n inductively on $n \geq 1$: the homotopy commutativity relative A of (2.5) without the dotted arrow induces a map of fibres in $\mathcal{T}$, $\hat{\sigma}_n : E^n(\Omega(X,K)) \to E^{m+n}(\Omega(X,L))$.

$$\begin{array}{ccccc} E^n(\Omega(X,K)) & \xrightarrow{p_n^{\Omega(X,K)}} & P^{n-1}(\Omega(X,L)) & \xrightarrow{e_{n-1}^{(X,K)}} & X \\ \Big\downarrow{\scriptstyle \hat{\sigma}_n} & & \Big\downarrow{\scriptstyle \sigma_{n-1}} & & \Big\downarrow{\scriptstyle \mathrm{id}_X} \\ E^{m+n}(\Omega(X,L)) & \xrightarrow{p_{m+n}^{\Omega(X,L)}} & P^{m+n-1}(\Omega(X,L)) & \xrightarrow{e_{m+n-1}^{(X,L)}} & X. \end{array} \tag{2.5}$$

A standard argument shows that the homotopy commutativity of the left square implies the existence of $\sigma_n : P^n(\Omega(X,L)) \to P^{m+n}(\Omega(X,L))$ which makes (2.4) commutative up to homotopy relative A.

3. Properties of a new relative L-S category. Here we prove Theorem 1.14 and some consequences. For that we need

LEMMA 3.1. $\mathrm{cat}(X;K,L{:}A) \le m$ *if and only if the inclusion* $g : K \hookrightarrow X$ *is compressible into* $P^m(\Omega(X,L)) \subset P^\infty(\Omega(X,L)) \simeq X$ *relative* A *as* $\sigma : K \to P^m(\Omega(X,L))$ *the structure map for* $\mathrm{cat}(X;K,L{:}A) \le m$.

Proof. Let us assume that $\mathrm{cat}(X;K,L{:}A) \le m$. Then by the definition of the relative category, the diagonal map $\Delta^{m+1}|_K : K \hookrightarrow X \to \prod^{m+1} X$ is compressible relative A into $\mathrm{T}^{m+1}(X,L)$. This implies that there exists a map σ from K to $P^m(\Omega(X,L))$, which is a compression relative A of the inclusion $g : K \hookrightarrow X$. Conversely, we assume that there is a compression relative A of the inclusion $g : K \hookrightarrow X$ into $P^m(\Omega(X,L))$. Composing with $q_m : P^m(\Omega(X,L)) \to \mathrm{T}^{m+1}(X,L)$, we obtain a compression relative A of the diagonal map $\Delta^{m+1}|_K : K \hookrightarrow X \to \prod^{m+1} X$ into $\mathrm{T}^{m+1}(X,L)$. The following propositions complete the proof of Theorem 1.14.

PROPOSITION 3.2. *Assume* $X = K \supset L = A \supset *$. *Then* $\mathrm{cat}(X{:}A) = \mathrm{cat}(X;X,A{:}A) = \mathrm{cat}^{\mathrm{FH}}(X,A)$.

Proof. By Lemma 3.1 with $X = K$ and $L = A$, $\mathrm{cat}(X;X,A{:}A) \le m$ if and only if there is a right homotopy inverse of $e_m^{(X;X:A)} : P^m(\Omega(X{:}A)) \to X$ relative A, which is equivalent to $\mathrm{cat}^{\mathrm{FH}}(X,K) \le m$.

PROPOSITION 3.3. *Assume* $X \supset K \supset L = A = *$. *Then* $\mathrm{cat}(X;K) = \mathrm{cat}(X;K,*{:}*) = \mathrm{cat}^{\mathrm{BG}}(X,K)$.

Proof. By Lemma 3.1 with $A = *$, $\mathrm{cat}(X;K) \le m$ if and only if the inclusion $K \hookrightarrow X$ is compressible into $P^m(\Omega(X))$, which is equivalent to $\mathrm{cat}^{\mathrm{BG}}(X,K) \le m$.

PROPOSITION 3.4. *Assume* $X = K \supset L \supset A = *$. *Then* $\mathrm{cat}(X,L) = \mathrm{cat}(X;X,L{:}*) = \mathrm{cat}^{\mathrm{AL}}(X,L)$.

Proof. By Lemma 3.1 with $X = K$ and $A = *$, $\mathrm{cat}(X,L) = \mathrm{cat}(X;X,L{:}*) \le m$ if and only if there is a right homotopy inverse of $e_m^{(X;X,L)} : P^m(\Omega(X,L)) \to X$, which is equivalent to $\mathrm{cat}^{\mathrm{AL}}(X,L) \le m$.

For relative L-S categories, one has:

THEOREM 3.5. *(1) Let $(X;K,L{:}A)$ be a triad in $\mathcal{T}^A$. Then*

$$\mathrm{cat}(X;K,L{:}A) \le \mathrm{cat}(X;K{:}A) \le \mathrm{cat}(X;L{:}A) + \mathrm{cat}(X;K,L{:}A),$$
$$\mathrm{cat}(X;K,L{:}A) \le \mathrm{cat}(X,L{:}A) \le \mathrm{cat}(X,K{:}A) + \mathrm{cat}(X;K,L{:}A).$$

More generally, for any maps $f:(L{:}A) \to (X{:}A)$ and $g:(K{:}A) \to (X{:}A)$,

$$\mathrm{cat}(g,f) \le \mathrm{cat}(g,*_A) \le \mathrm{cat}(f,*_A) + \mathrm{cat}(g,f),$$
$$\mathrm{cat}(g,f) \le \mathrm{cat}(1_{(X:A)},f) \le \mathrm{cat}(1_X,g) + \mathrm{cat}(g,f),$$

where $1_X : (X{:}A) = (X{:}A)$ denotes the identity and $*_A : (A{:}A) \hookrightarrow (X{:}A)$ denotes the trivial inclusion.

(2) If $(X',L'{:}A) \supset (X,L{:}A)$ and $(K'{:}A') \subset (K{:}A)$, then

$$\mathrm{cat}(X';K',L'{:}A') \le \mathrm{Min}\{\mathrm{cat}(X';K,L'{:}A), \mathrm{cat}(X;K',L{:}A')\}$$
$$\le \mathrm{Max}\{\mathrm{cat}(X';K,L'{:}A), \mathrm{cat}(X;K',L{:}A')\} \le \mathrm{cat}(X;K,L{:}A).$$

More generally, for any maps $f':(L'{:}A) \to (X'{:}A)$, $f:(L{:}A) \to (X{:}A)$, $g:(K{:}A) \to (X{:}A)$, $h:(X{:}A) \to (X'{:}A)$, $k:(K'{:}A') \to (K{:}A)$ and $\ell:(L{:}A) \to (L'{:}A)$ which satisfy the relation $f'\circ\ell = h\circ f$, we have

$$\mathrm{cat}(h\circ g\circ k, f') \le \mathrm{Min}\{\mathrm{cat}(h\circ g, f'), \mathrm{cat}(g\circ k, f)\}$$
$$\le \mathrm{Max}\{\mathrm{cat}(h\circ g, f'), \mathrm{cat}(g\circ k, f)\} \le \mathrm{cat}(g,f).$$

The following corollaries are immediate consequences of Theorem 3.5:

COROLLARY 3.6. *(1) For a triad $(X;K,L{:}*)$ in $\mathcal{T}_*$, we have*

$$\mathrm{cat}(X;K,L) \le \mathrm{cat}(X;K) = \mathrm{cat}^{\mathrm{BG}}(X,K)$$
$$\le \mathrm{cat}(X;L) + \mathrm{cat}(X;K,L) = \mathrm{cat}^{\mathrm{BG}}(X,L) + \mathrm{cat}(X;K,L),$$
$$\mathrm{cat}(X;K,L) \le \mathrm{cat}(X,L) = \mathrm{cat}^{\mathrm{AL}}(X,L)$$
$$\le \mathrm{cat}(X,K) + \mathrm{cat}(X;K,L) = \mathrm{cat}^{\mathrm{AL}}(X,K) + \mathrm{cat}(X;K,L).$$

