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THE L^2-INVARIANTS AND MORSE NUMBERS

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Abstract. We study the homotopy invariants of free cochain complexes and Hilbert complexes. These invariants are applied to calculation of exact values of Morse numbers of smooth manifolds.

1. Introduction. Let W^n be a smooth manifold. By definition the i-th Morse number $\mathcal{M}_i(W^n)$ of W^n is the minimal number of critical points of index i taken over all Morse functions on W^n.

It is known [2, 10, 19] that for closed smooth manifolds of dimension greater than 6 the i-th Morse numbers are invariants of the homotopy type. There is a very complicated unsolved problem: *find exact values of Morse numbers for every i* (see [19] for more details).

In [20] using new homotopy invariants $\mathbb{D}^i(W^n)$ of free cochain complexes and Hilbert complexes of non simply-connected manifolds W^n we proved the following theorem.

THEOREM. *Let W^n $(n \geq 8)$ be a smooth closed manifold with $\pi = \pi_1(W^n)$. Then for $4 \leq i \leq n-4$ the following equality holds true:*

$$\mathcal{M}_i(W^n) = \mathbb{D}^i(W^n) + \widehat{S}^i_{(2)}(W^n) + \widehat{S}^{i+1}_{(2)}(W^n) + dim_{N(\mathbb{Z}[\pi])}(H^i_{(2)}(W^n, \mathbb{Z})).$$

The Morse number $\mathcal{M}(W^n)$ of a manifold W^n is the minimum of the total number of critical points over all Morse functions on W^n. In this paper we prove the following theorem.

THEOREM. *Let $(W^n, V_0^{n-1}, V_1^{n-1})$ $(n \geq 6)$ be a compact smooth manifold with boundary $\partial W^n = V_0^{n-1} \cup V_1^{n-1}$ and $\pi = \pi_1(W^n)$ be the fundamental group of the manifold W^n. Suppose that $\pi(V_i^{n-1}) \to \pi_1(W^n)$ is an isomorphism, $Wh(\pi) = 0$, where $Wh(\pi)$ is the*

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Whitehead group of π*, and*

$$\widehat{\mathbb{D}}_r^0(W^n) = \widehat{\mathbb{D}}_l^{n-1}(W^n) = \mathbb{D}_r^0(W^n) = \mathbb{D}_l^{n-2}(W^n) = \mathbb{D}^i(W^n) = 0$$

for all i*. Then*

$$\mathcal{M}(W^n) = 2\sum_{i=2}^{n-3} \widehat{S}^i_{(2)}(W^n) + \widehat{S}^{n-2}_{(2)}(W^n) + \sum_{i=1}^{n-2} dim_{N[\pi]}(H^i_{(2)}(W^n, \mathbb{Z})) + 2\mu(H^{n-1}(\widetilde{W}^n, \mathbb{Z}[\pi])) - dim_{N[\pi]}(H^{n-1}_{(2)}(W^n, \mathbb{Z})).$$

2. Stable invariants of finitely generated modules and L^2-modules. Denote the ring of integers by $\mathbb{Z}$ and the field of complex numbers by $\mathbb{C}$. Let G be a discrete group. Denote its integer group ring by $\mathbb{Z}[G]$ and the group ring over the field $\mathbb{C}$ by $\mathbb{C}[G]$. In the group ring there exists an augmentation epimorphism $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ ($\varepsilon : \mathbb{C}[G] \to \mathbb{C}$) acting by the rule $\varepsilon(\sum_i \alpha_i g_i) = \sum_i \alpha_i$. Denote the kernel of the epimorphism ε by $\mathbb{I}[G]$. In the ring $\mathbb{C}[G]$ there exists an involution $* : \mathbb{C}[G] \to \mathbb{C}[G]$, $(\sum_i \alpha_i g_i)^* = \sum_i \overline{\alpha}_i g_i^{-1}$, where $\overline{\alpha}$ denotes the conjugation in $\mathbb{C}$. This involution satisfies the following conditions:

a) $(r^*)^* = r$;
b) $(\alpha r_1 + \beta r_2)^* = \overline{\alpha} r_1^* + \overline{\beta} r_2^*$ $(\alpha, \beta \in \mathbb{C})$;
c) $(r_1 r_2)^* = r_2^\star r_1^\star$.

We can define the trace $tr : \mathbb{C}[G] \to \mathbb{C}$ by the rule $tr(\sum_i^k \alpha_i g_i) = \alpha_1$, where α_1 is the coefficient at the unit $g_1 = e$ of G. It is obvious that the trace satisfies the following conditions:

a) $tr(e) = 1$;
b) tr is a $\mathbb{C}$-linear mapping;
c) $tr(r_1 r_2) = tr(r_2 r_1)$;
d) $tr(rr^*) \geq 0$, and if $tr(rr^*) = 0$, then $r = 0$.

In what follows, M will be a finitely generated left module over a certain associative ring Λ with unit e. Rings for which the rank of the free module is uniquely defined are called IBN-rings. It is known that the group rings $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ are IBN-rings. Denoting the minimum number of the generators of the module M by $\mu(M)$, we get $\mu(M \oplus F_n) < \mu(M) + n$, where F_n is a free module of rank n. There exist examples of stably free modules when the strict inequality holds. Recall that a Λ-module M is called *stably free* if the direct sum of M and a certain free Λ-module F_k is free. We assume that $\mu(M) = 0$ for zero module $M = 0$.

