DIMENSION VS. GENUS:
A SURFACE REALIZATION OF THE LITTLE k-CUBES
AND AN E_∞ OPERAD

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Abstract. We define a new E_∞ operad based on surfaces with foliations which contains E_k suboperads. We construct CW models for these operads and provide applications of these models by giving actions on Hochschild complexes (thus making contact with string topology), by giving explicit cell representatives for the Dyer-Lashof-Cohen operations for the 2-cubes and by constructing new Ω spectra. The underlying novel principle is that we can trade genus in the surface representation vs. the dimension k of the little k-cubes.

Introduction. The fact [Ka1] that the cacti operad introduced in [V] has an E_2 suboperad has been instrumental for the considerations of string topology [CS, S]. In terms of algebraic actions this particular E_2 operad has been useful in describing actions on the Hochschild cochains of an associative algebra [Ka2]. All these considerations have some form of physical 1+1 dimensional field theoretical inspiration or interpretation, which for a mathematician essentially means that one is dealing with maps of surfaces. In particular the E_2 structure of the little discs and cacti is at home in such a 2-dimensional geometry.

In this context, the natural question arises if the higher order E_k operads can also be realized on surfaces. According to the yoga of string theory, two dimensional structures should be enough. In particular one should be able to describe higher dimensional objects, like branes, with strings. In our setting this translates to the expectation that there should be surface realizations for E_k operads. The fulfillment of this expectation is exactly what we accomplish. The novel feature is that these surfaces are of arbitrary genus and not only of genus zero. Now, as soon as one introduces genus into an operadic structure,

[241]

the genus ceases to be bounded. This is why we first construct an E_∞ operad using surfaces with boundaries of all genera. The way we identify E_∞ structure is to invoke Berger–Fiedorowicz's theory [B, F] of E_n and E_∞ operads. Hence we obtain a filtration of our E_∞ operad by E_n operads. This filtration is roughly by genus and it exhibits an interesting periodicity. The E_{2k} and E_{2k+1} operads are both realized basically by genus k surfaces with boundaries. More precisely, their operadic degree 2 components are realized on a surface of the indicated genus. The specific difference between the operads E_{2k} and E_{2k+1} is identified to be the possibility to twist on one particular boundary, namely the "out boundary".

The method we use for the construction is based on the $\mathcal{A}rc$ operad, whose formalism we briefly review. Just like for cacti there will be a certain tree condition. Since although arc graphs are for us the most natural language, the language of ribbon graphs is more widely used, we provide an Appendix with a dual description in terms of ribbon graphs. If one so wishes using this dictionary one can translate all the results into this purely combinatorial language thereby sacrificing their geometric origin.

In order to produce the operads, we will have to use a new technique of "stabilizing". It is clear that some identifications have to be made, since we know from representations or better algebras over the operad $H_*(E_2)$ that the Gerstenhaber bracket does not always vanish and likewise neither does the string bracket. In other words we should not be able to find a homotopy which "kills off" the cell for the bracket in the usual formalism of arcs and moduli spaces [K3, K4]. In fact, for a Frobenius algebra, we know from [K4] that the obstruction to "kill" the bracket is the Euler element of the algebra. On the geometric level we can force the homotopy, by identifying boundary components of a cellular compactification with cells comprised of lower genus surfaces. This is what our stabilization procedure formalizes. In the algebraic setting this stabilization is possible in the case that the algebra is semi-simple and has a particularly simple metric.

Our constructions can be generalized to the full $\mathcal{A}rc$ spaces and will yield a new way to stabilize moduli spaces. In the future, we also expect to find explicit formulas for the higher Dyer-Lashof-Cohen operations using the new geometric insights from the surface formalism.

The paper is organized as follows: In the first section, we review the basic setup for the arc operad to make the paper more self contained. The second section introduces the stabilization for the various tree operads. The straightforward construction yields operads without a 0-term, just like cacti. In order to obtain a 0-term for the operad we consider a thickening of the construction. Without the thickening we can add a 0-term, but then the associativity will hold only up to homotopy. In the third section, we show that stabilization and thickening yield a cellular E_∞ operad in the sense of Berger. There is a filtration on the E_∞ operad giving rise to a new surface representation for E_k operads. Without the 0-terms we can omit the thickening step. The fourth section passes to the chain level and gives cellular models as well as operations, such as $\cup_i$ and the Dyer-Lashof-Cohen operations. For the chain level, as we show, one can omit the thickening procedure as the induced structure of the stabilization is already an operad even if one includes a 0-term. The last section contains applications to string topology

and Hochschild actions as well as the construction of a new Ω spectrum. We also discuss the generalizations to the $\mathcal{A}rc$ operad and the Sullivan PROP. The Appendix contains the dual ribbon graph picture.

1. Reviewing the $\mathcal{A}rc$ operad. In order to be more self-contained, we begin with reviewing the constructions of the $\mathcal{A}rc$ operad of [KLP]. Ultimately we will specialize to a suboperad in this paper, but the gluing procedure is of course still the same. Furthermore, the more general point of view will allow for some generalizations in the future. We will follow [K3] for this abbreviated exposition. The reader familiar with these constructions may skip ahead only consulting §1.2.4 for the definition of the new suboperads we will consider.

1.1. *Spaces of graphs on surfaces.* Fix an oriented surface $F_{g,r}^s$ of genus g with s punctures and r boundary components which are labeled from 0 to $r-1$, together with marked points on the boundary, one for each boundary component. We call this data F for short if no confusion can arise.

The piece of the $\mathcal{A}rc$ operad supported on F will be an open subspace of a space $\mathcal{A}_{g,r}^s$. The latter space is a CW complex whose cells are indexed by graphs on the surface $F_{g,r}^s$ up to the action of the pure mapping class group PMC which is the group of orientation preserving homeomorphisms of $F_{g,r}^s$ modulo homotopies that pointwise fix the set which is the union of the set of the marked points on the boundary and the set of punctures. A quick review in terms of graphs is as follows.

1.1.1. *Embedded graphs.* By an embedding of a graph Γ into a surface F, we mean an embedding $i : |\Gamma| \to F$ with the conditions

    i) Γ has at least one edge.
    ii) The vertices map bijectively to the marked points on the boundaries.
    iii) No images of two edges are homotopic to each other, by homotopies fixing the endpoints.
    iv) No image of an edge is homotopic to a part of the boundary, again by homotopies fixing the endpoints.

Two embeddings are equivalent if there is a homotopy of embeddings of the above type from one to the other. Note that such a homotopy is necessarily constant on the vertices.

The images of the edges are called arcs, and the connected components of $F \setminus i(\Gamma)$ are called complementary regions.

Changing representatives in a class yields natural bijections of the sets of arcs and connected components of $F \setminus i(\Gamma)$ corresponding to the different representatives. We can therefore associate to each equivalence class of embeddings its sets of arcs together with their incidence conditions and connected components—strictly speaking of course the equivalence classes of these objects.

DEFINITION 1.1. By *a graph γ on a surface* we mean a triple $(F, \Gamma, [i])$ where $[i]$ is an equivalence class of embeddings of Γ into that surface. We will denote the isomorphism class of the set of complementary regions by $Comp(\gamma)$. We will also set $|\gamma| = |E_\Gamma|$, where

E_Γ is the set of edges of Γ. Fixing the surface F, we will call the set of graphs on a surface $\mathcal{G}(F)$.

1.1.2. *A linear order on arcs.* Notice that due to the orientation of the surface the graph inherits an induced linear order of all the flags at every vertex $F(v)$ from the embedding. Furthermore there is even a linear order on all flags by enumerating the flags first according to the boundary components on which their vertex lies and then according to the linear order at that vertex. This induces a linear order on all edges by enumerating the edges by the first appearance of a flag of that edge.

1.1.3. *The poset structure.* The set of such graphs on a fixed surface F is a poset. The partial order is given by calling $(F, \Gamma', [i']) \prec (F, \Gamma, [i])$ if Γ' is a subgraph of Γ with the same vertices and $[i']$ is the restriction of $[i]$ to Γ'. In other words, the first graph is obtained from the second by deleting some arcs.

We associate a simplex $\Delta(F, \Gamma, [i])$ to each such graph. Δ is the simplex whose vertices are given by the set of arcs/edges enumerated in their linear order. The face maps are then given by deleting the respective arcs. This allows us to construct a CW complex out of this poset.

DEFINITION 1.2. Fix $F = F_{g,r}^s$. The space $\mathcal{A}_{g,r}'^s$ is the space obtained by gluing the simplices $\Delta(F, \Gamma', [i'])$ for all graphs on the surface according to the face maps.

The pure mapping class group naturally acts on $\mathcal{A}_{g,r}'^s$ and has finite isotropy [KLP].

DEFINITION 1.3. We let $\mathcal{A}_{g,r}^s := \mathcal{A}_{g,r}'^s / PMC$ be the quotient space and call its elements arc families.

1.1.4. *CW structure of $\mathcal{A}_{g,r}^s$*

DEFINITION 1.4. Given a graph on a surface, we call its PMC orbit its *arc graph*. If γ is a graph on a surface, we denote by $\bar{\gamma}$ its arc graph or PMC orbit. We denote the set of all arc graphs of a fixed surface F by $\overline{\mathcal{G}}(F)$. A graph is called *exhaustive* if there are no vertices v with valence $val(v) = 0$. This condition is invariant under PMC and hence we can speak about exhaustive arc graphs. The set of all exhaustive arc graphs on F is denoted by $\overline{\mathcal{G}}^e(F)$.

Since the incidence conditions are preserved, we can set $|\bar{\gamma}| = |\gamma|$ where γ is any representative and likewise define $Comp(\bar{\gamma})$. We call an arc graph exhaustive if and only if it contains no isolated vertices, that is, vertices with $val(v) = 0$.

Now by construction it is clear that $\mathcal{A}_{g,r}^s$ is realized as a CW complex which has one cell of dimension $|\gamma| - 1$ for each arc graph $\bar{\gamma}$. Moreover the cell for a given class of graphs is actually a map of a simplex whose vertices correspond to the arcs in the order discussed above. The attaching maps are given by deleting edges and identifying the resulting face with its image. Due to the action of PMC some of the faces may become identified by these maps, so that the image will not necessarily be a simplex. The open part of the cell will however be homeomorphic to an open simplex, which can be taken as one of its preimages. The PMC action acts on the graph and hence acts simplicially. Let $C(\bar{\alpha})$ be the image of the cell and $\dot{C}(\bar{\alpha})$ be its interior, then

$$\mathcal{A}_{g,r}^s = \bigcup_{\bar{\alpha}\in\overline{\mathcal{G}}(F_{g,r}^s)} C(\bar{\alpha}), \quad \mathcal{A}_{g,r}^s = \coprod_{\bar{\alpha}\in\overline{\mathcal{G}}(F_{g,r}^s)} \dot{C}(\bar{\alpha}) \tag{1}$$

Let Δ^n denote the standard n-simplex and $\dot{\Delta}$ its interior. Then $\dot{C}(\gamma) = \mathbb{R}_{>0}^{|E_G|}/\mathbb{R}_{>0} = \dot{\Delta}^{|E_\Gamma|-1} =: C(\Gamma)$ which only depends on the underlying graph Γ of γ. This also means that the space $\mathcal{A}_{g,r}^s$ is filtered by the cells of dimension less than or equal to k.

1.1.5. *Elements of the $\mathcal{A}_{g,r}^s$ as projectively weighted graphs.* Using barycentric coordinates for the open part of the cells the elements of $\mathcal{A}_{g,r}^s$ are given by specifying an arc graph together with a map w from the edges of the graph E_Γ to $\mathbb{R}_{>0}$ assigning a weight to each edge such that the sum of all weights is 1.
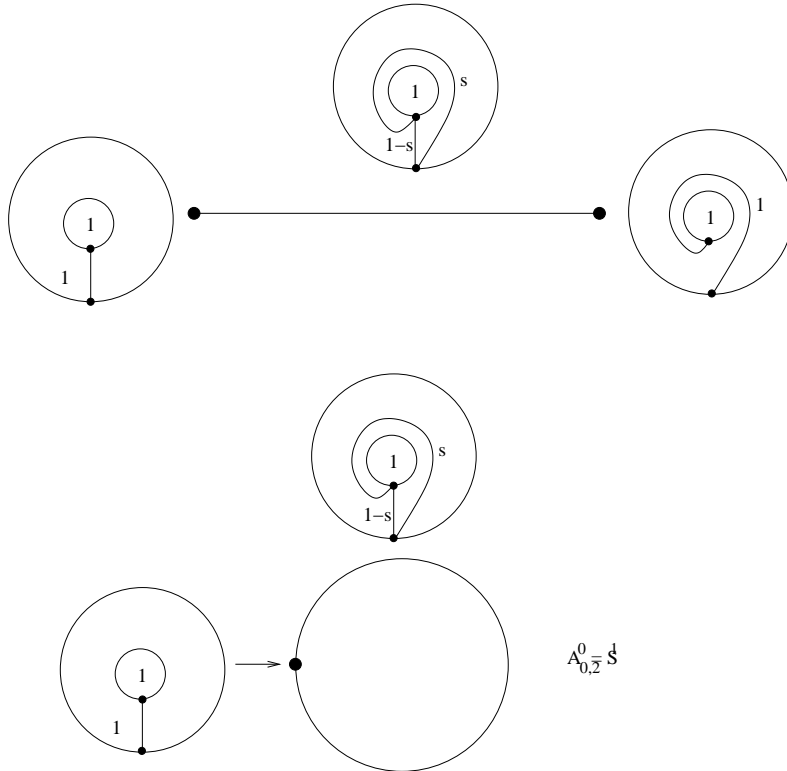

Fig. 1. The space $\mathcal{A}_{0,2}^0$ is given as the CW decomposition of S^1 with one 0-cell and one 1-cell. It can be thought of as the quotient of the interval in which the endpoints are identified by the action of the pure mapping class group. The generator of $CC_*(S^1)$ is called Δ.