(2) For a pair $(X,L{:}A)$ in $\mathcal{T}^A$, we have

$$\mathrm{cat}(X,L{:}A) \le \mathrm{cat}(X{:}A) = \mathrm{cat}^{\mathrm{FH}}(X,A)$$
$$\le \mathrm{cat}(X;L{:}A) + \mathrm{cat}(X,L{:}A) \le \mathrm{cat}(X;L{:}A) + \mathrm{cat}^{\mathrm{FH}}(X,L).$$

If we further assume that $A = *$, then

$$\mathrm{cat}(X,L) \le \mathrm{cat}(X) \le \mathrm{cat}(X;L) + \mathrm{cat}(X,L).$$

(3) For maps $f : L \subset X$, $f' : * \subset Y$, $g = 1_X : X \to X$, $h : X \to Y$, $k = 1_X : X \to X$ and $\ell : L \to *$ in $\mathcal{T}_*$ with $h|_L = \ell$, we have

$$\mathrm{cat}^{\mathrm{BG}}(h) = \mathrm{cat}(h,*) = \mathrm{cat}(h\circ g, f') \le \mathrm{cat}(g,f) = \mathrm{cat}^{\mathrm{AL}}(X,L).$$

In Definition 1.6, we have $\mathrm{cat}(X;F_i,F_{i-1}{:}A) \le 1$ for the filtration $\{F_i\}$. Hence we have:

COROLLARY 3.7. $\mathrm{cat}(X;F_i,A{:}A) \le i$ *for every i.*

Proof of Theorem 3.5. The case of maps is left to the reader, and we concentrate on the case of spaces.

Firstly, we show (1) for a triad $(X;K,L{:}A)$ in $\mathcal{T}^A$:

To show $\mathrm{cat}(X;K,L{:}A) \le \mathrm{cat}(X;K{:}A)$, we assume that $\mathrm{cat}(X;K{:}A) = m$. By Lemma 3.1 for the triad $(X;K,A{:}A)$, $\mathrm{cat}(X;K{:}A) = \mathrm{cat}(X;K,A{:}A) \le m$ if and only if there is a compression $\sigma : K \to P^m(\Omega(X{:}A))$ relative A of the inclusion $K \hookrightarrow X$. By Lemma 2.1 for the inclusion $(X;K,A{:}A) \hookrightarrow (X;K,L{:}A)$, the composition $P^m(\Omega(f|_{X:A}))\circ\sigma : K \to P^m(\Omega(X,L))$ gives a compression of the inclusion $K \hookrightarrow X$, which implies $\mathrm{cat}(X;K,L{:}A) \le m = \mathrm{cat}(X;K{:}A)$.

To show $\mathrm{cat}(X;K,L{:}A) \le \mathrm{cat}(X,L{:}A)$, we assume that $\mathrm{cat}(X,L{:}A) = m$. By Lemma 3.1 for the triad $(X;X,L{:}A)$, $\mathrm{cat}(X,L{:}A) \le m$ if and only if there is a compression $\sigma : X \to P^m(\Omega(X,L))$ relative A of the identity 1_X. By restricting σ to K, we obtain a compression $\sigma|_K : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$, which implies $\mathrm{cat}(X;K,L{:}A) \le m = \mathrm{cat}(X.L{:}A)$.

To show the inequality $\mathrm{cat}(X;K{:}A) \le \mathrm{cat}(X;L{:}A) + \mathrm{cat}(X;K,L{:}A)$, we assume that $\mathrm{cat}(X;L{:}A) = m$ and $\mathrm{cat}(X;K,L{:}A) = n$. By Lemma 3.1 for the triad $(X;L,A{:}A)$, $\mathrm{cat}(X;L{:}A) \le m$ if and only if there is a compression $\sigma : L \to P^m(\Omega(X{:}A))$ relative A of the inclusion $L \hookrightarrow X$. Then by Lemma 2.2 for the triad $(X;L,A{:}A)$, we have the following commutative ladder with $\sigma_0 = \sigma$ up to homotopy relative A:

$$\begin{array}{ccccc} P^{n-1}(\Omega(X,L)) & \hookrightarrow & P^n(\Omega(X,L)) & \xrightarrow{e_n^{(X,L)}} & X \\ \downarrow{\scriptstyle \sigma_{n-1}} & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \mathrm{id}_X} \\ P^{m+n-1}(\Omega(X{:}A)) & \hookrightarrow & P^{m+n}(\Omega(X{:}A)) & \xrightarrow{e_{m+n}^{(X:A)}} & X. \end{array}$$

Again by Lemma 3.1 for the triad $(X;K,L{:}L)$, $\mathrm{cat}(X;K,L{:}A) \le n$ if and only if there is a compression $\tau : K \to P^n(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$. Then the composition $\sigma_n\circ\tau : K \to P^{m+n}(\Omega(X{:}A))$ gives a compression relative A of the inclusion $K \hookrightarrow X$, which implies that $\mathrm{cat}(X;K{:}A) \le m+n = \mathrm{cat}(X;L{:}A) + \mathrm{cat}(X;K,L{:}A)$.

To show the inequality $\mathrm{cat}(X,L{:}A) \le \mathrm{cat}(X,K{:}A) + \mathrm{cat}(X;K,L{:}A)$, we assume that $\mathrm{cat}(X;K,L{:}A) = m$ and $\mathrm{cat}(X,K{:}A) = n$. By Lemma 3.1 for the triad $(X;K,L{:}A)$, $\mathrm{cat}(X;K,L{:}A) \le m$ if and only if there is a compression $\tau : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$. Then by Lemma 2.2 for the triad $(X;K,L{:}A)$, we have the following commutative ladder with $\tau_0 = \tau$ up to homotopy relative A:

$$\begin{array}{ccccc} P^{n-1}(\Omega(X,K)) & \hookrightarrow & P^n(\Omega(X,K)) & \xrightarrow{e_n^{(X,K)}} & X \\ \downarrow{\scriptstyle \tau_{n-1}} & & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \mathrm{id}_X} \\ P^{m+n-1}(\Omega(X,L)) & \hookrightarrow & P^{m+n}(\Omega(X,L)) & \xrightarrow{e_{m+n}^{(X,L)}} & X. \end{array}$$

Again by Lemma 3.1 for the triad $(X;X,K{:}A)$, $\mathrm{cat}(X,K{:}A) \le n$ if and only if there is a compression $\rho : X \to P^n(\Omega(X,K))$ relative A of the identity $1_X : X \to X$. Then the composition $\tau_n\circ\rho : X \to P^{m+n}(\Omega(X,L))$ gives a compression relative A of the identity $1_X : X \to X$, which implies that $\mathrm{cat}(X,L{:}A) \le m+n = \mathrm{cat}(X,K{:}A) + \mathrm{cat}(X;K,L{:}A)$.