Definition 2.1 ([19]). For a finitely generated module M over an IBN-ring Λ define the following function

$$\mu_s(M) = \lim_{n\to\infty} (\mu(M \oplus F_n) - n))$$

called the *stable minimal number of generators of the module* M.

If ring Λ is Hopfian then for any Λ-module M the equality

$$\mu_s(M) = 0,$$

holds if and only if $M = 0$. Recall that a ring Λ is called *Hopfian* if every epimorphism of a free Λ-module F_n on itself is an isomorphism.

From the theorems of Kaplansky and Cockroft it follows that the group rings $\mathbb{Z}[G]$ and $\mathbb{C}[G]$ are Hopfian. It is clear that for any non-zero module M we have $0 < \mu_s(M) \leq \mu(M)$. The difference

$$\mu(M) - \mu_s(M)$$

shows how many times one can add a free module of rank one to the modules $M \oplus k\Lambda$ $(k = 0, 1 \ldots)$ so that the number $\mu(M \oplus k\Lambda)$ does not increase. For every finite generated module M over an IBN-ring Λ there is a natural number n such that for the module $N = M \oplus n\Lambda$ and all $m \geq 0$ we have that $\mu(N \oplus m\Lambda) = \mu(N) + m$.

In the ring $\mathbb{C}[G]$ there is an inner product $\langle \sum_i \alpha_i g_i, \sum_i \beta_i g_i \rangle = \sum_i \alpha_i \overline{\beta}_i$, so the norm for an element $r \in \mathbb{C}[G]$ is defined by $|r| = tr(rr^*)^{1/2}$. Consider a completion of the ring $\mathbb{C}[G]$ with respect to this norm and denote it by $L^2(G)$. Then $L^2(G)$ is a Hilbert space (the inner product is given by the same formula as for the group ring $\mathbb{C}[G]$). The Hilbert space $L^2(G)$ has an orthonormal basis consisting of all elements of the group G. Now $\mathbb{C}[G]$ acts faithfully and continuously on $L^2(G)$ by multiplication from the left, so we may regard $\mathbb{C}[G] \subseteq \mathbf{B}(L^2(G))$, where $\mathbf{B}(L^2(G))$ denotes the set of bounded linear operators on $L^2(G)$.

Let $N[G]$ denotes the (reduced) group of von Neumann algebra of G: thus by definition $N[G]$ is a week closure of $\mathbb{C}[G]$ in $\mathbf{B}(L^2(G))$. Therefore the map $w \to w(e)$ allows us to identify $N[G]$ with a subspace of $L^2(G)$, where $w \in N[G]$ and e is the unit element of the group G. Thus algebraically we have $\mathbb{C}[G] \subset N[G] \subset L^2(G)$. The involution and the trace map on $N[G]$ can be defined exactly as for the ring $\mathbb{C}[G]$. For the set $M_n(N[G])$ of $n \times n$ matrices over von Neumann algebra $N[G]$ the trace map can be extended by setting $tr(W) = \sum_{i=1}^n w_{ii}$, where $W = (w_{ij})$ is a matrix with entries in $N[G]$.

Let $L^2(G)^n$ denote the Hilbert direct sum of n copies of $L^2(G)$, so $L^2(G)^n$ is a Hilbert space. The von Neumann algebra $N[G]$ acts on $L^2(G)^n$ from the left, so $L^2(G)^n$ is a left $N[G]$-module called a *free Hilbert $N[G]$-module of rank n. The left Hilbert $N[G]$-module* M is a closed left $\mathbb{C}[G]$-submodule of $L^2(G)^n$ for some n. By definition a *Hilbert $N[G]$-submodule* of M is a closed left $\mathbb{C}[G]$-submodule of M, an $L^2(G)$-ideal is an Hilbert $N[G]$-submodule of $L^2(G)$, and a *homomorphism* $f : M \to N$ between Hilbert $N[G]$-modules is a continuous left $\mathbb{C}[G]$-map [4].

Let M be a Hilbert $N[G]$-module and let $p : L^2(G)^n \to L^2(G)^n$ be the orthogonal projection onto $M \subset L^2(G)^n$. The *von Neumann dimension* of the Hilbert $N[G]$-module M is the following number: $dim_{N[G]}(M) = tr(p) = \sum_{i=1}^n \langle p(e_i), e_i \rangle_{L^2(G)^n}$. Here $e_i = (0, \ldots, g, \ldots, 0)$ is the standard basis in $L^2(G)^n$. It is known that $dim_{N[G]}(V)$ is a non-negative real number [12].

DEFINITION 2.2. Let M be a finitely generated $\mathbb{Z}[G]$-module. Consider the Hilbert $N[G]$-module $L^2(G) \otimes_{\mathbb{Z}[G]} M$ and define the following number

$$S(M) = \mu_s(M) - dim_{N[G]}(L^2(G) \otimes_{\mathbb{Z}[G]} M).$$

LEMMA 2.3. *For any finitely generated $\mathbb{Z}[G]$-module M the number $S(M)$ is non-negative.*

The proof is given in [20].