Alternatively, we can regard the map $w : E_\Gamma \to \mathbb{R}_{>0}$ as an equivalence class under the equivalence relation of, i.e. $w \sim w'$ if $\exists \lambda \in \mathbb{R}_{>0}$ such that $\forall e \in E_\Gamma : w(e) = \lambda w'(e)$. That is, w is a projective metric. We call the set of $w(e)$ the projective weights of the edges. In the limit, when the projective weight of an edge goes to zero, the edge/arc is deleted, see [KLP] for more details. For an example, see Figure 1, which is discussed below in Example 1.5.

An element $\alpha \in \mathcal{A}_{g,r}^s$ can be described by a tuple $\alpha = (F, \Gamma, \overline{[i]}, w)$ where F and Γ are as above, $\overline{[i]}$ is a PMC orbit of an equivalence class of embeddings and w is a projective metric for Γ. Alternatively it can be described by a tuple $(\bar{\gamma}, w)$ where $\bar{\gamma} \in \overline{\mathcal{G}}(F)$ and w is a projective metric for the underlying abstract graph Γ.

EXAMPLE 1.5. $\mathcal{A}_{0,2}^0 = S^1$. Up to PMC there is a unique graph with one edge and a unique graph with two edges. The former gives a zero–cell and the latter gives a one-cell whose source is a 1-simplex. Its two subgraphs with one edge that correspond to the boundary lie in the same orbit of the action of PMC and thus are identified to yield S^1. The fundamental cycle is given by Δ of Figure 1. Identifying S^1 with $\mathbb{R}/\mathbb{Z}$ we define T_s to be the element corresponding to $s \in S^1$ as depicted in Figure 1.

1.1.6. *Drawing pictures for arcs.* There are several pictures one can use to view elements of $\mathcal{A}$. In order to draw elements it is useful to expand the marked point on the boundary to an interval called window, and let the arcs end on this interval according to the linear order. Equivalently, one can mark one point of the boundary and let the arcs end in their linear order anywhere *but* on this point. We will mostly depict arc graphs in the latter manner. See Figure 2 for an example of an arc graph—all arcs running to the marked points—and its alternate depiction with *none* of the arcs hitting the marked point and all arcs having *disjoint endpoints*.
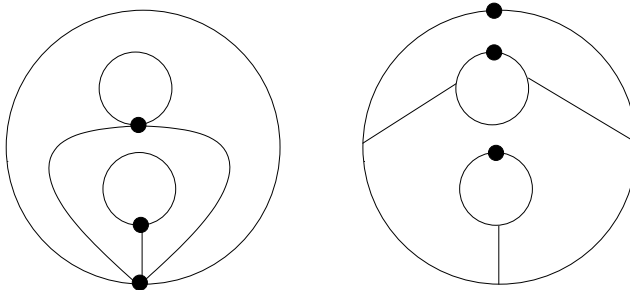

Fig. 2. An arc graph and its alternate depiction with disjoint arcs not hitting the marked points on the boundary.

NOTATION 1.6. Since in the following we will always be dealing with arc graphs, we will now omit the over-line in the notation. Hence we will write $\gamma \in \overline{\mathcal{G}}(F)$. We also fix that $\Gamma(\gamma)$ is the underlying graph. Furthermore elements of $\mathcal{A}_{g,r}^s$ will usually be called α and β. If $\alpha \in \mathcal{A}_{g,r}^s$ we fix that $\gamma(\alpha)$, $\Gamma(\alpha)$ and $w(\alpha)$ are the underlying arc graph, its underlying graph and the projective metric, respectively.

1.2. *Topological operad structure*

1.2.1. *The spaces $\mathcal{A}rc(n)$*

DEFINITION 1.7. We define $\mathcal{A}rc_g^s(n) \subset \mathcal{A}_{g,n+1}^s$ to be the subset of those weighted arc graphs whose arc graph is exhaustive. We define $\mathcal{A}rc(n) := \coprod_{s,g \in \mathbb{N}} \mathcal{A}rc_g^s(n)$.

1.2.2. *Topological description of the gluing.* We shall only give a short recap. The full details are in [KLP]. To give the composite $\alpha \circ_i \alpha'$ for two arc families $\alpha = (F, \Gamma, \overline{[i]}, w) \in \mathcal{A}rc(m)$ and $\alpha' = (F', \Gamma', \overline{[i']}, w') \in \mathcal{A}rc(n)$ one most conveniently chooses metrics on F and F'. The construction does not depend on the choice. With this metric, one produces a partially measured foliation in which the arcs are replaced by bands of parallel leaves (parallel to the original arc) of width given by the weight of the arc. For this we choose the window representation and also make the window tight in the sense that there is
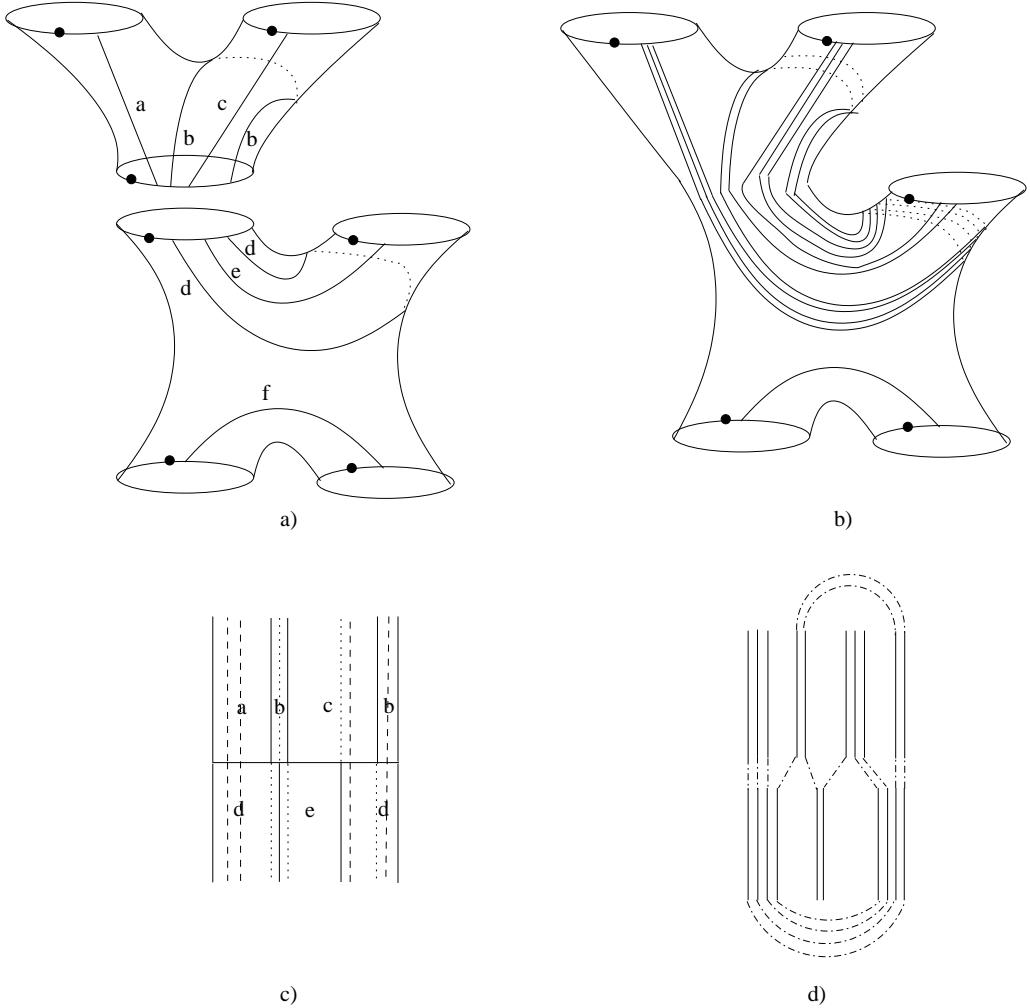

Fig. 3.   Example of gluing the top arc family to the bottom arc family. a) The arc graphs which are to be glued assuming the relative weights a,b,c,d and e as indicated by the solid lines in c). b) The result of the gluing (the weights are according to c). c) The combinatorics of cutting the bands. The solid lines are the original boundaries, the dotted lines are the first cuts and the dashed lines represent the recursive cuts. d) The combinatorics of splitting, and joining flags.

no space between the bands and between the end-points of the window and the bands. Finally, we put in the separatrices. The normalization we choose is that the sum of the weights at boundary i of α coincides with the sum of the weights at the boundary 0, we can also fix them both to be one. Now when gluing the boundaries, we match up the windows, which have the same width, and then just glue the foliations. This basically means that we glue two leaves of the two foliations if they end on the same point. We then delete the separatrices. Afterwards, we collect together all parallel leaves into one band. In this procedure, some of the original bands might be split or "cut" by the separatrices. We assign to each band one arc with weight given by the width of the consolidated band. If arcs occur, which do not hit the boundaries, then we simply delete these arcs. We call these arcs or bands "closed loops" and say that "closed loops appear in the gluing".

Notice that after gluing there will be no parallel arcs, since all parallel leaves are collected into *one* band and the condition of being parallel is PMC invariant—before and after gluing.

THEOREM 1.8 ([KLP]). *Together with the gluing operations above, the spaces $\mathcal{A}rc$ form a cyclic operad.*

Another way to see the gluing is in terms of duplicating arcs and gluing the complementary regions. The duplication occurs when inserting the separatrices or equivalently cutting the bands; see Figure 3 for an example.

1.2.3. *Cutting: "co-operad structure".* We will often be interested in the dual structure to gluing, that of cutting. In order to cut a surface into two components such that their operadic composition is the original surface we have to specify the following data: a separating curve c and a point p on c. The point p can actually be arbitrary. In order to cut, we simply cut along c and make the images of p the marked points on the two now boundaries. Notice that when we glue, p and c just disappear.

1.2.4. *Subspaces.* We would like to recall and introduce the following notation for subspaces.

Subspace	Condition
$\mathcal{A}rc^s_{\#\ g}(n) \subset \mathcal{A}rc^s_g(n)$	Complementary regions are polygons or once punctured polygons.
$\mathcal{G}Tree(n) \subset \mathcal{A}rc(n)$	$s = 0$ and all arcs run only from boundary 0 to some boundary $i \neq 0$.
$\mathcal{C}\mathcal{G}Tree(n) \subset \mathcal{G}Tree(n)$	The *cyclic* order of the arcs at the boundary 0 is anti-compatible with the linear order at each other boundary. I.e. if $\prec_i$ is the linear order at i then $e \prec_i e'$ is equivalent to $e' \prec_0 e$.
$\mathcal{L}\mathcal{G}Tree(n) \subset \mathcal{G}Tree(n)$	The *linear* order of the arcs at the boundary 0 is anti-compatible with the linear order at each other boundary.
$\mathcal{C}orol$	Exactly one arc for each boundary $i \neq 0$ which runs to boundary 0.

We will use the subscript cp to signify $g = s = 0$ and use the notation $\mathcal{T}ree :=$ $\mathcal{GT}ree_{cp}$ and $\mathcal{LT}ree := \mathcal{LGT}ree_{cp}$. Notice that $\mathcal{CGT}ree_{cp} = \mathcal{T}ree$, since the condition is guaranteed by the condition $g = s = 0$.

NOTATION 1.9. For a collection of subspaces $\mathcal{S}(n)$ as above we will write $\mathcal{S}_g(n)$ to indicate that the genus and the number of boundary components are fixed, of course $\mathcal{S}(n) = \coprod_g \mathcal{S}_g(n)$. The symbol $\mathcal{S}$ as a space will stand for $\coprod_{g,n} \mathcal{S}_g(n)$ and as an operad for the collection $\{\mathcal{S}(n)\}$.

1.2.5. *De-projectivized arcs.* In order to get *isomorphisms* with cacti [Ka2, V] one needs to include a factor of $\mathbb{R}_{>0}$ in these operads. The process was called de-projectivizing in [KLP]. Skipping this step one still obtains equivalences.

DEFINITION 1.10. Let $\mathcal{DA}rc_{g,r}^s := \mathcal{A}rc_{g,r}^s \times \mathbb{R}_{>0}$.

The elements of $\mathcal{DA}rc$ are graphs on surfaces with a metric, i.e. a function $w : E_\Gamma \to \mathbb{R}_{>0}$. Furthermore $\mathcal{DA}rc$ is a cyclic operad equivalent to $\mathcal{A}rc$ [KLP]. The operad structure on $\mathcal{DA}rc$ is given as follows. Let α, α' be elements of $\mathcal{DA}rc$, if the total weight at the boundary i of α is λ and the total weight at the boundary 0 of α' is μ, then first scale the metric w of α to μw and likewise scale the metric w' of α' to $\lambda w'$ and afterwards glue as above.

NOTATION 1.11. Any collection of subspaces $\mathcal{S}$ of $\mathcal{A}rc$ defines a collection of subspaces $\mathcal{DS} := \mathcal{S} \times \mathbb{R}_{>0}$ of $\mathcal{DA}rc$.

PROPOSITION 1.12. *For any suboperad $\mathcal{S}$ of $\mathcal{A}rc$ there are* isomorphisms *of operads $\mathcal{DS}/\mathbb{R}_{>0} \simeq \mathcal{S}$ where $\mathbb{R}_{>0}$ acts by scaling on the right factor $\mathbb{R}_{>0}$ of $\mathcal{DA}rc$. And these isomorphisms induce equivalences of operads: $\mathcal{DS} \sim \mathcal{S}$.* ∎

THEOREM 1.13 ([KLP, Ka1, Ka2, K3]). *$\mathcal{GT}ree, \mathcal{LGT}ree$ and $\mathcal{C}orol$ as well as their restrictions to $g = s = 0$ are suboperads (not cyclic) of the cyclic operad $\mathcal{A}rc$. The same holds for their versions in $\mathcal{DA}rc$ defined above. The spaces $\mathcal{A}rc_{\#,g}^0(n)$ form a rational suboperad and $\mathcal{DA}rc_{\#,g}^0(n)$ is a rational suboperad of $\mathcal{DA}rc$. (Here rational means that the compositions only need to be defined on a dense open set.) Furthermore the following relations hold, where the first line only holds on the level of rational operads.*

Suboperad	isomorphic operad		equivalent operad	
$\mathcal{DA}rc_{\#,g}^0(n)$	$M_{g,n+1}^{1^{n+1}}$	[K3]		
$\mathcal{DT}ree$	$\mathcal{C}acti$	[KLP]	fD_2	[Ka1]
$\mathcal{DLT}ree$	$\mathcal{C}act$	[KLP]	D_2	[Ka1]
$\mathcal{DC}orol_{cp}$	$\mathcal{SCC}$	[Ka1]	A_∞	[Ka1]

Additionally $\mathcal{CGT}ree$ is also a suboperad.