Secondly, we show (2) for a triad $(X;K,L{:}A)$ with spaces $X'\supset X$, $(K'{:}A')\subset(K{:}A)$ and $(L'{:}A')\subset(L{:}A)$, which is sufficient to show that $\mathrm{cat}(X';K,L'{:}A)\le\mathrm{cat}(X;K,L{:}A)$ and $\mathrm{cat}(X;K',L{:}A')\le\mathrm{cat}(X;K,L{:}A)$:

To show $\mathrm{cat}(X';K,L'{:}A)\le\mathrm{cat}(X;K,L{:}A)$, we assume that $\mathrm{cat}(X;K,L{:}A)=m$. By Lemma 3.1 for the triad $(X;K,L{:}A)$, $\mathrm{cat}(X;K,L{:}A)\le m$ if and only if there is a compression $\sigma:K\to P^m(\Omega(X,L))$ relative A of the inclusion $K\hookrightarrow X$. Since $X'\supset X$, we have the inclusion of triads : $(X;K,L{:}A)\hookrightarrow(X';K,L'{:}A)$. Then by Lemma 2.1 for the map of triads $j:(X;K,L{:}A)\hookrightarrow(X';K,L'{:}A)$, we have the following commutative ladder up to homotopy relative A:

$$\begin{array}{ccccc}
P^{m-1}(\Omega(X,L)) & \hookrightarrow & P^m(\Omega(X,L)) & \xrightarrow{e_m^{(X,L)}} & X\\
\downarrow{\scriptstyle j_{m-1}} & & \downarrow{\scriptstyle j_m} & & \downarrow{\scriptstyle j|_X}\\
P^{m-1}(\Omega(X',L')) & \hookrightarrow & P^m(\Omega(X',L')) & \xrightarrow{e_m^{(X',L')}} & X'
\end{array}$$

with $j_0=\mathrm{id}_L$ and $j_k=P^k(\Omega(j|_{(X,L)}))$, $1\le k\le m$. Thus the map $j_m\circ\sigma$ gives a compression relative A of the inclusion $K\hookrightarrow X\subset X'$, and hence $\mathrm{cat}(X';K,L'{:}A)\le m$ $=\mathrm{cat}(X;K,L{:}A)$.

To show $\mathrm{cat}(X;K',L{:}A')\le\mathrm{cat}(X;K,L{:}A)$, we may assume that $A=A'$, since it is clear by definition that $\mathrm{cat}(X;K,L{:}A')\le\mathrm{cat}(X;K,L{:}A)$ if $A'\subset A$: let us assume that $\mathrm{cat}(X;K,L{:}A)=m$. By Lemma 3.1 for the triad $(X;K,L{:}A)$, $\mathrm{cat}(X;K,L{:}A)\le m$ if and only if there is a compression $\sigma:K\to P^m(\Omega(X,L))$ relative A of the inclusion $K\hookrightarrow X$. Hence the restriction $\sigma|_{K'}$ of the map σ to K' gives a compression relative A of the inclusion $K'\hookrightarrow X$, and hence $\mathrm{cat}(X;K',L{:}A)\le m=\mathrm{cat}(X;K,L{:}A)$.

4. A higher Hopf invariant for a triad. Let us consider the following exact sequences of abelian groups and algebraic loops:

$$0\to[\Sigma V,E^{m+1}(\Omega(X,L))]\xrightarrow{p^{(X,L)}_{m+1\,*}}[\Sigma V,P^m(\Omega(X,L))]\xrightarrow{e^{(X,L)}_{m\,*}}[\Sigma V,X]\to 0,\tag{4.1}$$

$$1\to[V,E^{m+1}(\Omega(X,L))]\xrightarrow{p^{(X,L)}_{m+1\,*}}[V,P^m(\Omega(X,L))]\xrightarrow{e^{(X,L)}_{m\,*}}[V,X].\tag{4.2}$$

Since the fibre $\Omega(X)$ of a fibration $p^{(X,L)}_{m+1}$ is contractible in the total space $E^{m+1}(\Omega(X,L))$ of $p^{(X,L)}_{m+1}$, we know $e^{(X,L)}_{m\,*}:[\Sigma V,P^m(\Omega(X,L))]\to[\Sigma V,X]$ is an epimorphism of abelian groups and $p^{(X,L)}_{m+1\,*}:[\Sigma V,E^{m+1}(\Omega(X,L))]\to[\Sigma V,P^m(\Omega(X,L))]$ is a monomorphism of abelian groups. Similarly, $p^{(X,L)}_{m+1\,*}:[V,E^{m+1}(\Omega(X,L))]\to[V,P^m(\Omega(X,L))]$ is a monomorphism of algebraic loops. Thus we obtain the following proposition:

Proposition 4.1. *(1) $e^{(X,L)}_{m\,*}:[\Sigma V,P^m(\Omega(X,L))]\to[\Sigma V,X]$ is an epimorphism of abelian groups.*

(2) $p^{(X,L)}_{m+1\,*}:[\Sigma V,E^{m+1}(\Omega(X,L))]\to[\Sigma V,P^m(\Omega(X,L))]$ is a monomorphism of abelian groups.

(3) $p^{(X,L)}_{m+1\,*}:[V,E^{m+1}(\Omega(X,L))]\to[V,P^m(\Omega(X,L))]$ is a monomorphism of algebraic loops.

We give here a definition of higher Hopf invariants in a slightly different form: let $(X;K,L{:}A)$ be a triad in $\mathcal{T}^A$, let V be a co-loop co-H-space, and let $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. We assume that $\operatorname{cat}(X;K,L{:}A) \leq m$. Then by Lemma 3.1 for the triad $(X;K,L{:}A)$, $\operatorname{cat}(X;K,L{:}A) \leq m$ implies that the inclusion $i : K \hookrightarrow X$ is compressible into $P^m(\Omega(X,L))$ relative A as a map $\sigma : K \to P^m(\Omega(X,L))$. Since $e^{(X,L)}_m \circ\sigma\circ\alpha \sim i\circ\alpha$ is trivial in $\hat{K} \subset X$, we obtain a unique lift $H^\sigma_m(\alpha) : V \to E^{m+1}(\Omega(X,L)) \simeq \Omega(X,L) * \Omega(X) * \overset{m}{\cdots} * \Omega(X)$ of $\sigma\circ\alpha$.

Definition 4.2. We define $H^{(X;K,L:A)}_m(\alpha)$ as follows:

$$H^{(X;K,L:A)}_m(\alpha) = \left\{ [H^\sigma_m(\alpha)] \;\middle|\; \begin{array}{l}\sigma : K \to P^m(\Omega(X,L)) \text{ is a compression rela-}\\ \text{tive } A \text{ of the inclusion } K \hookrightarrow X\end{array} \right\}$$
$$\subset [V, \Omega(X,L) * \Omega(X) * \overset{m}{\cdots} * \Omega(X)].$$

Now let $(K,L{:}A)$ be a pair in $\mathcal{T}^A$ and let $\alpha : V \to K$ a map in $\mathcal{T}$. We assume that $\operatorname{cat}(K,L{:}A) \leq m$. By Lemma 3.1 for the triad $(K;K,L{:}A)$, $\operatorname{cat}(K,L{:}A) \leq m$ implies that the identity $1_K : K \to K$ is compressible into $P^m(\Omega(K,L))$ relative A as a map $\sigma : K \to P^m(\Omega(K,L))$. By Lemma 2.1 for the inclusion $j : (K;K,*{:}*) \hookrightarrow (K;K,L{:}*)$, the following ladder is commutative up to homotopy:

$$\begin{array}{ccccccc}
* & \hookrightarrow & \Sigma\Omega(K) & \hookrightarrow & P^m(\Omega(K)) & \xrightarrow{e^K_m} & K \\
\big\downarrow & & \big\downarrow{\scriptstyle j_1} & & \big\downarrow{\scriptstyle j_m} & & \big\downarrow{\scriptstyle \mathrm{id}_K} \\
L & \hookrightarrow & P^1(\Omega(K,L)) & \hookrightarrow & P^m(\Omega(K,L)) & \xrightarrow[e^{(K,L)}_m]{} & K,
\end{array}$$

where $e^K_1 = e^K_m|_{\Sigma\Omega(K)} : \Sigma\Omega(K) \to K$ is given by the evaluation map (see Ganea [9] or [14]). Since V is a co-loop co-H-space, the evaluation map $e^V_1 : \Sigma\Omega(V) \to V$ admits a right homotopy inverse, say the co-H-structure map $\rho^V : V \to \Sigma\Omega(V)$ for V, by Ganea [10]. Then we have $e^K_1\circ\Sigma\Omega(\alpha)\circ\rho^V \sim \alpha\circ e^V_1\circ\rho^V \sim \alpha$, and hence $e^{(K,L)}_1\circ j_1\circ\Sigma\Omega(\alpha)\circ\rho^V \simeq \mathrm{id}_K\circ\, e^K_1\circ\Sigma\Omega(\alpha)\circ\rho^V \sim \alpha$. Since both maps $e^{(K,L)}_1\circ\sigma\circ\alpha$ and $e^{(K,L)}_1\circ j_1\circ\Sigma\Omega(\alpha)\circ\rho^V$ are homotopic to α, the difference $d(\alpha) = \sigma\circ\alpha - j_1\circ\Sigma\Omega(\alpha)\circ\rho^V$ is trivial in K. Thus we obtain a unique lift $H^\sigma_m(\alpha) : V \to E^{m+1}(\Omega(K,L)) \simeq \Omega(X,L) * \Omega(X) * \overset{m}{\cdots} * \Omega(X)$ of $d(\alpha)$.