3. Stable invariants of homomorphisms. The next results can be found in [20]. Consider a Λ-homomorphism $f : F_k \to F_t$, where F_k and F_t are free modules over ring Λ of ranks k and t respectively. The homomorphism f is a *splitting along a submodule* $\overline{F}_p \subseteq F_k$ if there exists a presentation of f of the form

$$f = f_p \oplus f_t : \overline{F}_p \oplus \overline{F}_{k-p} \to \widetilde{F}_p \oplus \widetilde{F}_{t-p},$$

such that

$$f|_{\overline{F}_p \oplus 0} = f_p : \overline{F}_p \to \widetilde{F}_p, \qquad f|_{0 \oplus \overline{F}_{k-p}} = f_t : \overline{F}_{k-p} \to \widetilde{F}_{t-p},$$

where f_p is an isomorphism. From now on, in this situation we will assume that the submodules $\overline{F}_p$, $\overline{F}_{k-p}$, $\widetilde{F}_p$, $\widetilde{F}_{t-p}$ are free.

Definition 3.1. The number p above is called the *rank* of the splitting $f = f_p \oplus f_t$. The *rank* $R(f)$ of a homomorphism f is the maximal value of possible ranks of splittings of f.

Definition 3.2. *Stabilization* of a homomorphism $f : F_k \to F_t$ by a free module F_p is a homomorphism

$$f_{st}(p) : F_k \oplus F_p \to F_t \oplus F_p$$

such that

$$f_{st}(p)|_{F_k \oplus 0} = f, \qquad f_{st}(p)|_{0 \oplus F_p} = Id.$$

A thickening of $f : F_k \to F_t$ by free modules F_m and F_n is the homomorphism

$$f_{th}(m,n) : F_k \oplus F_m \to F_t \oplus F_n,$$

such that

$$f_{th}(m,n)|_{F_k \bigoplus 0} = f, \qquad f_{th}(m,n)|_{0 \bigoplus F_m} = 0.$$

Definition 3.3. The *stable rank* $Sr(f)$ of a homomorphism $f : F_k \to F_t$ is the limit

$$Sr(f) = \lim_{m,n,p \to \infty} (R(f_{th}(m,n)_{st}(p)) - p).$$

Since $Sr(f) \leq \min(k,t)$, this limit always exists. There are examples of stably free modules with $Sr(f) > R(f)$.

Lemma 3.4. *For any homomorphism $f : F_k \to F_t$ the following equality holds:*

$$Sr(f_{st}(v)) = Sr(f) + v.$$

Remark 3.5. For every homomorphism $f : F_k \to F_t$ there exists a number n_0 such that the stable rank $Sr(f)$ of the homomorphism f can be calculated by the formula

$$Sr(f) = R(f_{th}(m,n)_{st}(p)) - p$$

for any $m \geq n_0, n \geq n_0, p \geq n_0$.

For a homomorphism f define the following numbers [20]:

$$\mathbb{D}_r(f) = Sr(f) - Sr_r(f), \quad \mathbb{D}_l(f) = Sr(f) - Sr_l(f).$$

It is clear that $\mathbb{D}_r(f) = \mathbb{D}_r(f_{st}(p))$ and $\mathbb{D}_l(f) = \mathbb{D}_l(f_{st}(p))$ for any integer p.

Consider a composition of homomorphisms of free modules

$$F_m \xrightarrow{\ f\ } F_n \xrightarrow{\ g\ } F_t,$$

such that

$$g \cdot f = 0. \qquad (\partial)$$

We say that the homomorphisms f and g are splitting along submodules $\overline{F}_p \subseteq F_m$ and $\overline{F}_q \subseteq F_n$ if there are presentations of f and g of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{F}_p & \xrightarrow{f_1} & \widetilde{F}_p & \longrightarrow & 0 & & \\
 & & \oplus & & \oplus & & & & \\
 & & F_{m-p} & \xrightarrow{f_2} & F_{n-p-q} & \xrightarrow{g_2} & F_{t-q} & & \\
 & & & & \oplus & & \oplus & & \\
 & & 0 & \longrightarrow & \overline{F}_q & \xrightarrow{g_1} & \widetilde{F}_q & \longrightarrow & 0
\end{array}$$

such that

$$f|_{\overline{F}_p \oplus 0} = f_1, \qquad g|_{0 \oplus 0 \oplus \overline{F}_q} = g_1.$$

We allow the modules $\overline{F}_p$ or $\overline{F}_q$ to be zero. In the sequel we will suppose that the submodules $\overline{F}_p$, $\overline{F}_q$, F_{m-p} , F_{t-q} , F_{n-p-q} are free.

Definition 3.6. The number $p+q$ will be called the *common rank* of the splitting of the homomorphisms f and g along the submodules $\overline{F}_p \subseteq F_m$ and $\overline{F}_q \subseteq F_n$. The *common rank* $Cr(f, g)$ of the homomorphisms f and g is the maximal value of common ranks of the splitting of f and g.

Definition 3.7. The *stabilization* of a composition of homomorphisms of free modules

$$F_m \xrightarrow{f} F_n \xrightarrow{g} F_t,$$

satisfying the condition (∂) by free modules F_p and F_q is the following composition of homomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_p & \xrightarrow{id} & F_p & \longrightarrow & 0 & & \\
 & & \oplus & & \oplus & & & & \\
 & & F_m & \xrightarrow{f} & F_n & \xrightarrow{g} & F_t & & \\
 & & & & \oplus & & \oplus & & \\
 & & 0 & \longrightarrow & F_q & \xrightarrow{id} & F_q & \longrightarrow & 0.
\end{array}$$

We will denote it by $(f_{st}(p), g_{st}(q))$.