Proof. The only statement not contained in the references is the one about $\mathcal{CGT}ree$. This follows, however, in a straightforward fashion from the gluing procedure. Alternatively one can use Proposition 2.18 below. ∎

REMARK 1.14. Although the first line only deals with rational operads, it induces an isomorphism of true operads on the chain level [K3]. Here $M_{g,n+1}^{1^{n+1}}$ is the moduli space of

genus g curves with n marked points and a tangent vector at each of these marked points. The operads in the second column are as follows: $\mathcal{C}acti$ is the operad of cacti introduced in [V], $\mathcal{C}act$ is the operad of spineless cacti [Ka1] and $\mathcal{S}CC$ is the suboperad of spineless cacti with only one vertex. The operads in the third column are the familiar ones, that is, D_2 is the E_2 operad of little discs, A_∞ is the E_1 operad of little intervals and fD_2 is the framed little discs operad. The inclusion of $\mathcal{GT}ree_{cp} \subset \mathcal{A}rc_\#$ thus gives an BV_∞ (BV up to homotopy) structure to a cell model of moduli which includes an A_∞ structure.

We will deal with $\mathcal{GT}ree$, $\mathcal{LGT}ree$ and $\mathcal{CGT}ree$ in the following.

1.3. *Extended gluing.* The gluing procedure above was defined when gluing together two boundaries which have the same width of the foliations. The space $\mathcal{A}rc$ was chosen to guarantee that the boundaries are hit and hence can be scaled to agree. The extension of the gluing we wish to make is to sometimes allow gluing on a boundary with no incident arcs. In this case we glue the surface and *delete* all the arcs incident to the boundary we are gluing onto.

There will also be a gluing, where we will remember the deleted arcs. This is described in detail in §2.5.1. In this gluing we allow gaps in the foliation of a given width at the boundary 0.

As an alternative to scaling the whole surface as in $\mathcal{D}\mathcal{A}rc$ we will consider scaling only those arcs incident to the boundaries to be glued. There are three types of scalings which provide glueable foliations. Homogeneously scaling the arcs (1) at boundary i or (2) at the boundary 0 of the other surface or (3) symmetrically scaling. We will use the version (1) where we scale the arcs of the boundary i.

2. The operad $\mathcal{StGT}ree$

ASSUMPTION. From here on, we will assume that there are no punctures. Consequently we will set $s = 0$ and *drop the superscript* 0 from the terminology of [KLP], e.g. we write $\mathcal{A}rc_g(n)$ for $\mathcal{A}rc_g^0(n)$.

2.1. *Technical setup*

2.1.1. *Euler characteristic and quasi-filling arc graphs*

DEFINITION 2.2. We define the *Euler characteristic* of an element $\alpha \in A_{g,r}^s$ to be $\chi(\alpha) = |Comp(\alpha)| - |E_{\Gamma(\alpha)}|$.

PROPOSITION 2.3 ([K3]). *The following inequality holds:*

$$\chi(\alpha) \geq \chi(F(\alpha)) \tag{2}$$

and equality holds if and only if the complementary regions are polygons.

The difference $\chi(\alpha) - \chi(F(\alpha))$ measures the defect of the surface.

DEFINITION 2.4. We set $\epsilon(\alpha) = \sum_{R \in Comp(\gamma(\alpha))} (\chi(R) - 1)$ and call it *the Euler defect*. If the Euler defect is 0 we call the elements *quasi-filling*. Otherwise the element is called *unstable*.

EXAMPLE 2.5. The elements T_a have Euler defect 0 and for the graphs in Figure 7: H_a has Euler defect -1 and G has Euler defect -2.

LEMMA 2.6. *The Euler defect defines an operadic filtration on $\mathcal{A}rc$ by $\mathcal{A}rc^{(i)}$ where these are the elements of at most defect $-i$.*

Proof. It is clear that the defect may only drop, since $\chi - 1$ is additive under gluing the complementary regions, if there is no self-gluing; and if there is self-gluing then the Euler characteristic decreases. For a careful analysis of all the combinatorics that can occur see [K3]. ∎

2.1.2. *Twisting at the boundary*

DEFINITION 2.7. We define the *twisting at the boundary $i \neq 0$ of α by an angle a* to be the composition $\alpha \circ_i T_a$ and at the boundary 0 we define the twist to be $T_{1-a} \circ_1 \alpha$.

Notice $T_a \circ T_b = T_{a+b}$; here we calculate in $\mathbb{R}/\mathbb{Z}$. The effect of a twist is to move the boundary point by the angle a measured in units of 2π.

DEFINITION 2.8. An arc graph is called *twisted at the boundary i* if the first and last edges at that boundary become homotopic, if one allows the endpoint on the boundary i to vary considering the marked point of the boundary i as part of the boundary.

An arc graph is called *untwisted* if it is not twisted at any boundary. It is called *possibly twisted at 0* if it is untwisted at all boundaries i with $i \neq 0$.

An element of $\mathcal{A}$ is called *twisted or untwisted* at a boundary if the underlying graph is. And likewise possibly twisted at 0 if its arc graph is.

LEMMA 2.9. *An element $\alpha \in \mathcal{A}rc$ twisted at the boundary i can be decomposed as $\alpha' \circ_i \tau$ if $i \neq 0$ or $\tau \circ_1 \alpha'$ for some $\tau \in \mathcal{A}rc_0(1)$ and α' not twisted at the boundary i.*

Proof. If a boundary is twisted, then it becomes untwisted by moving the boundary point through one of the two parallel bands. This corresponds to a composition with T_a for some a. Since $T_{a-1} \circ_1 T_a = T_0$, if we assume that $i \neq 0$ we see that $\alpha = \alpha' \circ_i T_{a-1}$ with $\alpha' = \alpha \circ_i T_a$ and analogously for $i = 0$. ∎

Let Γ as usual denote the full operadic composition as opposed to the pseudo operadic compositions $\circ_i$.

COROLLARY 2.10. *Any $\alpha \in \mathcal{A}rc(n)$ can be written as $\Gamma(\Gamma(\tau_0, \alpha'), \tau_1, \ldots, \tau_n)$ with $\tau_i \in \mathcal{A}rc_0(1)$ and α' untwisted.*

REMARK 2.11. Note that this decomposition is not canonical in general.

LEMMA 2.12. *When cutting an element into two elements, we can always choose the point p on the cutting curve in such a manner that one of the new boundaries is untwisted.*

Proof. For this we first consolidate all bands that become parallel after cutting on one of the two surfaces. Now we choose the point p not to lie inside any of these consolidated bands. ∎

2.2. *The structure of $\mathcal{GT}ree$.* In this section, we show that elements in $\mathcal{GT}ree$ have a standard decomposition in terms of twists, an unstable element in $\mathcal{GT}ree(1)$—which can be decomposed into canonical elements from $\mathcal{GT}ree_1(1)$—and an untwisted quasi-filling element; here we used Notation 1.9.

2.2.1. *Twisting in $\mathcal{GT}ree$*

LEMMA 2.13. *If $\alpha, \beta \in \mathcal{GT}ree$ are both untwisted then for every possible i, $\alpha \circ_i \beta$ is untwisted.*

If $\alpha, \beta \in \mathcal{GT}ree$ are both possibly twisted at zero, then for every possible i $\alpha \circ_i \beta$ is possibly twisted at 0.

Proof. The first statement is immediate. For the second statement, we have to use the fact that all arcs run to 0. Thus after gluing, if two arcs would become parallel after allowing the endpoints to vary across the marked point on a boundary different from 0, then they would have to be parallel starting at the boundary 0 up to the separating curve which was the glued boundary and furthermore the marked point on that curve would have had to lie between them. Hence the two arcs in question have to be continued by parallel arcs, contrary to the assumption. ■

COROLLARY 2.14. *The subspaces of untwisted elements and those of possibly twisted at 0 are suboperads. The former will be given a superscript $'$ and the latter a superscript 0.*

Particular examples are $\mathcal{CGT}ree^0 = \mathcal{LGT}ree$; and the suboperads $\mathcal{LGT}ree'$ and $\mathcal{GT}ree^0$ and $\mathcal{GT}ree'$.

LEMMA 2.15. *Any element $\alpha \in \mathcal{CGT}ree(n)$ can be canonically written as $\alpha = \Gamma(\alpha', \tau_1, \ldots, \tau_n)$ with $\tau_i \in \mathcal{Arc}_0(1)$ and $\alpha' \in \mathcal{LGT}ree(n)$.*

Proof. Just like in [Ka1] the main point is that the space $\mathcal{CGT}ree(n)$ is a trivial $(S^1)^{\times n}$ bundle over $\mathcal{CGT}ree^0(n) = \mathcal{LGT}ree$. The fact that forgetting the marked points on the boundaries different from zero is an $(S^1)^{\times n}$ bundle is clear. The section is given as follows: all the bands hit 0 and the cyclic orders are compatible. This means that going around the boundary zero for each boundary there is a first band that appears. The first leaf of this band defines a canonical point on the i-th boundary. Now the marked point on this boundary is then determined by the distance (using the partial measure on the foliation) from this point. Since this map depends continuously on the marked point at 0 and the other marked points, it gives global co-ordinates and a global trivialization. In particular, the element α' above is the element where the canonical points are marked and the τ_i are the elements T_{a_i} which twist by the distance. ■

REMARK 2.16. In general on $\mathcal{GT}ree$ the section constructed above is actually only piecewise linear and may become discontinuous as soon as the genus is bigger or equal to one. The compatibility of the cyclic orders was key above. If a braiding occurs, the result ceases to be true.

This allows us to decode the structure of $\mathcal{CGT}ree$ as the generalization of $\mathcal{Cacti}$. Recall (see e.g. [Ka1]) that for any monoid M there is an operad $\mathcal{M}$ given by taking $\mathcal{M}(n) := M^{\times n}$ with the permutation action and the compositions given by using the diagonal embedding and the multiplication of the monoid.

PROPOSITION 2.17 ([Ka1]). *$\mathcal{T}ree$ is the suboperad of $\mathcal{Arc}$ generated by $\mathcal{LT}ree$ and $\mathcal{Arc}_0(1)$. Moreover it is a bi-crossed product of $\mathcal{LT}ree$ with the operad built on the monoid $\mathcal{Arc}_0(1) \simeq S^1$.*

PROPOSITION 2.18. $\mathcal{CGT}ree$ *is the suboperad of* $\mathcal{A}rc$ *generated by* $\mathcal{A}rc_0(1)$ *and* $\mathcal{LGT}ree$. *Moreover it is a bi-crossed product of* $\mathcal{LGT}ree$ *with the operad built on the monoid* $\mathcal{A}rc_0(1) \simeq S^1$.

For the definition of bi-crossed products see [Ka1].

Proof. The first part follows from Corollary 2.10. In view of Lemma 2.15 and its proof, the proof of the bi-crossed product part for $\mathcal{GT}ree$ is analogous to the argument given in [Ka1] for $\mathcal{T}ree$. ∎

2.2.2. *Classifying elements in* $\mathcal{GT}ree$

LEMMA 2.19.   *Any unstable element* $\alpha \in \mathcal{GT}ree(n)$ *can be written as* $\alpha_1 \circ_1 \alpha'$ *with* $\alpha_1 \in \mathcal{GT}ree(1)$ *unstable and* α' *quasi-filling.*

Proof. Since all the complementary regions border the boundary 0 we can decompose α as $\alpha_1 \circ_1 \alpha'$ with α' quasi-filling and α_1 unstable by "sliding down" the defects and cutting with a separating curve. ∎
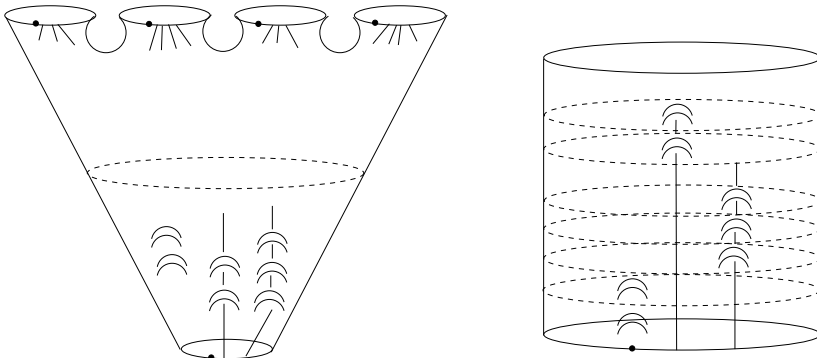
An example of this procedure is given in Figure 4.

Fig. 4.   Decomposing an unstable arc graph into standard form. First "slide down" the defects, viz. decompose α into α', the top part, and α_1 the bottom part. Second further decompose α_1 by cutting so that there is one defect H or G per "ring".