Definition 4.3. We define $H^{(K,L:A)}_m(\alpha)$ as follows:

$$H^{(K,L:A)}_m(\alpha) = \{[H^\sigma_m(\alpha)] \mid \sigma \text{ is a compression relative } A \text{ of } 1_K\}$$
$$\subset [V, \Omega(K,L) * \Omega(K) * \overset{m}{\cdots} * \Omega(K)].$$

Proof of Theorem 1.15. Let $(X;K,L{:}A)$ be a triad in $\mathcal{T}^A$, V be a co-loop co-H-space and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. Assuming $\operatorname{cat}(X;K,L{:}A) \leq m$ and $H^{(X;K,L:A)}_m(\alpha) = 0$, we show $\operatorname{cat}(X;\hat{K},L{:}A) \leq m$: by the assumption, there is a compression $\sigma : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$ such that $\sigma\circ\alpha \sim p^{(X,L)}_{m+1}\circ H^\sigma_m(\alpha) \sim *$, and hence there is a map $\hat{\sigma} : \hat{K} \to P^m(\Omega(X,L))$ whose restriction to K is σ. Since $e^{(X,L)}_m\circ\sigma$ and the inclusion $K \hookrightarrow X$ are homotopic relative A, the difference between $e^{(X,L)}_m\circ\hat{\sigma}$ and the inclusion $\hat{K} \hookrightarrow X$ is given by an element $[\delta] \in [\Sigma V, X]$. By Proposition 4.1 (1), we have a map $\hat{\delta} : \Sigma V \to P^m(\Omega(X,L))$ such that $e^{(X,L)}_m\circ\hat{\delta} \sim \delta$. By subtracting $\hat{\delta}$ from $\hat{\sigma}$, we obtain a genuine compression

$\sigma' = \hat{\sigma} - \hat{\delta} : \Sigma V \to P^m(\Omega(X, L))$ of the inclusion $\hat{K} \to P^m(\Omega(X, L))$ relative A, where the subtraction is given by the co-action of ΣV under $K \cup_\alpha C^2V = \hat{K}$ the mapping cone of α. This implies that $\mathrm{cat}(X; \hat{K}, L{:}A) \leq m$.

We describe here the relationship among higher Hopf invariants. The following definition is essentially due to Berstein and Hilton [3]:

DEFINITION 4.4. Let $(X; K, L{:}A)$ and $(X'; K', L'{:}A)$ be triads in $\mathcal{T}^A$, V be a co-loop co-H-space, and $s : K \to \mathrm{T}^{m+1}(X, L)$ and $s' : K' \to \mathrm{T}^{m+1}(X', L')$ be compressions of $\Delta^{m+1}{\circ}i : K \hookrightarrow \prod^{m+1} X$ and $\Delta^{m+1}{\circ}i' : K' \hookrightarrow \prod^{m+1} X'$ relative A, respectively, so that $\mathrm{cat}(X; K, L{:}A) \leq m$ and $\mathrm{cat}(X'; K', L'{:}A) \leq m$. A map $f : (X; K, L{:}A) \to (X'; K', L'{:}A)$ of triads in $\mathcal{T}^A$ is called m-primitive (with respect to s and s'), if $s'{\circ}f|_K \sim \mathrm{T}^{m+1}(f|_{(X',L')}){\circ}s$.

Let $(X; K, L{:}A)$ and $(X'; K', L'{:}A)$ be triads in $\mathcal{T}^A$, and let $\mathrm{cat}(X; K, L{:}A) \leq m$ and $\mathrm{cat}(X'; K', L'{:}A) \leq m$ with compressions $s : K \to \mathrm{T}^{m+1}(X, L)$ and $s' : K' \to \mathrm{T}^{m+1}(X', L')$ of $\Delta^{m+1}{\circ}i : K \hookrightarrow \prod^{m+1} X$ and $\Delta^{m+1}{\circ}i' : K' \hookrightarrow \prod^{m+1} X'$ relative A, respectively. By using the lower right square of the diagram (2.2), we obtain structure maps σ, σ' for $\mathrm{cat}(X; K, L{:}A) \leq m$ and $\mathrm{cat}(X'; K', L'{:}A) \leq m$ corresponding to s and s', respectively by $s \sim q_m^{(X,L)}{\circ}\sigma$ and $s' \sim q_m^{(X',L')}{\circ}\sigma'$ relative A.

LEMMA 4.5. *Let $f : (X; K, L{:}A) \to (X'; K', L'{:}A)$ be a map of triads in $\mathcal{T}^A$. Then f is m-primitive with respect to s and s', if and only if $\sigma'{\circ}f|_K \sim P^m(\Omega(f|_{(X,L)})){\circ}\sigma$ relative A for the corresponding structure maps σ and σ'.*

Proof. Assume that f satisfies that $\sigma'{\circ}f|_K \sim P^m(\Omega(f|_{(X,L)})){\circ}\sigma$. By composing $q_m^{(X',L')} : P^m(\Omega(X', L')) \to \mathrm{T}^{m+1}(X', L')$ with both sides, we obtain

$$\begin{aligned} s'{\circ}f|_K &\sim q_m^{(X',L')}{\circ}\sigma'{\circ}f|_K \sim q_m^{(X',L')}{\circ}P^m(\Omega(f|_{(X,L)})){\circ}\sigma \\ &\sim \mathrm{T}^{m+1}(f|_{(X,L)}){\circ}q_m^{(X,L)}{\circ}\sigma \sim \mathrm{T}^{m+1}(f|_{(X,L)}){\circ}s \end{aligned}$$

relative A, and hence f is m-primitive with respect to s and s'. Conversely assume that f is m-primitive with respect to s and s'. Then the naturality of the lower right square of the diagram (2.2) immediately induces the homotopy $\sigma'{\circ}f|_K \sim P^m(\Omega(f|_{(X,L)})){\circ}\sigma$ relative A.

THEOREM 4.6. *Let $(X; K, L{:}A)$ and $(X'; K', L'{:}A)$ be triads in $\mathcal{T}^A$, V be a co-loop co-H-space, and $s : K \to \mathrm{T}^{m+1}(X, L)$ and $s' : K' \to \mathrm{T}^{m+1}(X', L')$ be compressions of the inclusions $i : K \hookrightarrow X$ and $i' : K' \hookrightarrow X'$ relative A, respectively, so that* $\mathrm{cat}(X; K, L{:}A) \leq m$ *and* $\mathrm{cat}(X'; K', L'{:}A) \leq m$*, respectively. Let $f : (X; K, L{:}A) \to (X'; K', L'{:}A)$ be a map of triads in $\mathcal{T}^A$ and let $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$ and $X' \supset \hat{K}' = K' \cup_{f|_K \circ \alpha} CV \supset K$. If f is m-primitive with respect to s and s', then*

$$E^{m+1}(\Omega(f|_{(X,L)}))_{\#}{\circ}H_m^{(X;K,L:A)}(\alpha) \subset H_m^{(X';K',L':A)}(f|_K{\circ}\alpha).$$