Definition 3.8. Consider a composition of homomorphisms f and g

$$F_m \xrightarrow{f} F_n \xrightarrow{g} F_t,$$

satisfying the condition (∂). The *thickening* of this composition by free modules F_p and F_q is the following composition of homomorphisms

$$F_m \oplus F_p \xrightarrow{f_{th}(p)} F_n \xrightarrow{g_{th}(q)} F_t \oplus F_q,$$

such that

$$f_{th}(p)|_{F_m \oplus 0} = f, \quad f_{th}(p)|_{0 \oplus F_p} = 0, \quad g_{th}(q) = g.$$

It will be denoted by $(f_{th}(p), g_{th}(q))$.

Definition 3.9. The *stable common rank $Scr(f,g)$ of the composition of homomorphisms* of free modules

$$F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$$

satisfying the condition (∂) is the limit

$$Scr(f,g) = \lim_{p,q,v,w\to\infty} (Cr(f_{th}(p)_{st}(v), g_{th}(q)_{st}(w)) - v - w).$$

Since $Scr(f,g) \leq n$, this limit always exists. There are examples of stably free modules showing that $Scr(f,g) \geq Cr(f,g)$.

Lemma 3.10. *For any homomorphisms*

$$F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$$

satisfying the condition (∂) the following equality holds true:

$$Scr(f_{st}(x), g_{st}(y)) = Scr(f,g) + x + y.$$

Remark 3.11. For every composition of homomorphisms f and g satisfying the condition (∂) there exists a number n_0 such that the stable common rank $Sr(f)$ can be calculated by the following formula:

$$Scr(f,g) = Cr(f_{th}(p)_{st}(v), g_{th}(q)_{st}(w)) - v - w$$

for any $p \geq n_0, q \geq n_0, v \geq n_0, w \geq n_0$.

Definition 3.12. The stable common rank from the left (from the right) $Scr_l(f,g)$ ($Scr_r(f,g)$) of the composition of homomorphisms of free modules

$$F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$$

satisfying condition (∂) is the following limit of values of common ranks:

$$Scr_l(f,g) = \lim_{p,v,w\to\infty} (Cr(f_{th,l}(p)_{st}(v), g_{st}(w)) - v - w)$$

$$(Scr_r(f,g) = \lim_{q,v,w\to\infty} (Cr(f_{st}(v), g_{th,r}(q)_{st}(w)) - v - w)).$$

Remark 3.13. For the stable common rank from the left (from the right) $Scr_l(f,g)$ ($Scr_r(f,g)$) of the composition of the homomorphisms satisfying the condition (∂) the analogues of Lemma 3.10 and Remark 3.11 hold true.

Definition 3.14. The defect $\mathbb{D}(f,g)$ of the composition of homomorphisms of free modules

$$F_m \xrightarrow{f} F_n \xrightarrow{g} F_t$$

satisfying condition (∂) is the following number:

$$\mathbb{D}(f,g) = Sr(f) + Sr(g) - Scr(f,g).$$

Remark 3.15. a) For arbitrary composition of homomorphisms f and g satisfying the condition (∂) there exists a number n_0 such that defect $\mathbb{D}(f,g)$ can be calculated by the formula

$$\mathbb{D}(f,g) = R(f_{th}(p,w)_{st}(v)) + R(g_{th}(v,q)_{st}(w)) + Cr(f_{th}(p)_{st}(v), g_{th}(q)_{st}(w))$$

for any $p \geq n_0, q \geq n_0, v \geq n_0, w \geq n_0$;

b) If in the composition of the homomorphisms f and g the module $F_n/f(F_m)$ is stably free, but not free, then $\mathbb{D}(f,g) > 0$.

LEMMA 3.16. *Consider two compositions of homomorphisms of free modules*

$$F_m \xrightarrow{\ f\ } F_n \xrightarrow{\ g\ } F_t$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & F_v & \xrightarrow{id} & F_v & \longrightarrow & 0 \\
 & & \oplus & & \oplus & & \oplus \\
 & & F_m \oplus F_p & \xrightarrow{f_{th,l}(p)} & F_n & \xrightarrow{g_{th,r}(q)} & F_t \oplus F_q \\
 & & \oplus & & \oplus & & \oplus \\
 & & 0 & \longrightarrow & F_w & \xrightarrow{id} & F_w & \longrightarrow & 0
\end{array}$$

satisfying the condition (∂) *(the numbers* p, q, v, w *are nonnegative). Then*

$$\mathbb{D}(f,g) = \mathbb{D}(f_{th,l}(p)_{st}(v), g_{th,r}(q)_{st}(w)).$$

DEFINITION 3.17. The defect from the left (from the right) $\mathbb{D}_l(f,g)$ ($\mathbb{D}_r(f,g)$) of a composition of homomorphisms of free modules

$$F_m \xrightarrow{\ f\ } F_n \xrightarrow{\ g\ } F_t,$$

satisfying condition (∂) is the number

$$\mathbb{D}_l(f,g) = Sr_l(f) + Sr(g) - Scr_l(f,g)$$
$$(\mathbb{D}_r(f,g) = Sr(f) + Sr_r(g) - Scr_r(f,g)).$$

REMARK 3.18. For the defect from the left (from the right) $\mathbb{D}_l(f,g)$ ($\mathbb{D}_r(f,g)$) of a composition of homomorphisms f and g satisfying condition (∂) the analogues of Lemma 3.16 and Remark 3.11 hold true.