LEMMA 2.20.   *In the above decomposition, we can furthermore decompose* α' *as* $\alpha'' \circ \alpha_0$ *where* $\alpha'' \in \mathcal{GT}ree(1)$ *is quasi-filling and* α_0 *is in* $\mathcal{GT}ree_0(n)$ *with either* α_0 *not twisted at 0 or* α'' *not twisted at 1.*

Proof. As above we can choose a cutting curve which separates the surfaces as stipulated. The additional condition about being untwisted follows from Lemma 2.12 while the fact that both α_0 and α'' have to be quasi-filling follows from Lemma 2.6. ∎

LEMMA 2.21.   *If* α *is an untwisted unstable element of* $\mathcal{GT}ree_1(1)$ *then up to twists* α *is either of the form* G *or* H_a *as depicted in Figure 7.*

Proof. The proof is a straightforward consideration in geometric topology. Since we are working up to a twist, we will omit the marked points in the consideration. If there is only one arc then up to the action of PMC the element is G. Say we have several arcs.

We cut along the first arc which after PMC action we can assume to be as in Figure 5. The resulting surface will have one boundary component and genus 1; see Figure 5. If there is a second arc, we can put it into the position as in Figure 6. After cutting along this second arc, the situation is as in the last part of Figure 6. But in this figure any arc running from a piece of the boundary marked by 0 to a piece of the boundary marked by 1 will cut the surface into a polygon. ∎
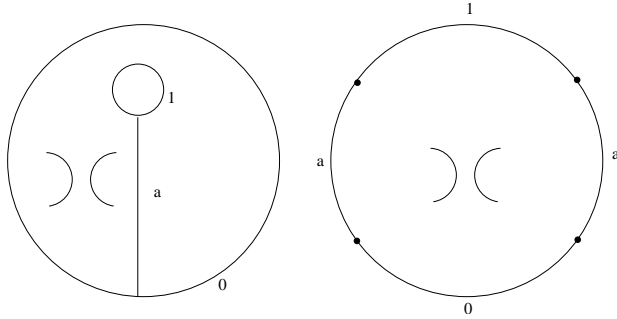

Fig. 5.   An unstable graph in $\mathcal{GT}ree_1(1)$ with one arc

We will call the sum of genera of the complementary regions the genus defect and the sum of the number of boundaries minus one of the complementary regions the boundary defect (see the Appendix for more details).
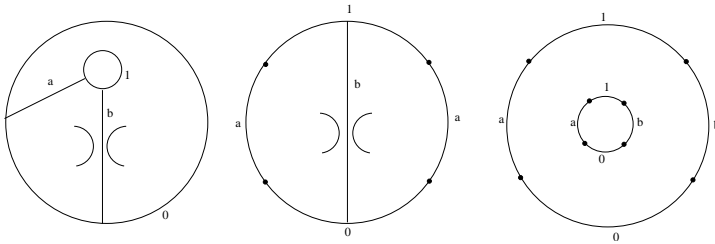

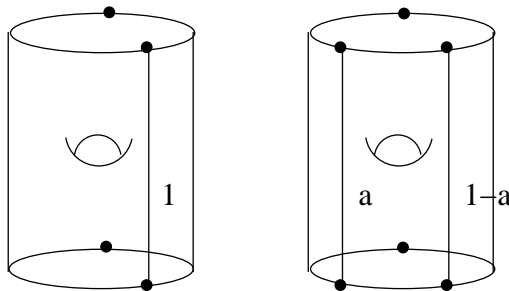Fig. 6.   An unstable graph in $\mathcal{GT}ree_1(1)$ with two arcs

Fig. 7.   The two basic unstable arc graphs G and H

LEMMA 2.22. *Any element α in $\mathcal{GT}ree(1)$ can be written as*

$$\alpha = T_{a_0} \circ_1 G \circ_1 T_{a_1} \circ_1 \cdots \circ_1 G \circ_1 T_{a_k} \circ_1 H_{b_1} \circ_1 T_{a_{k+1}} \cdots \circ_1 H_{b_l} \circ_1 T_{a_{k+l}} \circ_1 \alpha' \quad (3)$$

with α' quasi-filling and not twisted at 0. Furthermore k is the sum of the genus defects of the complementary regions and l is the sum of boundary defects.

REMARK 2.23. There are no free boundary defects in $\mathcal{GT}ree$, since all boundaries get hit.

Proof. After separating off the quasi-filling part, using Lemma 2.19, we can use separating curves to cut α so that there is at most one handle in each piece. We can furthermore arrange the handles with no curve passing through to be cut first. ∎

An example of this procedure is given in Figure 4.

PROPOSITION 2.24. *We have the following identities in $\mathcal{GT}ree$: $G \circ_1 H_a = H_a \circ_1 G$ and $H_a \circ_1 H_b = H_{b-1} \circ_1 H_{a-1}$ and furthermore for any $\alpha \in \mathcal{GT}ree : \alpha \circ_i G = G \circ_1 \alpha$ and there is some b such that $\alpha \circ_i H_a = H_b \circ_1 \alpha$.*

If $\alpha = \alpha_1 \circ_1 \alpha'$ and $\beta = \beta_1 \circ \beta'$ as in Lemma 2.19 then $\alpha \circ_i \beta = \gamma_1 \circ_1 \gamma'$ in the same notation with $\gamma' = \alpha' \circ_i T_b \circ_i \beta'$ where b is the sum of all the twists in β_1.

Proof. The first part is straightforward. For the relations for G and H we notice that we can "pull down" the handle and cut it off just like before. Then the last part follows since the intermediate twists will all add up. ∎

2.3. *Stabilizing at 0.* Notice that the compositions $\alpha \mapsto T_{-a} \circ_1 H_b \circ_1 T_a \circ_1 \alpha$ and $\alpha \mapsto T_{-a} \circ_1 G \circ_1 T_a \circ_1 \alpha$ give maps: $st_H^g(a,b) : \mathcal{GT}ree_g(n) \to \mathcal{GT}ree_{g+1}(n)$ and $st_G^g(a) : \mathcal{GT}ree_g(n) \to \mathcal{GT}ree_{g+1}(n)$.

DEFINITION 2.25. We define $\mathcal{StGT}ree(n) := \text{colim}_{\mathcal{S}} \mathcal{GT}ree(n)$ where the colimit is taken over the system of maps $\mathcal{S}$ generated by $st_G^g(a)$ and $st_H^g(b,c)$ with $a,b \in [0,1)$ and $c \in (0,1)$. We will denote the image of a subspace by the prefix $\mathcal{St}$, e.g. $\mathcal{StLGT}ree$.

We could of course also use that $G = H_0 = H_1$, but the above is maybe more natural. An example of stabilization is given in Figure 8.

PROPOSITION 2.26. *The image of an element $\alpha \in \mathcal{GT}ree$ is given as follows. Let α be decomposed as in equation (3). Then $[\alpha] = [T_b \circ_1 \alpha'] \in \mathcal{StGT}ree$ with b the sum of all the twists.*

Proof. Decompose equation (3):

$$(T_{a_0} \circ_1 G \circ_1 T_{-a_0}) \circ_1 (T_{a_0+a_1} \circ_1 G \circ_1 T_{-a_0-a_1}) \circ_1 \cdots \circ_1 (T_{a_0+\cdots+a_{k-1}} \circ_1 G \circ_1 T_{-(a_0+\cdots+a_{k-1})}) \circ_1$$

$$(T_{a_0+\cdots+a_k} \circ_1 H_{b_1} \circ_1 T_{a_0+\cdots+a_k}) \circ_1 \cdots \circ_1 (T_{\sum_0^{k+l-1} a_i} H_{b_l} \circ_1 T_{-\sum_0^{k+l-1} a_i}) \circ_1 T_{\sum_0^{k+l} a_i} \circ_1 \alpha'$$

$$= st_G(\bar{a}_0) \circ \cdots \circ st_G(\bar{a}_{k-1}) \circ st_H(\bar{a}_k, b_1) \circ st_H(\bar{a}_{k+l-1}, b_l)(T_{\bar{a}} \circ_1 \alpha') \quad (4)$$

where $\bar{a}_j := \sum_{i=0}^{j} a_i$. ∎

COROLLARY 2.27. *As spaces $\mathcal{StLGT}ree(n) = \mathcal{LGT}ree_{\#}(n)$, that is, the quasi-filling elements of $\mathcal{LGT}ree$.*
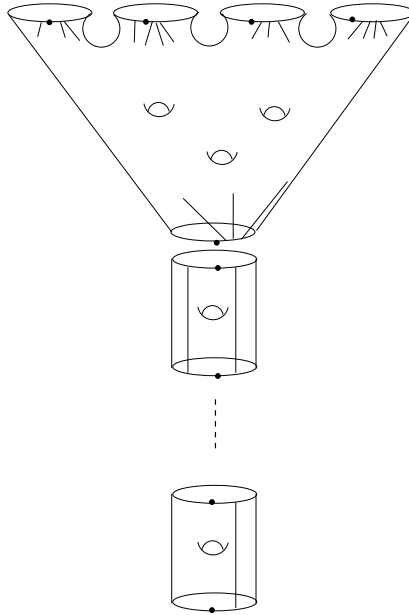
Fig. 8.   Stabilizing

THEOREM 2.28. *The operad structure of $\mathcal{GT}ree$ descends to $\mathcal{StGT}ree$. Moreover $\mathcal{StLGT}ree$ and $\mathcal{StLGT}ree'$ are suboperads.*

Proof. The fact that the operad structure descends is a direct consequence of Proposition 2.26 and Proposition 2.24. Since the stabilization adds a net twist of zero, the claims for the suboperads hold true. ∎

2.4. *Degeneracies and thickening $\mathcal{StLGT}ree$*

2.4.1. *Preoperads and weak unital operads.* Recall that a preoperad is given by a collection $\mathcal{O}(n), n > 0$ together with $\mathbb{S}_n$ actions and degeneracy maps $s_i : \mathcal{O}(n) \to \mathcal{O}(n-1)$ (see e.g. [MSS, B] for details) which are $\mathbb{S}_n$ equivariant and satisfy the usual relations.

According to the language of [M2] an operad $\mathcal{O} = \{\mathcal{O}(n)\}, n \geq 0$ is unital if $\mathcal{O}(0) = *$, that is, it is a point. Any unital operad gives a preoperad by forgetting all the structure maps, except the composition with $\mathcal{O}(0)$ and the identity $\mathbf{1}$ in $\mathcal{O}(1)$: $s_i(a) :=$ $\Gamma(a; \mathbf{1}, \ldots, \mathbf{1}, *, \mathbf{1}, \ldots, \mathbf{1})$ where $*$ is in the i-th position.

We would also like to consider the new notion of a *weak unital operad* which is given by an operad $\mathcal{O} = \{\mathcal{O}(n)\}, n > 0$ together with a preoperad structure on the collection $\mathcal{O}$. Any unital operad yields a weak unital operad by forgetting $\mathcal{O}(0)$ but retaining the induced degeneracy maps. Finally we call a weak unital topological operad a quasi-unital operad if $\mathcal{O}(n), n > 0$ form an operad $\mathcal{O}(0) = *$ and the $\mathcal{O}(n), n \geq 0$ form a quasi-operad (viz. homotopy associative) and defines a preoperad structure. The degeneracies only need to commute with the other operadic compositions up to homotopy.

NOTATION 2.29. For a preoperad, we define $\phi_{ij}^* : \mathcal{O}(n) \to \mathcal{O}(2), 1 \leq i < j \leq n$ by using the degeneracy maps in all entries *everywhere except* at i and j.

For an operad $\mathcal{O}$ with a fixed element $* \in \mathcal{O}(0)$ ϕ_{ij}^* is the map that glues in $*$ *everywhere except* at i and j, viz $a \mapsto \Gamma(a; *, \ldots, *, \mathbf{1}, \ldots, *, \mathbf{1}, * \ldots, *)$, where $\mathbf{1}$ is in the i-th and j-th position.

2.4.2. *Adding degeneracies.* The operad $\mathcal{GT}ree$ does not have a 0-th space. We can add a 0-th space $\mathcal{GT}ree(0)$ consisting of all surfaces $F_{g,1}$, that is genus g with one boundary component that has a marked point and the empty foliation. The operadic compositions are given by the extended gluing, which erases arcs.

There is a special element $*$ which is the disc D with a marked point on the boundary and without any arcs. For any arc family α we define $s_i(\alpha)$ to be the arc family resulting from gluing in D into the i-th boundary using the extended gluing of §1.3.

PROPOSITION 2.30. *Adding $\mathcal{GT}ree(0)$ gives $\mathcal{GT}ree$ the structure of a quasi-operad. Using $*$ to define degeneracies s_i gives $\mathcal{GT}ree(n), n > 0$ the structure of a preoperad. This structure descends to the stabilization, where $\mathcal{StGT}ree(0)$ is a point which is the image of D. The collection $\mathcal{StGT}ree(n), n \geq 0$ is a quasi-unital operad.*

Proof. First we notice that the spaces are stable under the extended gluing. The effect of gluing in a surface with one boundary and an empty foliation is that the boundary is filled in by the surface and all the arcs running to this boundary are deleted. First we notice that indeed this decreases the boundaries by one and secondly the result is still in $\mathcal{GT}ree$ if $n \geq 2$ as all arcs still run only from the boundaries $i \neq 0$ to 0. In case that $n = 1$ after gluing in the surface we erase all arcs and obtain a surface with an empty foliation.

Secondly although the gluing using deletion is not strictly associative, it is homotopy associative. The homotopy which is tedious to write out is given by increasing and decreasing the weights according to the erased weights. Another proof of this fact comes from the thickening construction below.

The associativity and $\mathbb{S}_n$ equivariance are clear. The fact that stabilization goes over well is straightforward.

Finally notice that we can write any of $F_{g,1}$ with the empty foliation as the composition of $G \circ_1 G \circ_1 \cdots \circ_1 G \circ_1 D$ with g factors of G. This proves the last statement. ∎

2.5. *Thickening.* Although $\mathcal{StLGT}ree$ can be made into a quasi-unital operad by simply adding a point which is the image of the disc as a 0-th component, the extended gluing will fail to be associative on the nose, however, and will only be associative up to homotopy. This would of course be enough for the homology level and is even enough for a cellular chain model (see §4), but in order to use the results of [B] and the recognition principle of [M1], we will have to have a *bona fide* unital operad.

In order to achieve this we will thicken our construction just enough to keep track of the homotopies involved.