Proof. By Lemma 2.1 for $f : (X; K, L{:}A) \to (X'; K', L'{:}A)$ a map of triads in $\mathcal{T}^A$, the following diagram is commutative up to homotopy relative A:

$$
\begin{array}{ccccc}
E^{m+1}(\Omega(X,L)) & \xrightarrow{p_m^{\Omega(X,L)}} & P^m(\Omega(X,L)) & \xrightarrow{e_m^{(X,L)}} & X \\
\Big\downarrow {\scriptstyle E^{m+1}(\Omega(f|_{(X,L)}))} & & \Big\downarrow {\scriptstyle P^m(\Omega(f|_{(X,L)}))} & & \Big\downarrow {\scriptstyle f|_X} \\
E^{m+1}(\Omega(X',L')) & \xrightarrow[p_m^{\Omega(X',L')}]{} & P^m(\Omega(X',L')) & \xrightarrow[e_m^{(X',L')}]{} & X'.
\end{array}
$$

Since f is m-primitive with respect to s and s', we have the homotopy relation relative A $P^m(\Omega(f|_{(X,L)}))\circ\sigma \sim \sigma'\circ f|_K$ for the corresponding compressions σ and σ' relative A of the inclusions $i : K \hookrightarrow X$ and $i' : K' \hookrightarrow X'$, resp. Thus we have the following homotopy relation:

$$
\begin{aligned}
p_m^{\Omega(X',L')}&\circ E^{m+1}(\Omega(f|_{(X,L)}))\circ H_m^{\sigma}(\alpha) \\
&\sim P^m(\Omega(f|_{(X,L)}))\circ p_m^{\Omega(X,L)}\circ H_m^{(X;K,L:A)}(\alpha) \\
&\sim P^m(\Omega(f|_{(X,L)}))\circ\sigma\circ\alpha \sim \sigma'\circ f|_K\circ\alpha \sim p_m^{\Omega(X',L')}\circ H_m^{\sigma'}(f|_K\circ\alpha).
\end{aligned}
$$

Hence we obtain $E^{m+1}(\Omega(f|_{(X,L)}))\circ H_m^{\sigma}(\alpha) \sim H_m^{\sigma'}(f|_K\circ\alpha)$, since $p_{m\,*}^{\Omega(X',L')}$ is monic by Proposition 4.1 (3). Thus we have

$$
E^{m+1}(\Omega(f|_{(X,L)}))_{\#}H_m^{(X;K,L:A)}(\alpha) \subset H_m^{(X';K',L':A)}(f|_K\circ\alpha).
$$

This completes the proof of Theorem 4.6.

THEOREM 4.7. *Let $(X,K,L{:}A)$ be a triple in $\mathcal{T}^A$, V be a co-loop co-H-space, and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_{\alpha} CV \supset K$. If* $\operatorname{cat}(K,L{:}A) \leq m$*, then*

$$
E^{m+1}(\Omega(j|_{(K,L)}))_{\#}\circ H_m^{(K,L:A)}(\alpha) \subset H_m^{(X;K,L:A)}(\alpha),
$$

where $j : (K;K,L{:}A) \to (X;K,L{:}A)$ is the inclusion.

COROLLARY 4.8. *For the filtration $\{F_i\}$ in Definition 1.6, we have*

$$
E^{m+1}(\Omega(j_i|_{(F_i,F_{i-1})}))_{\#}\circ H_i^{(F_i,F_{i-1}:A)}(\alpha) \subset H_i^{(X;F_i,F_{i-1}:A)}(\alpha)
$$

for every i, where $j_i : (F_i;F_i,F_{i-1}{:}A) \hookrightarrow (X;F_i,F_{i-1}{:}A)$ denote the inclusion.

Proof. Proof of Theorem 4.7 Let $(X,K,L{:}A)$ be a triple in $\mathcal{T}^A$, V be a co-loop co-H-space and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_{\alpha} CV \supset K$. Assuming $\operatorname{cat}(K,L{:}A) \leq m$, we show $E^{m+1}(\Omega(j|_{(K,L)}))_{\#}H_m^{(K,L:A)}(\alpha) \subset H_m^{(X;K,L:A)}(\alpha)$, where $j : (K;K,L{:}A) \to (X;K,L{:}A)$ denotes the inclusion: By Lemma 2.1 for $j : (K;K,L{:}A) \to (X;K,L{:}A)$ an inclusion map of triads in $\mathcal{T}^A$, the following diagram is commutative up to homotopy relative A:

$$
\begin{array}{ccccc}
E^{m+1}(\Omega(K,L)) & \xrightarrow{p_m^{\Omega(K,L)}} & P^m(\Omega(X,L)) & \xrightarrow{e_m^{(K,L)}} & K \\
\Big\downarrow {\scriptstyle E^{m+1}(\Omega(j|_{(K,L)}))} & & \Big\downarrow {\scriptstyle P^m(\Omega(j|_{(K,L)}))} & & \Big\downarrow {\scriptstyle j|_K} \\
E^{m+1}(\Omega(X,L)) & \xrightarrow[p_m^{\Omega(X,L)}]{} & P^m(\Omega(X,L)) & \xrightarrow[e_m^{(X,L)}]{} & X.
\end{array}
$$

From the definition of a higher Hopf invariant, we obtain $p_m^{\Omega(K,L)}\circ H_m^{\sigma}(\alpha) \sim \sigma\circ\alpha -$

$j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V$, and hence we have the homotopy relation

$$\begin{aligned} p_m^{\Omega(X,L)} \circ E^{m+1}(\Omega(j|_{(K,L)})) \circ H_m^{\sigma}(\alpha) &\sim P^m(\Omega(j|_{(K,L)})) \circ p_m^{\Omega(K,L)} \circ H_m^{\sigma}(\alpha) \\ &\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha - P^m(\Omega(j|_{(K,L)})) \circ j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V \\ &\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha - j_1 \circ \Sigma\Omega(j|_{(K,L)}) \Sigma\Omega(\alpha) \circ \rho^V \\ &\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha - j_1 \circ \Sigma\Omega(j|_{(K,L)} \circ \alpha) \circ \rho^V \\ &\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha, \end{aligned}$$

since $j|_{(K,L)} \circ \alpha \sim *$ in X. This implies that $E^{m+1}(\Omega(j|_{(K,L)})) \circ H_m^{\sigma}(\alpha)$ is homotopic to $H_m^{P^m(\Omega(j|_{(K,L)})) \circ \sigma}(\alpha)$, and hence $E^{m+1}(\Omega(j|_{(K,L)}))_{\#} \circ H_m^{(K,L:A)}(\alpha) \subset H_m^{(X;K,L:A)}(\alpha)$.

5. Categorical length. Let F_i^X, $0 \leq i \leq m$, and F_j^Y, $0 \leq j \leq n$, be categorical sequences for $(X{:}A) \in \mathcal{T}^A$ and $(Y{:}A) \in \mathcal{T}^A$, respectively. Then for a map $f : (X{:}A) \to (Y{:}A)$, we say that f preserves categorical sequences, if $f(F_i^X) \subset F_i^Y$ for all $i \geq 0$. We first show the following:

Lemma 5.1. *Let $(X{:}A) \in \mathcal{T}^A$ be dominated by $(Y{:}A) \in \mathcal{T}^A$ with a categorical sequence of length m. Then there is a categorical sequence for $(X{:}A)$ of length m compatible with the given categorical sequence for $(Y{:}A)$, i.e., the inclusion $i : (X{:}A) \hookrightarrow (Y{:}A)$ and the retraction $r : (Y{:}A) \to (X{:}A)$ preserve categorical sequences.*

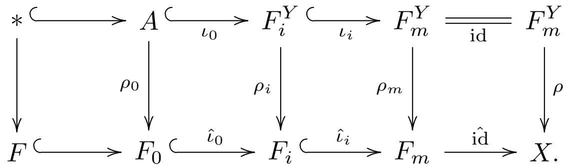
The above lemma implies the relationship between the L-S category and the categorical length.