4. Homotopy invariants of cochain complexes. The following statement can be found in [3].

PROPOSITION 4.1 (Cockroft-Swan). *Let* $f = f_n : (C,d) \to (\overline{C},\overline{d})$, $n \geq 0$ *be a cochain mapping between the free cochain complexes* (C,d) *and* $(\overline{C},\overline{d})$ *that induces an isomorphism in cohomology. Then there exist contractible free cochain complexes* (D,∂) *and* $(\overline{D},\overline{\partial})$ *such that the cochain complexes*

$$(C \oplus D, d \oplus \partial) \text{ and } (\overline{C} \oplus \overline{D}, \overline{d} \oplus \overline{\partial})$$

are cochain-isomorphic.

If $(C,d) : C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} C^n$ is a free cochain complex over a ring Λ, then the numbers $\mathbb{D}_r(d^0)$, $\mathbb{D}_l(d^{n-1})$, $\mathbb{D}_r(d^0,d^1)$, $\mathbb{D}_l(d^{n-2},d^{n-1})$, $\mathbb{D}(d^i,d^{i+1})$ are defined for $1 \leq i \leq n-3$. The next lemma shows that they are invariants of the homotopy type of a cochain complex (C,d).

LEMMA 4.2. *Let* $(C,d)_\Lambda$ *be the class of free cochain complexes over ring* Λ *homotopy equivalent to cochain complex* $(C,d) : C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} C^n$. *Then for any cochain* *complex* $(D,\partial): D^0 \xrightarrow{\partial^0} D^1 \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} D^n$ *belonging to the class* $(C,d)_\Lambda$ $(n \geq 4)$ *the following equalities hold:*

$$\begin{aligned} \mathbb{D}_r(d^0) &= \mathbb{D}_r(\partial^0), \\ \mathbb{D}_r(d^{n-1}) &= \mathbb{D}_r(\partial^{n-1}), \\ \mathbb{D}_r(d^0, d^1) &= \mathbb{D}_r(\partial^0, \partial^1), \\ \mathbb{D}_l(d^{n-2}, d^{n-1}) &= \mathbb{D}_l(\partial^{n-2}, \partial^{n-1}), \\ \mathbb{D}(d^i, d^{i+1}) &= \mathbb{D}(\partial^i, \partial^{i+1}) \end{aligned}$$

for $1 \leq i \leq n-3$.

Definition 4.3. Let (C,d): $C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{n-1}} C^n$ be a free cochain complex. Then the cochain complex $(C(i), d(i)) : C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{i-1}} C^i$ is called i-th skeleton of cochain complex (C,d).

Let $(C^*, d^*))$: $C_0 \xrightarrow{d_1} C_1 \to \dots \xrightarrow{d_n} C_n$ be a sequence of free Hilbert $N[G]$-modules and bounded $\mathbb{C}[G]$-map such that $d_{i+1} \circ d_i = 0$. This sequence is called a Hilbert complex. *The reduced cohomology of a Hilbert complex* (C^*, d^*) is the collection of the Hilbert $N[G]$-modules $\overline{H^i}_{(2)}(C^*, d^*) = Kerd^i / \overline{Imd^{i-1}}$.

Definition 4.4. Consider a free cochain complex over $\mathbb{Z}[G]$

$$(C^*, d^*) : C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{n-1}} C^n.$$

The Hilbert complex

$$(L^2(G) \otimes_{\mathbb{Z}[G]} C^*, Id \otimes_{\mathbb{Z}[G]} d^*) :$$

$$L^2(G) \otimes_{\mathbb{Z}[G]} C^0 \overset{Id \otimes_{\mathbb{Z}[G]} d^0}{\longrightarrow} L^2(G) \otimes_{\mathbb{Z}[G]} C^1 \longrightarrow \dots \overset{Id \otimes_{\mathbb{Z}[G]} d^{n-1}}{\longrightarrow} L^2(G) \otimes_{\mathbb{Z}[G]} C^n$$

of free Hilbert $N[G]$-modules is the *Hilbert complex generated by the* $\mathbb{Z}[G]$*-complex.*

Consider the i-th skeletons of these complexes

$$(C^*(i), d^*(i)) : C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{i-1}} C^i,$$

$$L^2(G) \otimes_{\mathbb{Z}[G]} C^0 \overset{Id \otimes_{\mathbb{Z}[G]} d^0}{\longrightarrow} L^2(G) \otimes_{\mathbb{Z}[G]} C^1 \to \dots \overset{Id \otimes_{\mathbb{Z}[G]} d^{i-1}}{\longrightarrow} L^2(G) \otimes_{\mathbb{Z}[G]} C^i,$$

Set $\Gamma^i = C^i / d^{i-1}(C^{i-1})$. It is clear that

$$\widehat{\Gamma^i} = L^2(G) \otimes_{\mathbb{Z}[G]} C^i / \overline{Id \otimes_{\mathbb{Z}[G]} d^{i-1}(L^2(G) \otimes_{\mathbb{Z}[G]} C^{i-1})}$$

is the i-th Hilbert $N[G]$-module of reduced cohomology of the i-th skeleton of the Hilbert complex

$$(L^2(G) \otimes_{\mathbb{Z}[G]} C^*(i), Id \otimes_{\mathbb{Z}[G]} d^*(i)).$$

Definition 4.5 ([18, 19]). For the cochain complex (C^*, d^*) over $\mathbb{Z}[G]$ set

$$\widehat{S}^i_{(2)}(C^*, d^*) = \mu_s(\Gamma^i) - dim_{N[G]} \widehat{\Gamma}^i.$$

Lemma 4.6. *The numbers* $\widehat{S}^i_{(2)}(C^*, d^*)$ *are non-negative for every* i. *If* (C^*, d^*) *and* (D^*, ∂^*) *are two homotopy equivalent free cochain complexes over the group ring* $\mathbb{Z}[G]$

then

$$\widehat{S}^i_{(2)}(C^*, d^*) = \widehat{S}^i_{(2)}(D^*, \partial^*).$$

DEFINITION 4.7. A free cochain complex $(C, d) : C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} C^n$ is called *minimal in dimension* i if for every free cochain complex $(D, \partial) : D^0 \xrightarrow{\partial^0} D^1 \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} D^n$ which is homotopy equivalent to (C, d) one has $\mu(C^i) \leq \mu(D^i)$, where $\mu(C^i)$ is the rank of the free module C^i. A free cochain complex (C, d) is called *minimal* if it is minimal in all dimensions.