2.5.1. *Thickening the operad.* As spaces we define $^{\text{Fat}}\mathcal{DGT}ree(n)$ for $n > 0$ to be given by pairs (α, gap) where α is a generalized weighted arc graph on a surface $F_{g,n+1}$ with marked points on the boundary and *gap* is a gap labelling function. In particular the graphs we consider are PMC orbits of finite exhaustive graphs on surfaces whose edges all run from the boundaries $i \neq 0$ to 0, where we now allow the edges to be parallel, and

gap is a map $gap : * \amalg E_\Gamma \to \mathbb{R}_{\geq 0}$, such that if e and e' with $e \prec_0 e'$ are parallel then $gap(e) > 0$. We think of the value of gap on an edge as the width of gap after this edge and the value $gap(*)$ as the gap before the first edge.

We let $|gap| := \sum_{e \in E_\Gamma} gap(e) + gap(*)$. Scaling (α, gap) by $\lambda \in \mathbb{R}_{\geq 0}$ means that we simultaneously scale all weights of α and scale gap to λgap where $(\lambda gap)(x) = \lambda \cdot gap(x)$. We call the value $gap(x)$ the width of the gap. The width may be zero. The total weight at zero $|gap| + \sum_{e \in \Gamma} w(e)$ will be positive.

We define $^{\mathrm{Fat}}\mathcal{DGT}ree(0)$ be the set of pairs (F', gap) where $F' = F_{g,1}$ is a surface with marked points on the boundary considered to have an empty foliation and $gap : * \to \mathbb{R}_{>0}$ is arbitrary. Notice that this makes the total weight at 0 positive.

2.5.2. *Thickened gluing.* The composition $(\alpha, gap) \circ_i (\alpha' \, gap')$ is defined to be the pair (α'', gap'') obtained as follows: first glue the surfaces as previously; secondly glue the foliations and gaps in the following perturbed way. As before and in [KLP] we fix a measure on the surface to turn edges with weights into bands of a foliation. As in [KLP], the construction does not depend on this choice.

1. Let w_1 be the sum $|gap|$ and the weights at 0 of α'. Let w_2 be the sum of the weights at i of α.
2. Scale (α, gap) by w_1 and (α', gap') by w_2.
3. Glue the foliations along an interval of width $w_1 w_2$ as follows. Arrange the foliations on the interval so that the ends of the scaled foliation at the boundary fill out the interval on one side and on the other side arrange the bands in the following way. The initial point of the interval corresponds to the marked points on the boundary. First, leave a gap of width $w_1 gap(*)$ then attach the first band corresponding to e_1 with width $w_1 w(e_1)$, then again leave a gap this time of width $w_1 gap(e_1)$ and so on. Now, a) fuse leaves which share the same endpoint and b) if leaves end on a gap erase the leaf, but mind the width of the gap and add its weight to the gap(s) at the boundary 0 to which the band of erased leaves are adjacent. This may result in the creation of new non-zero gaps or the consolidation of several gaps.
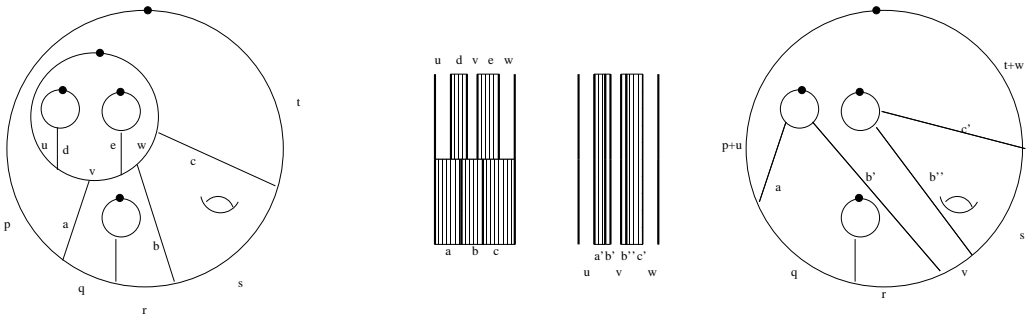4. Remove any closed leaves.

Fig. 9. Gluing with gaps. The weights at the glued boundaries are a, b, c and d, e while the weight of the gaps at the glued boundary 0 are u, v, w and the width of the gaps at the boundary 0 that is not glued are p, q, r, s, t. The new weights satisfy the equations: $a = a' + u, b = b' + v + b'', c = c' + w, d = a' + b', e = b'' + c'$.

Just like for $\mathcal{D}\mathcal{A}rc$ there is a transitive $\mathbb{R}_{>0}$ action given by scaling $\mathbb{R}_{>0} \times {}^{\text{Fat}}\mathcal{D}\mathcal{G}\mathcal{T}ree \rightarrow \mathcal{G}\mathcal{T}ree$ where $\lambda(\alpha, gap) := (\lambda\alpha, \lambda gap)$. We set ${}^{\text{Fat}}\mathcal{G}\mathcal{T}ree(n) := {}^{\text{Fat}}\mathcal{D}\mathcal{G}\mathcal{T}ree(n)/\mathbb{R}_{>0}$.

PROPOSITION 2.31. *The spaces* ${}^{\text{Fat}}\mathcal{D}\mathcal{G}\mathcal{T}ree(n), n \geq 0$ *form an operad using the gluings described above and the permutation action on the boundary labels. This operad structure descends to the collection* ${}^{\text{Fat}}\mathcal{G}\mathcal{T}ree(n), n \geq 0$.

Proof. This is a straightforward but tedious check. The basic reasoning is that instead of erasing the leaves of the foliation, we can leave them ending on the separating curve that is the image of the boundaries under the gluing. Gluing in this way is associative. Now we can erase the respective leaves after all the gluings are done and this coincides with the previously defined gluing. ∎

PROPOSITION 2.32. *There are operadic inclusions*

$$\{\mathcal{D}\mathcal{G}\mathcal{T}ree(n), n > 0\} \hookrightarrow {}^{\text{Fat}}\mathcal{D}\mathcal{G}\mathcal{T}ree \quad and \quad \{\mathcal{G}\mathcal{T}ree(n), n > 0\} \hookrightarrow {}^{\text{Fat}}\mathcal{G}\mathcal{T}ree.$$

Furthermore ${}^{\text{Fat}}\mathcal{D}\mathcal{G}\mathcal{T}ree(n), n \geq 0$ *retracts onto* $\mathcal{D}\mathcal{G}\mathcal{T}ree(n)$ *and* ${}^{\text{Fat}}\mathcal{G}\mathcal{T}ree(n), n \geq 0$ *onto* $\mathcal{G}\mathcal{T}ree(n)$.

Proof. The operadic inclusion is given by $\alpha \mapsto (\alpha, 0)$ where 0 is the constant map with value 0. For $n > 0$ the retraction is given by scaling *gap* to 0 and consolidating bands corresponding to parallel edges, by adding the weights of parallel edges and keeping only one edge per set of parallel edges. For $n = 0$ we just contract $\mathbb{R}_{>0}$ to the point 1. ∎

COROLLARY 2.33. *The operads* $\{{}^{\text{Fat}}\mathcal{D}\mathcal{G}\mathcal{T}ree(n), n > 0\}$ *and* $\{\mathcal{D}\mathcal{G}\mathcal{T}ree(n), n > 0\}$ *are equivalent.*

2.5.3. *Stabilizing* ${}^{\text{Fat}}\mathcal{G}\mathcal{T}ree$. Since we have established the inclusion, we have the system of maps $\mathcal{S}$ generated by $st_G(a)$ and $st_H(b, c)$. We also have the grading by genus of the underlying surface, which we again write as a subscript.

We set ${}^{\text{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree(n) := \text{colim}_{\mathcal{S}} {}^{\text{Fat}}\mathcal{G}\mathcal{T}ree(n)$. Notice that ${}^{\text{Fat}}\mathcal{S}t\mathcal{G}\mathcal{T}ree(0)$ is again a point. It can be given as $([D], [1])$ where $[D]$ is the image of the disc under stabilization and [1] is the orbit of the constant map $gap(*) = 1$ under the $\mathbb{R}_{>0}$ action.

In order for the operad structure to descend, we will need structure lemmata as before.

LEMMA 2.34. *For every* $\tau \in {}^{\text{Fat}}\mathcal{G}\mathcal{T}ree_0(1))$ *and* $\alpha \in {}^{\text{Fat}}\mathcal{L}\mathcal{G}\mathcal{T}ree(1)$: $\tau \circ_1 \alpha = \alpha \circ \tau$.

Proof. For this we first thicken the edges e by adding the weight $gap(e)$ and we also add $gap(*)$ to the first edge. We think of the new leaves in the bands as ending on the respective gaps. Notice that since we are in ${}^{\text{Fat}}\mathcal{L}\mathcal{G}\mathcal{T}ree(1)$ the order of the edges is preserved and all edges run from 0 to 1. Hence we have a homeomorphism between the two windows of the surface. Now we cut off a cylinder on the boundary 1. We fix the new marked point to be the translate of the base point along the foliation of the old boundary 0. After cutting off the cylinder, we create the gaps by deleting the leaves which used to end on the gaps, before thickening them. ∎

LEMMA 2.35. *Any element* α *in* ${}^{\text{Fat}}\mathcal{G}\mathcal{T}ree(n)$ *can be written as*

$$\alpha = T_{a_0} \circ_1 G \circ_1 T_{a_1} \circ_1 \cdots \circ_1 G \circ_1 T_{a_k} \circ_1 H_{b_1} \circ_1 T_{a_{k+1}} \cdots \circ_1 H_{b_l} \circ_1 T_{a_{k+l}} \circ_1 \tau \circ_1 \alpha' \quad (5)$$

with $\alpha' \in \mathcal{G}\mathcal{T}ree(n)$ *quasi-filling and not twisted at* 0, $\tau \in {}^{\text{Fat}}\mathcal{G}\mathcal{T}ree_0(1)$.

Proof. Any element α of $^{\mathrm{Fat}}\mathcal{GT}ree(n)$ can be decomposed as $\alpha = \tau \circ_1 \alpha'$ with $\tau \in {}^{\mathrm{Fat}}\mathcal{GT}ree_0(1)$ and $\alpha' \in \mathcal{GT}ree(n)$ by simply cutting off a small annulus around the boundary 0. Now we can decompose α' according to equation (3) and using Lemma 2.34 obtain the decomposition above. ∎

PROPOSITION 2.36. *The image of an element $\alpha \in {}^{\mathrm{Fat}}\mathcal{GT}ree$ is given as follows. Let α be decomposed as in equation (5). Then $[\alpha] = [T_b \circ_1 \tau \circ_1 \alpha'] \in \mathcal{StGT}ree$ with b the sum of all the twists.*

If $\alpha = \alpha_1 \circ_1 \tau \circ \alpha'$ and $\beta = \beta_1 \circ \tau' \circ \beta'$ as in Lemma 5 where we aggregated all the T, G, H terms into α_1 and β_1 then $\alpha \circ_i \beta = \gamma_1 \circ_1 \tau'' \circ_1 \gamma'$ in the same notation with $\tau'' \circ_1 \gamma' = \tau \circ_1 \alpha' \circ_i T_b \circ_i \tau' \circ_i \beta'$ where b is the sum of all the twists in β_1.

Proof. In view of Lemma 2.34 and Lemma 2.35 the first part is analogous to Proposition 2.26 and the second part is analogous to Proposition 2.24. ∎

We define $^{\mathrm{Fat}}\mathcal{LGT}ree$ and $^{\mathrm{Fat}}\mathcal{LGT}ree'$ analogously to their non-thickened counterparts. In view of the Proposition we obtain:

THEOREM 2.37. *The operad structure of $^{\mathrm{Fat}}\mathcal{GT}ree$ descends to $^{\mathrm{Fat}}\mathcal{StGT}ree$. $^{\mathrm{Fat}}\mathcal{StLGT}ree$ and $^{\mathrm{Fat}}\mathcal{StLGT}ree'$ are suboperads.* ∎

3. The E_k and E_∞ operad structures

3.1. *Berger's complete graphs operad.* Since we will use the Fiedorowicz-Berger criterion for E_∞ and E_n operads [B, F], we quickly recall the necessary definitions for the complete graph poset $\mathcal{K}$. We set $\mathcal{K}_p := \mathbb{N}^{\binom{p}{2}} \times \mathbb{S}_p$ and think of an element (μ, σ) as a collection of natural numbers $(\mu_{ij})_{1 \leq i < j < \leq p}$ and a permutation σ in the symmetric group on p letters $\mathbb{S}_p$.

The sets $\mathcal{K}_p$ form an operad under the compositions

$$\Gamma((\mu, \sigma); (\mu_1, \sigma_1), \ldots (\mu_p, \sigma_p)) = (\mu(\mu_1, \ldots, \mu_p), \sigma(\sigma_1, \cdots, \sigma_p)) \qquad (6)$$

where $\sigma(\sigma_1, \cdots, \sigma_p)$ is the usual block permutation and $(\mu(\mu_1, \ldots, \mu_p))$ is defined as follows: if i, j are in the same block, say r, then $(\mu_r)_{ij}$ is kept, if they belong to different blocks, say r and s, then one takes μ_{rs} keeping in mind the usual renumbering; see [B] for more details. The neutral element is $* = (\emptyset, \emptyset)$ and the map ϕ_{ij}^* in this case is given by $\phi_{ij}^*(\mu, \sigma) = (\phi_{ij}^*(\mu), \phi_{ij}^*(\sigma))$ and $\phi_{ij}^*(\sigma)$ is the restriction of the permutation to i and j where i is mapped to 1 and j to 2 and

$$\phi_{ij}^*(\mu) = \begin{cases} \mu_{ij} & \text{if } i < j, \\ \mu_{ji} & \text{if } j < i. \end{cases} \qquad (7)$$

Each set $\mathcal{K}_p$ is a poset with the order given by

$$(\mu, \sigma) \leq (\nu, \tau) \Leftrightarrow \forall i < j \text{ either } \phi_{ij}^*(\mu, \sigma) = \phi_{ij}^*(\nu, \tau) \text{ or } \mu_{ij} < \nu_{ij}. \qquad (8)$$

3.2. *Identifying the E_n and E_∞ operads.* We recall the following definition from [B]:

DEFINITION 3.2. Let A be a partially ordered set and X a topological space. A collection $(c_\alpha)_{\alpha \in A}$ of closed contractible subspaces (so-called "cells") of X are called a *cellular A decomposition* if the following three conditions hold:

1. $c_\alpha \subseteq c_\beta \Leftrightarrow \alpha \leq \beta$,
2. the cell inclusions are cofibrations,
3. $X = \varinjlim x_\alpha$, so X equals the union of the cells and the weak topology with respect to its cells.