Proof of Theorem 1.16. Assume $\mathrm{catlen}(X{:}A) = m$ with a categorical sequence $(F_i^X{:}A)$, $0 \leq i \leq m$ for $(X{:}A)$. Then by Corollary 3.7, we have $\mathrm{cat}(X{:}A) = \mathrm{cat}(X;X{:}A) = \mathrm{cat}(X;F_m^X{:}A) \leq m = \mathrm{catlen}(X{:}A)$. Hence we have $\mathrm{cat}(X{:}A) \leq \mathrm{catlen}(X{:}A)$. Conversely assume $\mathrm{cat}(X{:}A) = m$. Then the pair $(X{:}A)$ is dominated by $(P^m(\Omega(X{:}A)){:}A)$ which has the cone decomposition $(P^i(\Omega(X{:}A)){:}A)$, $0 \leq i \leq m$ as the canonical categorical sequence. Thus by Lemma 5.1, we have that $(X{:}A)$ has also a categorical sequence of length m, and hence that $\mathrm{catlen}(X{:}A) \leq m = \mathrm{cat}(X{:}A)$. This completes the proof of Theorem 1.16.

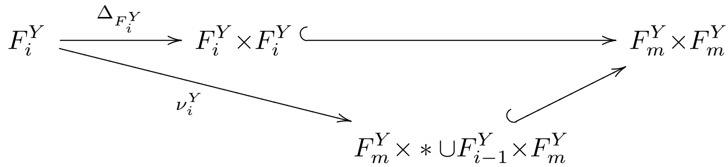
Proof of Lemma 5.1 Let $(F_i^Y{:}A)$, $0 \leq i \leq m$, be a categorical sequence for $(Y{:}A) \in \mathcal{T}^A$ and $\sigma : X \to Y$ and $\rho : Y \to X$ be maps such that $\rho \circ \sigma \sim 1_X$. Then we define F_i as the homotopy pullback of σ and the inclusion $\iota_i : F_i^Y \hookrightarrow F_m^Y$. Since the image of $\sigma|_A$ is the same as the inclusion $A \subseteq F_0^Y \hookrightarrow F_m^Y$, the space A is canonically embedded in F_0 and hence in $F_i \supset F_0$ for any $i \geq 0$.

$$\begin{array}{ccccccccc} F & \hookrightarrow & F_0 & \overset{\hat{\iota}_0}{\hookrightarrow} & F_i & \overset{\hat{\iota}_i}{\hookrightarrow} & F_m & \xrightarrow{\hat{\mathrm{id}}} & X \\ \downarrow & \mathrm{PB} & \downarrow{\scriptstyle \sigma_0} & \mathrm{PB} & \downarrow{\scriptstyle \sigma_i} & \mathrm{PB} & \downarrow{\scriptstyle \sigma_m} & \mathrm{HPB} & \downarrow{\scriptstyle \sigma} \\ * & \hookrightarrow & A & \underset{\iota_0}{\hookrightarrow} & F_i^Y & \underset{\iota_i}{\hookrightarrow} & F_m^Y & \underset{\mathrm{id}}{=\!=} & F_m^Y, \end{array}$$

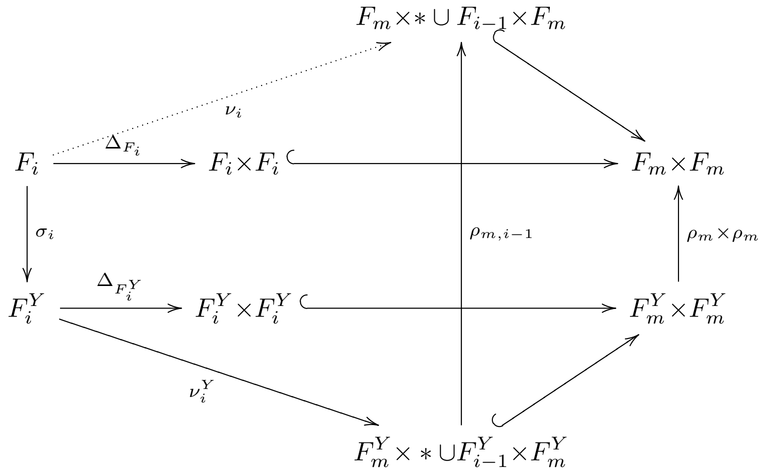
where F denotes the homotopy fibre of σ and F_m is the homotopy pullback of σ and the identity of F_m^Y. Since $\rho \circ \sigma \sim 1_X$, $\rho|_{F_i^Y}$ can be compressed into F_i and we have the following commutative diagram:



Then by the definition of categorical sequence, there is a compression $\nu_i^Y : F_i^Y \to F_m^Y \times * \cup F_{i-1}^Y \times F_m^Y$ of the diagonal map $\Delta_{F_i^Y} : F_i^Y \to F_i^Y \times F_i^Y \subseteq F_m^Y \times F_m^Y$ relative to F_{i-1}^Y:



By composing ρ_i and σ_i, we obtain a compression of the diagonal map $\Delta_{F_i} : F_i \to F_i \times F_i \subseteq F_m \times F_m$ as follows:



This implies $\mathrm{cat}(X'; X', F_{m-1}^X{:}A) \leq 1$, and hence $X' = F_m^X \supset F_{m-1} \supset \cdots \supset F_0 = A$ gives a categorical sequence for X.

The following lemma is our version of the result of Arkowitz and Lupton [1]:

LEMMA 5.2. *Let X be a space in $\mathcal{T}$ with $\mathrm{cat}(X) = m$ and $\{F_i\,;\,0 \leq i \leq m\}$ be a categorical sequence for X. Then there is a map $\mu : F_i \to F_m/F_{i-1} \vee F_m$ in $\mathcal{T}$ with axes $F_i \to F_m/F_{i-1}$ and the inclusion $F_i \hookrightarrow F_m$.*

Proof. By the definition of a categorical sequence, the diagonal map $\Delta : F_i \to F_i \times F_i \subseteq F_m \times F_m$ is compressible into $F_{i-1} \times F_m \cup F_m \times *$ as $F_i \xrightarrow{\hat\mu} F_{i-1} \times F_m \cup F_m \times * \subseteq F_m \times F_m$. Since $F_m/F_{i-1} \vee F_m$ can be regarded as the pushout of the second projection $\mathrm{pr}_2 : F_{i-1} \times F_m \to F_m$ and the canonical inclusion $\iota : F_{i-1} \times F_m \hookrightarrow F_{i-1} \times F_m \cup F_m \times *$, we have the following diagram:

$$\begin{array}{ccccc} & & F_i & & \\ & & \downarrow \hat{\mu} & \searrow \Delta & \\ F_{i-1}{\times}F_m & \hookrightarrow & F_{i-1}{\times}F_m \cup F_m{\times}* & \hookrightarrow & F_m{\times}F_m \\ \mathrm{pr}_2 \downarrow & \mathrm{PO} & \downarrow \hat{q}_i^{F_i} & \mathrm{PO} & \downarrow q_i^{F_i}\times \mathrm{id}_{F_i} \\ F_m & \xrightarrow{\mathrm{in}_2} & F_m/F_{i-1}\vee F_m & \xrightarrow{j} & F_m/F_{i-1}\times F_m, \end{array}$$

where $q_i^{F_i} : F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$ denotes the canonical collapsing map in $\mathcal{T}$. Let μ be the composition $\hat{q}_i^{F_i}\circ\hat{\mu} : F_i \to F_i/F_{i-1}\vee F_m$ so that $j\circ\mu$ is homotopic to $(q_i^{F_i}\times \mathrm{id}_{F_i})\circ\Delta$. Thus μ has axes $q_i^{F_i} : F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$ and the inclusion $F_i \hookrightarrow F_m$.

From this one immediately deduces corollary 1.17 of the introduction.