It is obvious that, for every i, in the homotopy class of any free cochain complex (C, d) there always exists a minimal free cochain complex in dimension i. However in the homotopy class of an arbitrary free cochain complex (C, d) there may exist no minimal free cochain complexes, because of the existence of stably free modules.

DEFINITION 4.8. The *i-th homotopy Morse number* of a cochain complex (C, d) over a ring Λ is the number $\mathcal{M}_i(C, d) = \mu(D_i)$, where

$$(D, \partial) : D^0 \xrightarrow{\partial^0} D^1 \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} D^n$$

is the minimal cochain complex in dimension i which is homotopy equivalent to (C, d).

The next result can be found in [20].

THEOREM 4.9. *Let (C, d): $C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{n-1}} C^n$ be a free cochain complex over a group ring $\mathbb{Z}[G]$ ($n \geq 4$). Its i-th homotopy Morse numbers satisfy the following equalities:*

$$\mathcal{M}_0(C, d) = \mathbb{D}_r(d^0) + \widehat{S}^1_{(2)}(C, d) + dim_{N[G]}(H^0(L^2(G) \otimes_{\mathbb{Z}[G]} C, id \otimes_{\mathbb{Z}[G]} d)),$$

$$\mathcal{M}_1(C, d) = \mathbb{D}_l(d^0, d^1) + \widehat{S}^1_{(2)}(C, d) + \widehat{S}^2_{(2)}(C, d) + dim_{N[G]}(H^1(L^2(G) \otimes_{\mathbb{Z}[G]} C, id \otimes_{\mathbb{Z}[G]} d))$$

$$\mathcal{M}_i(C, d) = \mathbb{D}(d^{i-1}, d^i) + \widehat{S}^i_{(2)}(C, d) + \widehat{S}^{i+1}_{(2)}(C, d) + dim_{N[G]}(H^i(L^2(G) \otimes_{\mathbb{Z}[G]} C, id \otimes_{\mathbb{Z}[G]} d)),$$

for $2 \leq i \leq n - 2$,

$$\mathcal{M}_{n-1}(C, d) = \mathbb{D}_l(d^{n-2}, d^{n-1}) + \widehat{S}^{n-1}_{(2)}(C, d) + \mu(H^n(C, d)) + dim_{N[G]}(H^{n-1}(L^2(G) \otimes_{\mathbb{Z}[G]} C, id \otimes_{\mathbb{Z}[G]} d)) - dim_{N[G]}(H^n(C \otimes_{\mathbb{C}[G]} C, id \otimes d)),$$

$$\mathcal{M}_n(C, d) = \mu(H^n(C, d)),$$

where $H^i(L^2(G) \otimes_{\mathbb{Z}[G]} C, id \otimes_{\mathbb{Z}[G]} d))$ is the cohomology of the cochain complex

$$(L^2(G) \otimes_{\mathbb{Z}[G]} C, id \otimes_{\mathbb{Z}[G]} d)).$$

REMARK 4.10. The number $\mathbb{D}(d^{i-1}, d^i)$ arises in this theorem because in the definition of the number $S^i(C, d)$ we take the number $\mu_s(\Gamma^i)$ but not the number $\mu(\Gamma^i)$. For example, if in Remark 3.15 the module $C^i/d^{i-1}(C^{i-1})$ is stably free but not free, then $\mathbb{D}(d^{i-1}, d^i) > 0$.

DEFINITION 4.11. The *Morse number* of a cochain complex

$$(C, d) : C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{n-1}} C^n$$

over a ring Λ is the number $\mathcal{M}(C, d) = \sum_{i=0}^{n} \mu(C_i)$.

Definition 4.12. The *homotopy Morse number* $\mathcal{M}_h(C,d)$ of a cochain complex (C,d) over a ring Λ is the minimum of Morse numbers taken over all cochain complexes homotopy equivalent to (C,d).

Theorem 4.13. *Let* (C,d): $C^0 \xrightarrow{d^0} C^1 \to \dots \xrightarrow{d^{n-1}} C^n$ *be a free cochain complex over a group ring* $\mathbb{Z}[G]$ $(n \geq 4)$ *such that*

$$\begin{aligned} \mathbb{D}_r(d^0) &= 0, \\ \mathbb{D}_r(d^{n-1}) &= 0, \\ \mathbb{D}_r(d^0, d^1) &= 0, \\ \mathbb{D}_l(d^{n-2}, d^{n-1}) &= 0, \\ \mathbb{D}(d^i, d^{i+1}) &= 0 \end{aligned}$$

for $1 \leq i \leq n-3$. *Then in the homotopy type of* (C,d) *there exists a minimal cochain complex and the homotopy Morse number of* (C,d) *is equal to*

$$\begin{aligned} \mathcal{M}_h(C,d) = \\ & 2\sum_{i=1}^{n-2}(\widehat{S}^i_{(2)}(C,d)) + \widehat{S}^{n-1}_{(2)}(C,d) + \sum_{i=0}^{n-1}(dim_{N[G]}(H^i(L^2(G)\otimes_{\mathbb{Z}[G]} C, id\otimes_{\mathbb{Z}[G]} d))) \\ & + 2\mu(H^n(C,d)) - dim_{N[G]}(H^n(L^2(G)\otimes_{\mathbb{Z}[G]} C, id\otimes_{\mathbb{Z}[G]} d)). \end{aligned}$$