LEMMA 3.3. $St\mathcal{LGT}ree(2)$ has a cellular $\mathcal{K}_2$ decomposition.

We use the term cellular here in the sense of Berger. Below, we also give a CW model where the "cells" are actual cells. In fact $St\mathcal{LGT}ree(2)$ is homeomorphic to $S^\infty \times \mathbb{R}_{>0}$ with the $\mathcal{K}_2$ being the hemispherical decomposition of Fiedorowicz and Berger [F, B] on the factor S^∞ and trivial in the $\mathbb{R}_{>0}$ factors. The cellular model will have cells exactly corresponding to the hemispherical decomposition.

Proof. By Corollary 2.27 the elements of $St\mathcal{LGT}ree(2)$ can be identified with the elements of $\mathcal{LGT}ree(2)$ that are quasi-filling. It is straightforward to verify that these elements are either of one of the two types of Figure 12 or with their labels 1 and 2 interchanged. We will call these graphs $\cup_i$ and $\tau_{12}\cup_i$. This means that the elements are indexed by the arc graphs which in turn can be indexed by $\mathbb{N} \times \mathbb{S}_2$. Here in a tuple $(i, \sigma) \in N \times \mathbb{S}_2$ i is the dimension, which is the total number of arcs of $\cup_i$ minus two and σ is either *id* or τ_{12}.

The spaces of elements of a fixed graph are contractible. For this we just decrease all the weights except on each first arc at each boundary to zero and then scale the remaining weights to the same value $1/(i+2)$. Now, the codimension 1 boundary strata of each cell are given by deleting one arc. We see from the alternating structure of the arcs that the result will be of codimension 2 unless we are deleting the first or last arc. In all other cases the element becomes unstable and after removing the degeneracy we are left with two parallel arcs which are combined into one arc hence decreasing the dimension. Contracting the first arc or the last arc, we obtain $\cup_{i-1}$ or $\tau_{12}\cup_{i-1}$.

We can actually say a little more since $\mathcal{LGT}ree(2) = \mathcal{LGT}ree^1(2) \times \mathbb{R}^2_{>0}$; see §4.1 below. The subspaces of $\mathcal{LGT}ree^1(2)$ indexed by $\cup_i$ are just $\Delta^{[(i+1)/2]} \times \Delta^{[i/2]} \sim_{\text{homeo}} B^i$ and the boundary maps glue the B^i to the S^{i-1} decomposed into two hemispheres. It is now straightforward to see that these cell inclusions are cofibrations and the topology is the weak, induced one. ∎
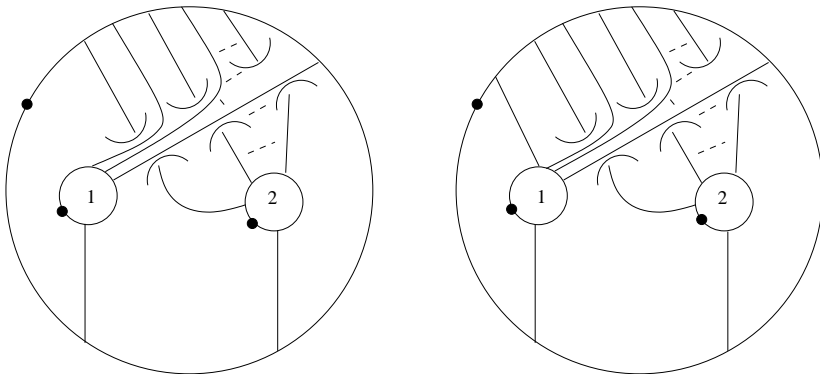

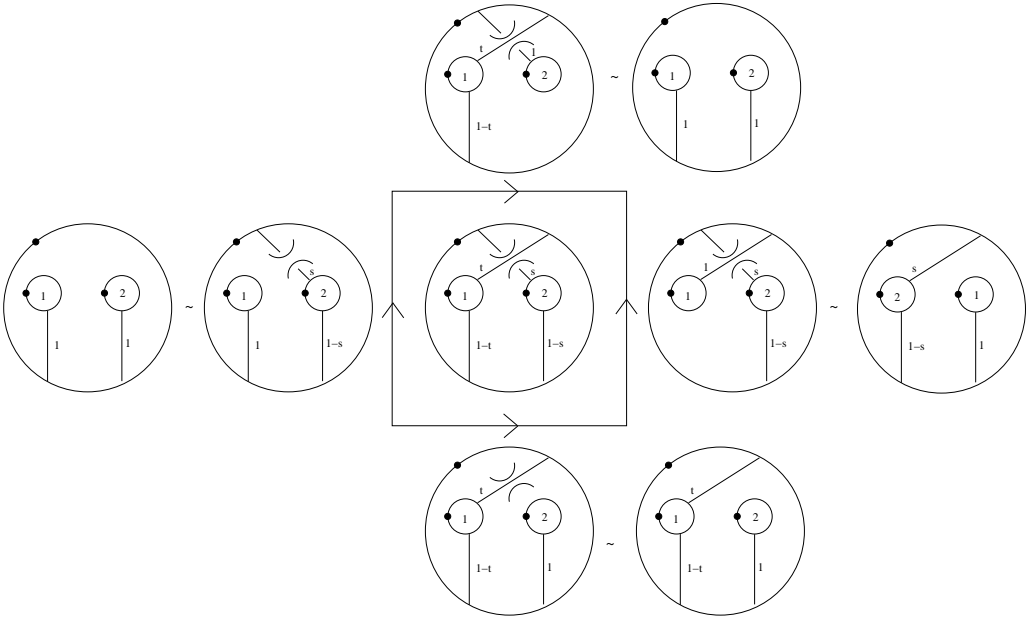Fig. 10. The $\cup_i$ operations for i even and i odd

Fig. 11. The $\cup_2$ operation and its boundary before stabilization and stable representatives of the boundary components

COROLLARY 3.4. $^{\mathrm{Fat}}\mathcal{S}t\mathcal{LGT}ree(2)$ *has a cellular* $\mathcal{K}_2$ *decomposition.*

Proof. Straightforward by contracting the gaps. ∎

We again repeat definitions of [B] extending them slightly to our setting.

DEFINITION 3.5. A topological preoperad $\mathcal{O}$ is called a *cellular* E_∞ *preoperad* if the $\mathbb{S}_2$ space $\mathcal{O}(2)$ admits a cellular $\mathcal{K}_2$ decomposition $(\mathcal{O}^{(\alpha)}(2))_{\alpha \in \mathcal{K}_2}$ compatible with the action of $\mathbb{S}_2$, such that

1. For each $p > 0$ and $\alpha \in \mathcal{K}_p$
$$\mathcal{O}(p)^{(\alpha)} := \bigcap_{1 \le i < j \le p} (\phi_{ij}^*)^{-1}(\mathcal{O}^{(\phi_{ij}^*(\alpha))}(2))$$
   is contractible, and for each $\alpha, \beta \in \mathcal{K}_p$ with $\alpha \le \beta$ the natural inclusion $\mathcal{O}^{(\alpha)}(p) \subseteq \mathcal{O}^{(\beta)}(p)$ is a cofibration.

2. Each $\mathbb{S}_p$ orbit of $\mathcal{O}(p)$ contains an ordered point, i.e. a point $x \in \mathcal{O}(p)$ whose projections $\phi_{ij}^*(x)$ belong to the interiors $\check{\mathcal{O}}^{(\mu,id)}(2)$ where $\check{c}_\alpha = c_\alpha \setminus \bigcup_{\beta < \alpha} c_\beta$.

Introduce the filtration
$$\mathcal{O}^{(n)}(p) = \bigcup_{\alpha \in \mathcal{K}_p^{(n)}} \mathcal{O}^{(\alpha)}(p) \tag{9}$$

DEFINITION 3.6. A (weak) unital operad $\mathcal{O}$ is called *a (weak) cellular* E_∞ *operad* if the underlying preoperad is a cellular E_∞ preoperad such that the operadic composition Γ
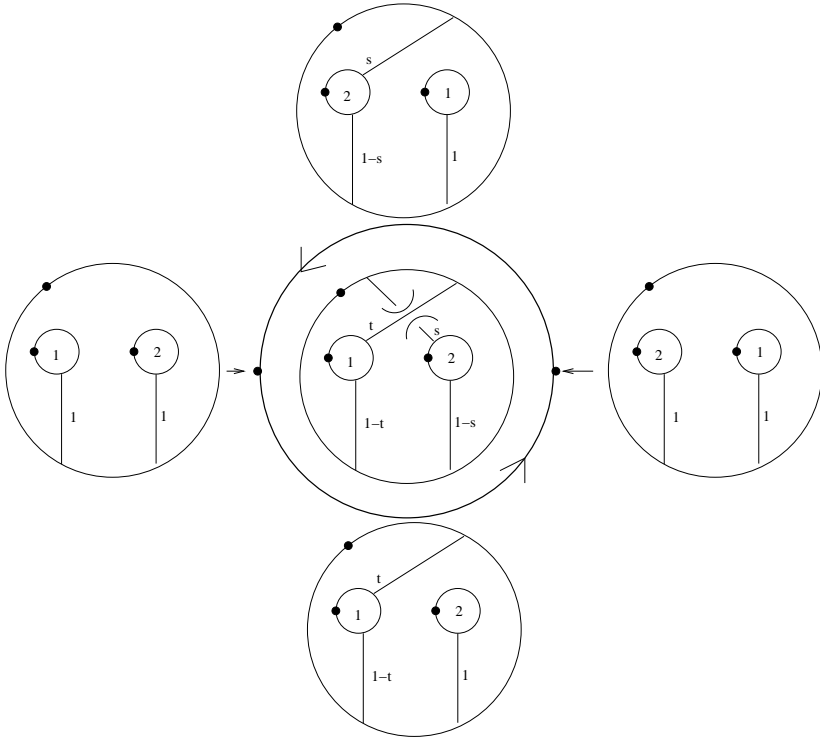
Fig. 12. The $\cup_2$ operation after stabilization

satisfies:

$$\Gamma : \mathcal{O}^{(\mu,\sigma)}(p) \times \mathcal{O}^{(\mu_1,\sigma_1)}(i_1) \times \cdots \times \mathcal{O}^{(\mu_p,\sigma_p)}(i_p) \subseteq \mathcal{O}^{(\mu(\mu_1,\ldots\mu_p),\sigma(\sigma_1,\ldots\sigma_p))}\Big(\sum_j i_j\Big). \quad (10)$$

The suboperads $\mathcal{O}^{(n)}$ are called (weak) cellular E_n operads.

We will use the weak unital operad $St\mathcal{LGT}ree$ and the unital operad ${}^{\mathrm{Fat}}St\mathcal{LGT}ree$.

PROPOSITION 3.7. $St\mathcal{LGT}ree$ *is a weak cellular* E_∞ *operad and the* $St\mathcal{LGT}ree^{(n)} = \{St\mathcal{LGT}ree^{(n)}(p)\}$ *are weak cellular* E_n *operads.* ${}^{\mathrm{Fat}}St\mathcal{LGT}ree$ *is a cellular* E_∞ *operad and the* ${}^{\mathrm{Fat}}St\mathcal{LGT}ree^{(n)} := \{St\mathcal{LGT}ree^{(n)}(p)\}$ *are cellular* E_n *operads.*

Proof. We have already shown that $St\mathcal{LGT}ree(2)$ admits a cellular $\mathcal{K}_2$ decomposition. We will now verify the rest of the conditions.

$St\mathcal{LGT}ree_p^{(\alpha)}(n)$ is contractible. This is analogous to the case with $n = 2$, we can decrease the weights of the arcs to zero of all but the first arc in a given boundary component, while at the same time scaling the weights of the first arcs to the same value. The fact that the "cell" inclusions are cofibrations is again analogous to the case $n = 2$.

The operad multiplication preserves the cellular structure. We see that gluing in arc families and then discs everywhere, there are only two situations that can arise. Either the two boundaries were in the same family or in two different ones. In both cases we

see that the result of gluing in discs is the same whether we do it before or after gluing. In the first case this means that we only look at the surface where both boundaries that are the pre-images of i, j lie, and in the second case we look at the surface into which we glue and only keep the boundaries r and s into which the two surfaces are glued. This is exactly how the composition in $\mathcal{K}$ was defined.

There is an ordered point in each cell. These points are given by the iterated gluings $\cup'_n \circ_2 \cup'_n \circ_3 \cdots \circ_{p-1} \cup'_n$ and their images under $\sigma \in \mathcal{S}_p$, where $\cup'_n$ is the element in $\cup_n$ whose weights on the arcs at each of the boundaries are all equal, i.e. $1/([(n+1)/2]+1)$ at the boundary 1 and $1/([n/2]+1)$ at the boundary 2.