6. Cup length and module weight for the relative L-S category. A computable lower estimate in L-S category theory is given by the classical cup-length. Here we give the definition for our new relative L-S category.

DEFINITION 6.1. For any two maps $f : (L{:}A) \subset (X{:}A)$ and $g : (K{:}A) \to (X{:}A)$ in $\mathcal{T}^A$, we define cup length for $(g, f) = (X; K, L{:}A)$:

(1) Let h be a multiplicative generalized cohomology theory.
$$\mathrm{cup}(g, f; h) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l} \forall\{v_0 \in h^*(X, L); v_1, \cdots, v_m \in h^*(X, A)\} \\ g^*(v_0{\cdot}v_1{\cdots}v_m) = 0 \text{ in } h^*(K, A) \end{array} \right\}.$$

(2) $$\mathrm{cup}(g, f) = \mathrm{Max}\left\{ \mathrm{cup}(g, f; h) \;\middle|\; \begin{array}{l} h \text{ is a multiplicative generalized} \\ \text{cohomology theory} \end{array} \right\}.$$

Then we have $\mathrm{cup}(g, f; h) \leq \mathrm{cup}(g, f) \leq \mathrm{cat}(g, f)$ for any multiplicative generalized cohomology h. When h is the ordinary cohomology with a coefficient ring R, we denote $\mathrm{cup}(g, f; h)$ by $\mathrm{cup}(g, f; R)$. This definition immediately implies the following.

REMARK 6.2. For $(g, f) = (X; K, L{:}A)$, using the arguments in [16], we have $\mathrm{cup}(g, f) = \mathrm{Min}\{m{\geq}0 \mid \tilde{\Delta}_K^{m+1} : K/A \to X/L \wedge \bigwedge^m X/A \text{ is stably trivial}\}$.

Let us recall that Rudyak [23] and Strom [26] introduced a homotopy theoretical version of Fadell-Husseini's category weight (see [6]). But unfortunately, we have not been able to give a version of category weight for our new relative L-S category. In this paper, we give instead a version of module weight which is a better computable lower estimate for our relative L-S category than cup length: let $f : (L{:}A) \subset (X{:}A)$ and $g : (K{:}A) \to (X{:}A)$ be maps in $\mathcal{T}^A$ and let h be a generalized cohomology theory.

DEFINITION 6.3 ([16]). A homomorphism $\phi : h^*(Y, L) \to h^*(K, A)$ of h_*-modules is called an (unstable) h-morphism if it preserves the action of any (unstable) cohomology operation on h^*.

DEFINITION 6.4. An (unstable) module weight $\mathrm{Mwgt}(g, f; h)$ of (g, f) with respect to h is defined as follows.

$$\mathrm{Mwgt}(g,f;h) = \mathrm{Min}\left\{ m\geq 0 \;\middle|\; \begin{array}{l}\text{There is a (unstable) } h\text{-morphism } \phi :\\ h^*(P^m(\Omega(X,L)),L) \to h^*(K,A) \text{ such that}\\ \phi\circ(e^X_m)^* = g^* : h^*(X,L) \to h^*(K,A).\end{array}\right\}.$$

When h is the ordinary cohomology theory with coefficients in a ring R, we denote $\mathrm{Mwgt}(g, f; h)$ by $\mathrm{Mwgt}(g, f; R)$.

REMARK 6.5. The invariants introduced in this paper satisfy the following inequality for any generalised cohomology theory h^*:

$$\mathrm{cup}(g, f; h) \leq \mathrm{Mwgt}(g, f; h) \leq \mathrm{cat}(g, f) = \mathrm{catlen}(g, f),$$

and hence for any ring R, we have

$$\mathrm{cup}(g, f; R) \leq \mathrm{Mwgt}(g, f; R) \leq \mathrm{cat}(g, f) = \mathrm{catlen}(g, f).$$

Similar to the above definition of $\mathrm{cup}(g, f)$, we define the following invariants.

DEFINITION 6.6. For any $(g, f) = (X; K, L{:}A)$, we define

$$\mathrm{Mwgt}(g,f) = \mathrm{Max}\left\{ m\geq 0 \;\middle|\; \begin{array}{l}\mathrm{Mwgt}(g,f;h) = m \text{ for some generalized}\\ \text{cohomology theory } h\end{array}\right\}.$$

REMARK 6.7. $\mathrm{cup}(g, f) \leq \mathrm{Mwgt}(g, f) \leq \mathrm{cat}(g, f) = \mathrm{catlen}(g, f)$.

7. Examples of categorical sequences. In [3], Berstein and Hilton showed that the L-S category of the cell complex $Q(\alpha) = S^r \cup_\alpha e^{q+1}$, $\alpha \in \pi_q(S^r)$, is determined by the Hopf invariant $H_1(\alpha) \in \pi_{q+1}(S^r{\times}S^r, S^r{\vee}S^r)$ ($\cong \pi_q(\Omega(S^r) * \Omega(S^r))$ by Ganea). We can easily observe that $F_0 = *$, $F_1 = S^r$ and $F_2 = Q(\alpha)$ give a cone decomposition of $Q(\alpha)$ of length 2. If $H_1(\alpha) = 0$, then by Theorem 1.15, we obtain that $F'_0 = F_0 = *$, $F'_1 = F_1 \cup_\alpha e^{q+1} = F_2 = Q(\alpha)$ give a categorical sequence of length 1.

In [15], the author showed that the L-S category of total space $E(\beta) = Q(\beta) \cup_{\psi(\beta)} e^{q+r+1}$, $\beta \in \pi_q(S^r)$, $\psi(\beta) \in \pi_{q+r}(Q(\beta))$ is determined by $\Sigma^r H_1(\beta) \in \pi_{q+r}(\Omega(S^r) * \Omega(Q(\beta)) * \Omega(Q(\beta)))$, if $H_1(\beta) \neq 0$. We can easily observe that $F_0 = *$, $F_1 = S^r$, $F_2 = Q(\beta)$ and $F_3 = E(\beta)$ give a cone decomposition of $E(\beta)$ of length 3. If $\Sigma^r H_1(\alpha) = 0$, then by Theorem 1.15, we obtain that $F'_0 = F_0 = *$, $F'_1 = F_1 = S^r$, $F'_2 = F_2 \cup_{\psi(\beta)} e^{q+r+1} = F_3 = E(\beta)$ give a categorical sequence of length 2.

Let us denote by $Z^{(k)}$ the k-skeleton of a CW complex Z. To give an upper-bound for L-S category of the total space of a fibre bundle $F \hookrightarrow E \to B$, we need a refinement of results of Varadarajan [28] and Hardie [11], and the corresponding result for strong category of Ganea [9]:

THEOREM 7.1 ([28, 11, 9]). *(1)* $\mathrm{cat}(E){+}1 \leq (\mathrm{cat}(F){+}1){\cdot}(\mathrm{cat}(B){+}1)$.
(2) $\mathrm{Cat}(E){+}1 \leq (\mathrm{Cat}(F){+}1){\cdot}(\mathrm{Cat}(B){+}1)$.

In [18], Iwase-Mimura-Nishimoto gave a refinement in the case when the base space B is non-simply connected. On the other hand in the case when B is simply connected, Iwase-Kono [17] gave another refinement if the higher Hopf invariant of the characteristic map is 0.

By assuming the fibre F is of categorical length m, we obtain a further refinement using categorical sequence in place of cone decomposition:

THEOREM 7.2. *Let B be a $(d-1)$-connected finite dimensional CW complex ($d \geq 1$), whose cells are concentrated in dimensions $0, 1, \ldots, s \mod d$ for some s, $0 \leq s \leq d-1$. Let $F \hookrightarrow X \to B$ be a fibre bundle with fibre F whose structure group is a compact Lie group G. Then we have $\mathrm{cat}(X) \leq m + [\frac{\dim B}{d}]$, if F has a categorical sequence of length m with the following compatibility assumption for some $d \geq 1$:*

(1) $\psi|_{G^{(d\cdot(i+1)+s-1)}\times F_j} : G^{(d\cdot(i+1)+s-1)}\times F_j \to F$ is compressible into F_{i+j}, $0 \leq i, j \leq i+j \leq m$.