Proof. From the conditions of the theorem it follows that in the homotopy type of (C,d) any cochain complex

$$(D,d) : D^0 \xrightarrow{d^0} D^1 \to \dots \xrightarrow{d^{n-1}} D^n$$

satisfies the condition $\mu(D_i/d^{i-1}D_{i-1}) = \mu_s(D_i/d^{i-1}D_{i-1})$ for all i. From [19] it follows that in the homotopy type of (C,d) there exists a minimal cochain complex. The value of the Morse number of a minimal cochain complex is equal to the homotopy Morse number of (C,d). The value of the Morse number of a minimal cochain complex may be found by direct calculations. ∎

5. Applications. Let K be a topological space with the structure of a finite CW-complex and with a non-zero fundamental group $\pi = \pi_1(K)$. Consider the universal covering space $p : \widetilde{K} \to K$ of K. Using the map p, lift the structure of CW-complex from K to $\widetilde{K}$. On the universal covering space $\widetilde{K}$ there exists a cell structure preserving the free action of the fundamental group $\pi = \pi_1(K)$. This action turns each chain group $C_i(\widetilde{K},\mathbb{Z})$ into a left module over the group ring $\mathbb{Z}[\pi]$. It is evident that the resulting chain module $C_i(\widetilde{K},\mathbb{Z})$ is free and finitely generated by i-cells of K. As a result we obtain the following free chain complex over the ring $\mathbb{Z}[\pi]$:

$$C_*(\widetilde{K}) : C_0(\widetilde{K},\mathbb{Z}) \xleftarrow{d_1} C_1(\widetilde{K},\mathbb{Z}) \leftarrow \dots \xleftarrow{d_n} C_n(\widetilde{K},\mathbb{Z}).$$

Denote by $w : \pi_1(K) \to \mathbb{Z}_2$ the homomorphism of orientation (the first Stiefel-Whitney class). Define an involution on the group ring $\mathbb{Z}[\pi]$ by the formula $g \to w(g)g^{-1}$. This involution makes it possible to turn every right $\mathbb{Z}[\pi]$-module into a left $\mathbb{Z}[\pi]$-module. Making use of this involution, we turn the right $\mathbb{Z}[\pi]$-module

$$C^i(\widetilde{K},\mathbb{Z}) = Hom_{\mathbb{Z}[\pi]}(C_i(\widetilde{K},\mathbb{Z}),\mathbb{Z}[\pi])$$

into a left one and consider the free cochain complex

$$C^*(\widetilde{K}) : C^0(\widetilde{K},\mathbb{Z}) \xrightarrow{d^0} C^1(\widetilde{K},\mathbb{Z}) \to \dots \xrightarrow{d^{n-1}} C^n(\widetilde{K},\mathbb{Z}).$$

Cohomology modules of this cochain complex are the cohomology with compact supports of CW-complex $\widetilde{K}$. Taking the tensor product of $C^*(\widetilde{K})$ and $L^2(\pi)$ as $\mathbb{Z}[\pi]$-modules we obtain the Hilbert complex

$$C^*_{(2)}(\widetilde{K}) : L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^0(\widetilde{K},\mathbb{Z}) \overset{id\otimes d^0}{\longrightarrow} L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^1(\widetilde{K},\mathbb{Z}) \to \dots$$
$$\dots \overset{id\otimes d^{n-1}}{\longrightarrow} L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^n(\widetilde{K},\mathbb{Z}).$$

The $L^2(\pi)$-module of cohomology of this Hilbert complex are $L^2(\pi)$-module of cohomology of the space K. Since the cochain complex $C^*(\widetilde{K})$ is constructed from the cellular structure of the space $\widetilde{K}$, we see that the segments of cochain complexes

$$C^*(\widetilde{K})(i) : C^0(\widetilde{K},\mathbb{Z}) \xrightarrow{d^0} C^1(\widetilde{K},\mathbb{Z}) \to \dots \xrightarrow{d^{i-1}} C^i(\widetilde{K},\mathbb{Z});$$
$$C^*_{(2)}(\widetilde{K})(i) : L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^0(\widetilde{K},\mathbb{Z}) \overset{id\otimes d^0}{\longrightarrow} L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^1(\widetilde{K},\mathbb{Z}) \to \dots$$
$$\dots \overset{id\otimes d^{i-1}}{\longrightarrow} L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^i(\widetilde{K},\mathbb{Z})$$

are evidently the cochain complexes of the i-th skeleton of the cellular decomposition of $\widetilde{K}$ and K respectively. Therefore the $\mathbb{Z}[\pi]$-module

$$\widehat{\Gamma}^i(\widetilde{K}) = C^i(\widetilde{K},\mathbb{Z})/d^{i-1}(C^{i-1}(\widetilde{K},\mathbb{Z})),$$

resp. the $L^2(G)$-module

$$\Gamma^i(K) = L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^i(\widetilde{K},\mathbb{Z})/id\otimes d^{i-1}(L^2(\pi)\otimes_{\mathbb{Z}[\pi]} C^{i-1}(\widetilde{K},\mathbb{Z})),$$

can be interpreted as the i-th cohomology module with compact support (the i-th $L^2(\pi)$-module of cohomology) of the i-th skeleton of $\widetilde{K}$ (the i-th skeleton of K).