The claims about $^{\mathrm{Fat}}St\mathcal{LGT}ree$ as cellular E_∞ preoperad again follow by contracting $^{\mathrm{Fat}}St\mathcal{LGT}ree(n)$ to $St\mathcal{LGT}ree(n)$ by contracting the gaps. The cellular E_∞ operad structure follows from the above argument, since the gaps can be ignored for the filtrations. ∎

THEOREM 3.8. *The operads* $^{\mathrm{Fat}}St\mathcal{LGT}ree^{(k)}$ *are* E_k *operads and the operad* $^{\mathrm{Fat}}St\mathcal{LGT}ree$ *is an* E_∞ *operad.*

Proof. Immediate from Fiedorowicz's theorem [B, Theorem 1.16], which states that cellular E_k operads are E_k operads and cellular E_∞ operads are E_∞ operads. ∎

Using Corollary 2.33:

COROLLARY 3.9. *The operads* $\{St\mathcal{LGT}ree^{(k)}(n), n > 0\}$ *are equivalent to* $\{C_k(n), n > 0\}$ *where* C_k *are the little* k *cubes and the operad* $\{St\mathcal{LGT}ree, n > 0\}$ *is an* E_∞ *operad without a 0-term.*

4. CW models and explicit operations. There are CW models for $\mathcal{GT}ree$ with the suboperads being sub-CW-models. Moreover the cellular chains are also a model for $^{\mathrm{Fat}}\mathcal{GT}ree$, so that if we are interested only in the chain level, we can omit the thickening step. The main point is that even if we include the 0-th spaces which makes the topological level only associative up to homotopy, the structures are already operads on the chain level. Of course this is true for the homology level *a priori* and *a posteriori*. These CW-models are well behaved under the stabilization process and hence we obtain a combinatorial graph chain model for $St\mathcal{GT}ree$ and $^{\mathrm{Fat}}St\mathcal{GT}ree$.

4.1. *The CW-model* $\mathcal{GT}ree^1$. This construction is completely analogous to that of $\mathcal{C}acti^1$ of [Ka1]. For an arc graph γ which belongs to $\mathcal{GT}ree(n)$ let v_0 be the vertex corresponding to the boundary component 0. We define $\overline{C}(\gamma) := \prod_{v \in V(\gamma) \setminus \{v_0\}} \Delta^{val(v)-1}$. We identify the interior of this cell with all the elements of $\mathcal{DGT}ree(n)$ whose arc graph is γ and whose weights at all the boundaries except 0 are 1, the coordinates in $\Delta^{val(v)-1}$ being given by the barycentric coordinates corresponding to the weights of the incident arcs in their order. We identify the boundary of this cell by letting the weight of the arc corresponding to a vertex of the simplex go to 0 and erasing it when passing to the face.

We define $\mathcal{GT}ree^1(n)$ to be the CW complex formed from the cells with the attaching maps given above. We can add a 0-th term to the quasi operad $\mathcal{GT}ree^1$ where $\mathcal{GT}ree^1(0) = \mathcal{GT}ree^1(0)$.

4.2. *The quasi-operad $\mathcal{GT}ree^1$ and its induced cellular operad.* We define the operations $\alpha \circ_i \beta$ on $\mathcal{GT}ree^1$ by using the alternative gluing that we scale the weights on the boundary i of α to match those of β at 0 and then glue in.

PROPOSITION 4.2. *With the gluings above and the action of $\mathbb{S}_n$ permuting the labels the spaces $\{\mathcal{GT}ree^1(n)\}, n \geq 0$ form a homotopy associative operad (aka. topological quasi-operad) such that*

  i) *the induced quasi-operad structure on the cellular chain complex of the CW complex $CC_*(\mathcal{GT}ree^1(n))$ is an operad structure,*
  ii) *the induced operad structure on $H_*(\mathcal{GT}ree^1(n))$ is isomorphic to $H_*(\mathcal{GT}ree(n))$,*
  iii) *the cellular chain complex $CC_*(\mathcal{GT}ree^1(n))$ gives a chain model for the operad $^{\mathrm{Fat}}\mathcal{GT}ree$.*

  The subspace $\mathcal{LGT}ree^1$ given by the cells with graphs of $\mathcal{LGT}ree$ is a sub-quasi operad on the topological level and a suboperad on the cellular and homology level.

  As spaces $\mathcal{DGT}ree(n) = \mathcal{GT}ree^1(n) \times \mathbb{R}_{>0}^n$ and as quasi-operads $\mathcal{DGT}ree(n) = \mathcal{GT}ree^1(n) \rtimes \mathbb{R}_{>0}^n$. Modding out by the overall $\mathbb{R}_{>0}$ action, we obtain similar results for $\mathcal{GT}ree$.

For the definition of semi-direct products of operads see [SW] and for quasi-operads [Ka1].

Proof. We do not wish to go into the gory details. The proof is a straightforward adaption from that of $\mathcal{C}acti$ presented in [Ka1, Ka2]. For the semi-direct product the first homeomorphism is given by reading off the weights at each boundary and then taking the projective class of the weights at each boundary individually. The semi-direct product is given by first scaling, then inserting and finally scaling back.

The statement about $H_*(^{\mathrm{Fat}}\mathcal{GT}ree)$ follows from Proposition 2.32. ∎

THEOREM 4.3. *The cellular models carry over to the stabilized situation. In particular $\mathcal{StGT}ree$ and $\mathcal{StLGT}ree$ have operadic CW-models $\mathcal{StGT}ree^1$ and $\mathcal{StLGT}ree^1$ whose underlying spaces are topological-quasi operads. In the former, the cells are indexed by the quasi-filling arc graphs, while in the latter the graphs also satisfy the conditions of $\mathcal{LGT}ree$. Furthermore adding 0-components the analogous statements hold true and the cellular chains of $\mathcal{StGT}ree^1$ are a model for $^{\mathrm{Fat}}\mathcal{StGT}ree$.*

Proof. The indexing by quasi-filling graphs is clear in view of Corollary 2.27. The rest is straightforward using the usual techniques of [Ka1, K3]. ∎

REMARK 4.4. We wish to point out that the boundary of a cell $\overline{C}(\gamma)$ now consists of those graphs obtained by removing an arc from γ which is not the only arc incident to a boundary and then stabilizing. For an illustration see Figure 11 and consult the Appendix for further remarks and discussion.

4.3. *Explicit operations.* In view of the CW-models above we can easily write down cellular representatives for classical operations.

4.3.1. *The $\cup_i$ operations.* We have already identified the $\cup_i$ products in the hemispherical decomposition. The graphs are in Figure 10. Now we can take the same graphs and reinterpret them as generators of $CC_*(\mathcal{S}t\mathcal{G}\mathcal{T}ree^1(2))$.

REMARK 4.5. We wish to point out that these operations belong to the appropriate part of the filtration of the cellular E_∞ operad. More interestingly, the $\cup_i$ product is realized on a surface of genus $g = [i/2]$.

Moreover this periodicity also manifests itself in the fact that $\cup_{2k}$ is in the image of $\mathcal{L}\mathcal{G}\mathcal{T}ree'$ and $\cup_{2k+1}$ is twisted at zero. This means that the sequence is: twist at zero, add genus, twist at zero etc.

4.3.2. *Dyer-Lashof operations.* In this formalism we can also make the Dyer-Lashof-Cohen operations for double loop spaces explicit. By the general theory, see [Co], we need to find particular elements

$$\xi_1 \in H_{p-1}(C_2(p)/\mathbb{S}_p, \pm\mathbb{Z}/p\mathbb{Z}), \tag{11}$$

that is, homology classes of the little 2-cubes with values in the sign representation.

Now taking co-invariants on $CC_*(\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree^1(p)) = CC_*(\mathcal{C}act^1(p))$ which is a chain model for the little discs operad $D_2(p)$ we see that the p-th iteration of the product $\cup_1$ that is the operation given by

$$^p\cup_1 := \gamma(\gamma(\ldots(\gamma(\cup_1),\cup_1),\ldots,\cup_1),\cup_1) \tag{12}$$

gives a class that is the sum over all trees of the highest dimension where the partial order on the labeled vertices when considered in the usual tree partial order is compatible with the linear order on $\bar{n}$.

PROPOSITION 4.6. $^p\cup_1$ *represents the cohomology class ξ_1 in $H_{p-1}(\mathcal{S}t\mathcal{L}\mathcal{G}\mathcal{T}ree^{(2)}(p)/\mathbb{S}_p, \pm\mathbb{Z}/p\mathbb{Z})$ of equation (11).*

Proof. This is a tedious but fairly straightforward calculation of the boundary of said combination of cells. The actual calculation can be adapted from the proof of Tourtchine [Tou] using cells instead of operations on the Hochschild complex. The dictionary for this is provided by [Ka2]. ∎

The first example for $p = 2$ is given by the operation of $\cup_1$ which has boundaries in the multiplication and its opposite, cf. Figure 10, and the example for $p = 3$ is the hexagon of Figure 13 with $i = 1$.

REMARK 4.7. We wish to point out several interesting facts.

1. The class is solely induced by an operation for $p = 2$.
2. The resulting cell description is just the left iteration of $\cup_1$, whereas the right iteration of $\cup_1$ is the simple class given by a cube.
3. When acting on p times the same even element, the two iterations coincide. Furthermore the action factors through the coinvariants. When acting on p times the same odd element, the action factors through the coinvariants under sign representation. This is not surprising, but here we have a very geometric picture.
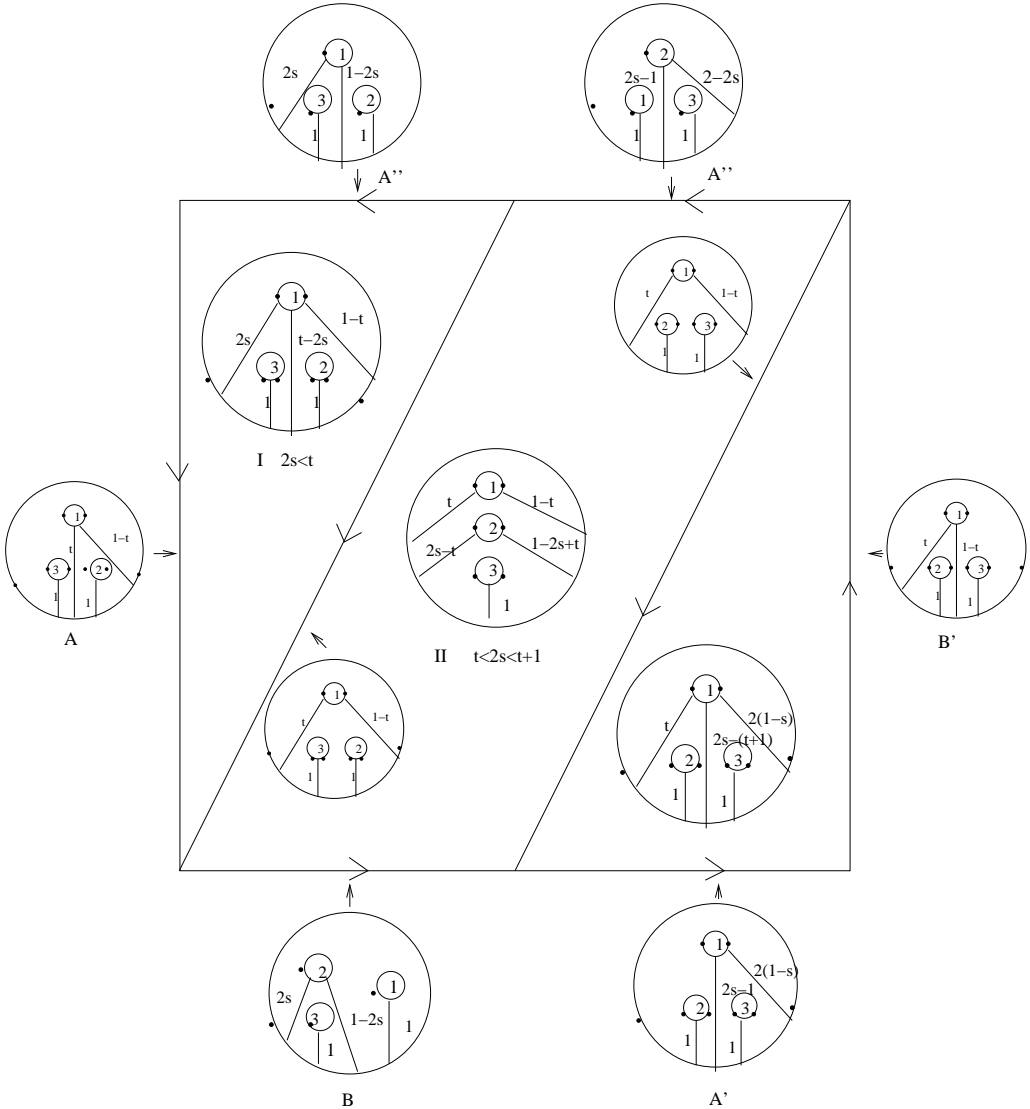
Fig. 13. The hexagon that gives the Dyer-Lashof operation. The cells A, A' and A'' become equivalent as well as B, B' and B'' become equivalent after passing to coinvariants using the sign representation.

For the actions, we can take the action of the chain operad on itself of the action on the Hochschild complex as defined in [Ka2]. Using the former action, we obtain a universal geometric point of view.

REMARK 4.8. Using the induced action on the Hochschild complex of [Ka2], we reproduce the results of Westerland [We] and Tourtchine [Tou] on representing the Dyer-Lashof-Cohen operations on the Hochschild co-chains.

4.4. *Relation to the McClure-Smith sequence operad*

DEFINITION 4.9. The *sequence* of an element $\alpha \in \mathcal{GT}ree(n)$ is the sequence $seq(\alpha)$: $\{1, \ldots, |E_{\Gamma(\alpha)}|\} \to \{1, \ldots, n\}$ defined as follows: using the linear order at the boundary 0 there is a natural bijection $\phi : \{1, \ldots, |E_{\Gamma(\alpha)}|\} \to E_{\Gamma(\alpha)}$ by counting off the edges using the linear order at the boundary zero. This map is a bijection since all edges are incident to zero. Now each edge e is incident to two boundaries, the boundary zero and a boundary $i =: boundary(e)$ with $i \neq 0$. The map $seq(\alpha)$ is now given as $seq(\alpha)(j) = boundary(\phi(j))$.