In [17], Kono and the author showed that there is a cone decomposition E_i, $0 \leq i \leq 8$ and E'_8 of Spin(9) of length 9, while the L-S category of Spin(9) is 8 by a combination of a higher Hopf invariant and the cone decomposition: we can easily see that the construction in §1 in [17] gives the following proposition:

PROPOSITION 7.3. *Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with a characteristic map $\alpha : \Sigma V \to Q$ a subspace of G. Then $\mathrm{catlen}(E) \leq m+n+1$ if G has a categorical sequence $* = F_0 \subset \cdots \subset F_m \simeq G$ with the following compatibility assumption for a positive integer n:*

(1) the restriction of the multiplication $\mu : G\times G \to G$ to the subspace $F_j\times Q \subseteq F_m\times F_m \simeq G\times G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j\geq 0$ as $\mu_j : F_j\times Q \to F_{j+n}$ such that $\mu_j|_{F_{j-1}\times Q} = \mu_{j-1}$.

REMARK 7.4. If we choose $n = m$, then the assumption (1) above is automatically satisfied and we always have $\mathrm{cat}(E) \leq 2\,\mathrm{cat}(G)+1$ which is a special case of a theorem of Hardie and Varadarajan [11, 28] (see Theorem 7.1 (1)).

Moreover, Lemma 1.1 in [17] implies that the higher Hopf invariant of the attaching map of the top cell of Spin(9) must vanish, since the structure map of $\mathrm{cat}(E'_8) = 8$ can be chosen to be compatible to the structure map of $\mathrm{cat}(E_8) = 8$ by the argument given in the proof of Lemma 1.1 in [17]. Hence by Theorem 1.15, we obtain that E_i, $0 \leq i \leq 7$ and E'_8 give the categorical sequence of length 8: we can easily see that the proof of Lemma 1.1 in [17] gives the following theorem:

PROPOSITION 7.5. *Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with a characteristic map $\alpha : \Sigma V \to Q$ a subspace of G. Then $\mathrm{cat}(E) \leq \mathrm{Max}\{m+n, m+2\}$ if G has a categorical sequence $* = F_0 \subset \cdots \subset F_m \simeq G$ with the following compatibility assumptions for a positive integer n:*

(1) the restriction of the multiplication $\mu : G\times G \to G$ to the subspace $F_j\times Q \subseteq F_m\times F_m \simeq G\times G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j\geq 0$ as $\mu_j : F_j\times Q \to F_{j+n}$ such that $\mu_j|_{F_{j-1}\times Q} = \mu_{j-1}$ and
(2) $H_n^{(E;Q\cup_\alpha C\Sigma V,Q;*)}(\alpha) = 0$.

These propositions imply the following result.

THEOREM 7.6. *Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with a characteristic map $\alpha : \Sigma V \to Q$, a subspace of G. Then $\mathrm{cat}(E) \leq \mathrm{Max}\{m+n+\mathrm{cat}(E; Q\cup_\alpha C\Sigma V, Q; *), m+2\}$*

if G has a categorical sequence $* = F_0 \subset \cdots \subset F_m \simeq G$ with the following compatibility assumption for a positive integer $n \geq 1$:

(1) the restriction of the multiplication $\mu : G{\times}G \to G$ to the subspace $F_j{\times}Q \subseteq F_m{\times}F_m \simeq G{\times}G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j{\geq}0$ as $\mu_j : F_j{\times}Q \to F_{j+n}$ such that $\mu_j|_{F_{j-1}\times Q} = \mu_{j-1}$.

Acknowledgments. Research of the author is supported by the Grant-in-Aid for Scientific Research #15340025 from Japan Society for the Promotion of Science.

References

[1] M. Arkowitz and G. Lupton, *Homotopy actions, cyclic and their duals*, Homology, Homotopy and Applications 7 (2005), 169–184.

[2] I. Berstein and T. Ganea, *The category of a map and of a cohomology class*, Fund. Math. 50 (1961/62), 265–279.

[3] I. Berstein and P. J. Hilton, *Category and generalised Hopf invariants*, Illinois J. Math. 12 (1968), 421–432.

[4] O. Cornea, *Some properties of the relative Lusternik-Schnirelmann category*, in: Stable and Unstable Homotopy (Toronto, 1996), Fields Inst. Commun. 19, 1998, 67–72.

[5] L. Fernández-Suárez, A. Gómez-Tato, J. Strom and D. Tanré, *The Lusternik-Schnirelmann category of $Sp(3)$*, Proc. Amer. Math. Soc. 132 (2004), 587–595.

[6] E. Fadell and S. Husseini, *Category weight and Steenrod operations*, Papers in honor of José Adem, Bol. Soc. Mat. Mexicana 37 (1992), 151–161.

[7] E. Fadell and S. Husseini, *Relative category, products and coproducts*, Rend. Sem. Mat. Fis. Milano 64 (1994), 99–115 (1996).

[8] R. H. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) 42 (1941), 333–370.

[9] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. 11 (1967), 417–427.

[10] T. Ganea, *Cogroups and suspensions*, Invent. Math. 9 (1970), 185–197.

[11] K. A. Hardie, *A note on fibrations and category*, Michigan Math. J. 17 (1970), 351–352.

[12] J. Harper, *A proof of Gray's conjecture*, in: Algebraic Topology (Evanston, 1988), Contemp. Math, 96, Amer. Math. Soc., Providence, 1989, 189–195.

[13] N. Iwase, *Ganea's conjecture on Lusternik-Schnirelmann category*, Bull. London Math. Soc. 30 (1998), 623–634.

[14] N. Iwase, *A_∞-method in Lusternik-Schnirelmann category*, Topology 41 (2002), 695–723.

[15] N. Iwase, *Lusternik-Schnirelmann category of a sphere-bundle over a sphere*, Topology 42 (2003), 701–713.

[16] N. Iwase, *The Ganea conjecture and recent developments on the Lusternik-Schnirelmann category*, Sūgaku 56 (2004), 281–296 (in Japanese).

[17] N. Iwase and A. Kono, *Lusternik-Schnirelmann category of* Spin(9), Trans. Amer. Math. Soc. 359 (2007), 1517–1526.

[18] N. Iwase, M. Mimura, T. Nishimoto, L-S categories of non-simply-connected compact simple Lie groups, Topology Appl. 150 (2005), 111–123.

[19] I. M. James, *The Topology of Stiefel Manifolds*, London Math. Soc. Lec. Notes 24, Cambridge University Press, Cambridge, 1976.

[20] I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology 17 (1978), 331–348.

[21] I. M. James, *Lusternik-Schnirelmann category*, in: Handbook of Algebraic Topology, North Holland, Amsterdam, 1995, 1293–1310.

[22] L. Lusternik and L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationnels*, Hermann, Paris, 1934.

[23] Y. B. Rudyak, *On category weight and its applications*, Topology 38 (1999), 37–55.

[24] M. Sakai, *A proof of the homotopy push-out and pull-back lemma*, Proc. Amer. Math. Soc. 129 (2001), 2461–2466.

[25] J. D. Stasheff, *Homotopy associativity of H-spaces, I, II*, Trans. Amer. Math. Soc. 108 (1963), 275–292, 293–312.

[26] J. Strom, *Essential category weight and phantom maps*, in: Cohomological Methods in Homotopy Theory (Bellaterra, 1998), Progr. Math. 196, Birkhäuser, Basel, 2001, 409–415.

[27] J. Strom, *Decomposition of the diagonal map*, Topology 42 (2003), 349–364.

[28] K. Varadarajan, *On fibrations and category*, Math. Z. 88 (1965), 267–273.

[29] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics 61, Springer, Berlin, 1978.