REMARK 4.10. In general for $\mathcal{GT}ree$ the information $seq(\alpha)$ does not contain all the information about α. This is different for $\mathcal{StLGT}ree$.

We already know that both the sequence operad of McClure and Smith [McCS1], which we will call $\mathcal{MS}$, and the operad $\mathcal{StLGT}ree$ are E_∞ operads and hence they are equivalent. The map above makes this explicit.

PROPOSITION 4.11. *The map* $\alpha \mapsto seq(\alpha)$ *induces a surjective morphism of operads* $\mathcal{StLGT}ree \to \mathcal{MS}$. *It maps the cellular filtration above to the filtration by complexity which was introduced in [McCS1].*

Proof. The fact that this is an operadic morphism that is surjective is only an unraveling of the definitions. Also it is straightforward from the definitions that the complexity of the sequence of boundary arcs corresponds exactly to the filtration induced by the $\mathcal{K}$-structure. ∎

REMARK 4.12. Probably the operads are even isomorphic, but for our purpose to make the equivalence explicit the result above is sufficient.

5. Applications and outlook

5.1. *Actions on Hochschild.* Let A be a commutative Frobenius algebra with a non-degenerate pairing $\langle \, , \, \rangle$ and unit 1. Set $\int a := \langle a, 1 \rangle$. Let e be the Euler element of A. Let Δ be the comultiplication which is the adjoint of the multiplication μ, then $e := \mu\Delta(1)$.

PROPOSITION 5.2. *The action of $CC_*(\mathcal{GT}ree)$ on the Hochschild co-chains $HC^*(A, A)$ as defined by restricting the action [K3, K4] passes to $CC_*(\mathcal{StGT}ree)$ if and only if the Euler element of A is the unit.*

Proof. We would only like to recall that the action is given by the product over local contributions where in particular there is one such contribution for each complementary region which are of the form $\int a_1 \ldots a_n e^{-\chi(R)+1}$, where e is the Euler element of A. The effect of stabilizing is to set the factor of $e^{\chi(R)-1}$ to 1, whence the claim. ∎

EXAMPLE 5.3. The condition is met in the case that A is semi-simple with a unital metric. I.e. there is a basis of idempotents $e_i e_j = \delta_{ij} e_i$ and $\int e_i = 1$.

REMARK 5.4. We again wish to make several remarks.

  1. In particular, if $A = H^*(X)$ with X a compact manifold, A is semi-simple only if X is a point. In all other cases the only way would be to formally invert the nilpotent element e which yields the zero algebra. This means that the stabilization is not

compatible with string topology, which is good, since otherwise the string bracket would vanish.

2. It is conjectured that for Fano varieties, which have a system of exceptional sheaves of appropriate length, the quantum cohomology is however semi-simple. This points to a connection with quantum cohomology.

3. We expect that the stabilization in our sense is related to the stabilization in the usual sense (see e.g. [H]), see below. In particular this gives a point of contact with Witten's τ function and higher Weil-Petersson volumes [KMZ, MZ].

4. In order to obtain actions in a wider setting one could try to use a conformal scaling or to alter the differential.

5.2. Ω *spectra.* Since $^{\mathrm{Fat}}\mathcal{St}\mathcal{LG}\mathcal{T}ree$ is an E_∞ operad it detects infinite loop spaces and so does any operad it is a suboperad of. Furthermore, since it acts on any operad, of which it is a suboperad, the respective operad will yield an Ω spectrum.

THEOREM 5.5. *The group completions of $\coprod_n {}^{\mathrm{Fat}}\mathcal{St}\mathcal{LG}\mathcal{T}ree(n)$, $\coprod_n {}^{\mathrm{Fat}}\mathcal{St}\mathcal{CG}\mathcal{T}ree(n)$ and $\coprod_n {}^{\mathrm{Fat}}\mathcal{St}\mathcal{G}\mathcal{T}ree(n)$ are infinite loop spaces and hence yield Ω spectra. Furthermore all the operads detect infinite loop spaces.* ∎

5.3. *Stabilized arcs and Sullivan PROP*

5.3.1. *Stabilizing the Sullivan PROP of [K3].* Of course $\mathcal{G}\mathcal{T}ree$ is also a suboperad of $\mathcal{A}rc$ and of the Sullivan PROP of [K3]. It would hence make sense to try and stabilize these two constructions. For the Sullivan PROP this is rather straightforward, since we again can choose to stabilize at the out boundaries only.

Without going into the full details, we define the stabilized Sullivan quasi-PROP to be the colimit over all maps in the system $\mathcal{S}$ where now we are allowed to glue to any out boundary.

5.3.2. *Stabilizing the $\mathcal{A}rc$ operad.* For the $\mathcal{A}rc$ operad we have to use a cyclic alternative. That is as a first approximation we would like to define the stabilization by gluing on the system $\mathcal{S}$ in all possible ways. This poses no problem. But, we will also have to deal with other types of degeneracies, which lead to disconnected graphs; see the Appendix for some more details.

Using this more careful analysis it will be possible to add a neutral element using thickenings.

CONJECTURE 5.6. *There is a suitable stabilization $^{\mathrm{Fat}}\mathcal{St}\mathcal{A}rc$ of a thickening of $\mathcal{A}rc$ whose 0-term is the disc and which contains $^{\mathrm{Fat}}\mathcal{St}\mathcal{G}\mathcal{T}ree$ as a suboperad and hence the group completion of $\coprod_{n\geq 0} {}^{\mathrm{Fat}}\mathcal{St}\mathcal{A}rc(n)$ is an infinite loop space.*

It will be interesting to figure out which Ω spectrum this is. Since stabilization by the element G embeds the moduli space $M_{g,n}^{1^n}$ as a piece of the boundary of the moduli space $M_{g+1,n}^{1^n}$, we expect that it will be closely related to the Segal-Tillmann picture [Ti1, Ti2].

5.4. *Outlook: Generalizing framed little discs and new decompositions.* One open question is the full role of the spaces $\mathcal{G}\mathcal{T}ree_g(1)$ that have been discussed below. In the case of genus one $\mathcal{A}rc_0(1)$ made the difference between framed little discs and little discs.

Similarly, $\mathcal{A}rc_0(1)$ leads to a bi-crossed product $\mathcal{CGT}ree$. One could wonder what the inclusion the space $\mathcal{GT}ree_g(1)$ signifies.

We can decode some of its structure.

PROPOSITION 5.7. *An element α of $\mathcal{GT}ree_g(1)$ which is untwisted has at most $3g + 1 + [(g-1)/3]$ arcs and this number is realized.*

Proof. We begin cutting along the arcs. The maximal number of non-separating cuts is $2g+1$, since the Euler characteristic of the underlying surface is $2-2g+2$ and each non-separating cut increases the Euler characteristic by 1. There are no more non-separating cuts when we are left with a disc of Euler characteristic 1. This disc will have a boundary made up of a $4(2g+1)$-gon. The sides are labeled by sequences where every 4th element is a part of the boundary 0, every $4n+2$ element is a part of the boundary 1 and the $4n+1$st and $4n+3$rd elements correspond to the cut edges—where each edge appears twice. For $g > 0$, we can insert a maximum of $[4(2g+1)/6] = g + [(g-1)/3]$ arcs, since each separating arc has to cut off at least an octagon. This is because each arc has to run from 0 to 1 and these cannot be only separated by one edge since otherwise the new arc and the arc represented by the edge would be parallel. ∎

COROLLARY 5.8. *The dimension of the top dimensional cells of $\mathcal{GT}ree_g^1(1)$, $g > 0$ is $3g + [(g-1)/3] + 2$.*

Proof. For the total count, we can add twists at both ends as soon as $g > 0$. Thus keeping in mind that the dimension is one less than the number of arcs, we arrive at the formula above. ∎

Thus $\mathcal{GT}ree$ cannot give the framed little discs as the dimension only grows linearly in g and not quadratically. On the other hand the dimensions fit with the dimension of spheres, so that it may look like a marked point on the boundary of the little cubes.

By Lemma 2.20, however, we see that we get a new decomposition for the E_n operads in terms of E_1 respectively E_2 and elements of $\mathcal{GT}ree_g(1)$. Notice though that if we decompose $\cup_2$ we can decompose it as $\cup_1$ and an element which is not in $\mathcal{CGT}ree(1)$. However this element contains exactly one braid. Furthermore, we see that the elements $\cup_i$ are generated by $\cup_1$ and particular braid elements of $\mathcal{GT}ree_{[(i-1)/2]}(1)$.

CONJECTURE 5.9. *For each n there is a suboperad of $\mathcal{StGT}ree(1)$ such that the operad $\mathcal{StLGT}ree^{(n)}$ (an E_n operad without 0-term) is a bi-crossed product of this suboperad and operad $\mathcal{LT}ree$ (an E_2 operad without 0-term).*

Appendix: Graphs, dual graphs and compactifications. In [Ka2] (see also [K3]) we introduced a dual graph for quasi-filling arc graphs. Here we extend this notion to all graphs. One upshot is that we can make contact with Kontsevich's stabilization in this way.

A.1. *Dual graph.* The dual graph of an arc graph α is the labeled graph (semi-stable labeled ribbon graph) $\Gamma(\alpha)$ whose vertices are the complementary regions of the arc graph. We will write $v(R)$ for the vertex corresponding to a region R. Edges correspond to the arcs of the arc graph and the vertices of an edge to the complementary region(s)

that are bordered by the respective arc. A flag will be a pair of an arc and a choice of orientation for it or equivalently a side of the arc. Notice that loops are allowed. Since the surface Σ was connected, the graph will be connected.

There is a bit more structure on these graphs, although they generally fall short of being ribbon graphs. To make this discussion more symmetric, we will use one of the equivalent versions for depicting arcs; see 1.1.6. This is, we move the end-points of the edges off the marked points on the boundary and move them apart along the orientation of the boundary, so that the arcs are all disjointly embedded, do not hit the endpoints, and their linear order from the marked ribbon graph coincides with the linear order given by counting them off starting at the marked point of a boundary and going around that boundary in its induced orientation. Fix a complementary region R. R is an oriented surface with boundary. There are two types of boundary components, those which contain arcs of the original graph and those who do not. The former are actually $2n$-gons whose sides alternate between pieces of the boundary and arcs, while the latter consist of a full boundary component of Σ. Let $b(R)$ be the number of boundary components of the former type, $f(R)$ the boundary components of the latter type and fix $g(R)$ to be the genus of R after gluing in discs into the boundary. We set $dt(R) = (g(R), b(R), f(R))$.

Now each boundary of R containing arcs has an induced orientation, hence we get a cyclic order on these arcs. Formally this means that at each vertex v of $\Gamma(\alpha)$ we have an action of $\mathbb{N}$. Each orbit corresponds to a set of flags stemming from one of the boundaries of R. In other words the set of flags $F(v)$ is partitioned into subsets $F(v) = F_1(v) \amalg \cdots \amalg F_k(v)$ and each $F_i(v)$ has a cyclic order. Moreover these cyclic orders fit together to give an action of $\mathbb{N}$ on the set of all flags $F(\Gamma(\alpha))$, by combining the previous action with the map $\imath$ as usual. The orbits of this map, which we call N are still called cycles of Γ.

Lastly there is a marking mk for each cycle of the graph. This is the first flag of the cycle which corresponds to the flag of the edge containing the marked point of the boundary.

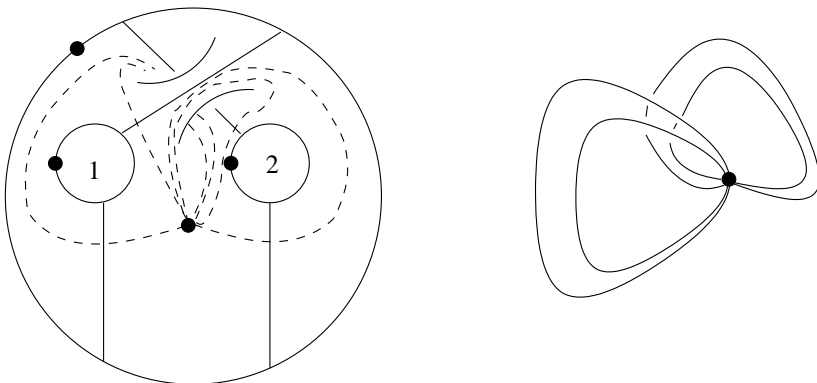DEFINITION A.2. *The dual graph of α is defined to be the graph (Γ, N, dt, mk).*

Fig. 14.   The arc graph for the $\cup_2$ product and its alternate depiction as a ribbon graph

An example of the dual graph is given in Figure 14. Unlike in the situation of $\mathcal{T}ree_{cp}$ where the dual graphs are cacti, the advantage of arc graphs may be more obvious.

The data of $b(R)$ is actually redundant, since $b(R)$ is the number of orbits of the cyclic action of $\mathbb{N}$ on the flags at that vertex.

REMARK A.3. In the case of $\mathcal{GT}ree$ we always have that $f(R) = 0$, since there are no boundaries which are not hit by arcs.

REMARK A.4. The use of the dual graph now gives a re-interpretation of Penner's compactification [P] in terms of Kontsevich's [Ko] and vice-versa.

A.2. *Stabilization.* The stabilization will then have the effect of setting the label $g(R)$ of a vertex to zero or in the case that $b(R) \neq 0$ the vertex will be split into the number of boundaries.

REMARK A.5. Although in general the graph can become disconnected this does not happen for $\mathcal{GT}ree$. The reason is that if it were disconnected, then there would be a separating curve which does not cut any of the arcs. This is impossible if all the arcs run to zero.
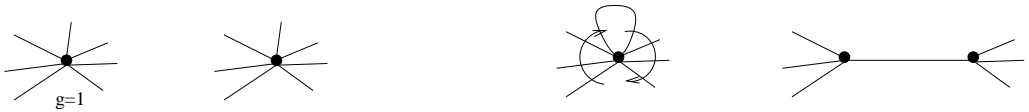

Fig. 15.   The stabilization of a vertex using G or H in the case of $\mathcal{GT}ree$

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