Definition 5.1. For a cell complex K set

$$\begin{aligned}
\widehat{S}^i_{(2)}(K) &= \mu_s(\widehat{\Gamma}^i(\widetilde{K})) - dim_{N[\pi]}(\Gamma^i(K)),\\
\widehat{\mathbb{D}}^0_r(K) &= \mathbb{D}_r(d^0),\\
\widehat{\mathbb{D}}^{n-1}_l(K) &= \mathbb{D}_l(d^{n-1}),\\
\mathbb{D}^0_r(K) &= \mathbb{D}_r(d^0,d^1),\\
\mathbb{D}^{n-2}_l(K) &= \mathbb{D}_l(d^{n-2},d^{n-1}),\\
\mathbb{D}^i(K) &= \mathbb{D}(d^{i-1},d^i)
\end{aligned}$$

for $1\le i\le n-2$.

It is well known that all chain complexes constructed from cellular decompositions of the topological space K have the same homotopy type. Therefore it follows directly either from the previous discussions or from [11, 19] that the numbers $\widehat{S}^i_{(2)}(W)$ and $\widehat{\mathbb{D}}^0_r(K)$, $\widehat{\mathbb{D}}^{n-1}_l(K)$, $\mathbb{D}^0_r(K)$, $\mathbb{D}^{n-2}_l(K)$, $\mathbb{D}^i(K)$ for $1\le i\le n-2$ are invariants of the homotopy type of the topological space K.

For a smooth manifold W there is a construction of cochain complex via Morse functions. The details can be found in [17].

Let $(W^n, V_0^{n-1}, V_1^{n-1})$ be a compact smooth manifold with boundary $\partial W^n = V_0^{n-1} \cup V_1^{n-1}$ (one of V_i^{n-1} or both may be empty). Let also $\pi = \pi_1(W^n)$ be the fundamental group of the manifold W^n. Denote by $p : (\widetilde{W^n}, \widetilde{V}_0^{n-1}, \widetilde{V}_1^{n-1}) \to (W^n, V_0^{n-1}, V_1^{n-1})$ the universal covering space. Here $\widetilde{V}_i^{n-1} = p^{-1}(V_i^{n-1})$. Let us choose on W^n an ordered Morse function

$$f : W^n \to [0, 1],\ f^{-1}(0) = V_0^{n-1},\ f^{-1}(1) = V_1^{n-1}$$

and a gradient-like vector field ξ [19]. Using the mapping p, lift f and ξ to $\widetilde{W^n}$, and denote a lifted function and a vector field by $\widetilde{f}$ and $\widetilde{\xi}$ respectively. Using f, ξ and $\widetilde{f}, \widetilde{\xi}$ construct chain complexes of abelian groups:

$$C_*(W^n, f, \xi) : C_0 \xleftarrow{d_1} C_1 \leftarrow \ldots \xleftarrow{d_n} C_n, ;$$

$$C_*(\widetilde{W^n}, \widetilde{f}, \widetilde{\xi}) : \widetilde{C}_0 \xleftarrow{\widetilde{d}_1} \widetilde{C}_1 \leftarrow \ldots \xleftarrow{\widetilde{d}_n} \widetilde{C}_n,$$

where

$$C_i = H_i(W_i, W_{i-1}, \mathbb{Z}), \qquad \widetilde{C}_i = H_i(\widetilde{W}_i, \widetilde{W}_{i-1}, \mathbb{Z});$$

and

$$\widetilde{W}_i = \widetilde{f}^{-1}[0, a_i], \qquad W_i = f^{-1}[0, a_i]$$

are submanifolds containing all critical points of indices less than or equal to i. For the generators of the chain groups C_i $(\widehat{C_i})$ one can take middle disks of critical points of index i constructed by the vector field ξ $(\widehat{\xi})$. The fundamental group $\pi = \pi_1(W^n)$ acts on manifolds $\widetilde{W^n}$. Making use of this actions, we can turn the chain group $\widetilde{C}_i$ into a finitely generated module over the ring $\mathbb{Z}[\pi]$. Making use of the involution, we turn the right $\mathbb{Z}[\pi]$-modules

$$C^{(i)} = Hom_{\mathbb{Z}[\pi]}(C_i, \mathbb{Z}[\pi]),$$

$\widehat{\mathbb{D}}_r^0(K)$, $\widehat{\mathbb{D}}_l^{n-1}(K)$, $\mathbb{D}_r^0(K)$, $\mathbb{D}_l^{n-2}(K)$, $\mathbb{D}^i(K)$ for $1 \le i \le n-2$ into left ones and construct the free cochain complex

$$C^*(\widetilde{W^n}, \widetilde{f}, \widetilde{\xi}) : \widetilde{C}^{(0)} \xrightarrow{\widetilde{d}^{(0)}} \widetilde{C}^{(1)} \to \ldots \xrightarrow{\widetilde{d}^{(n-1)}} \widetilde{C}^{(n)}.$$

Taking the tensor product of $C^*(\widetilde{W^n}, \widetilde{f}, \widetilde{\xi})$ and $L^2(\pi)$ as a $\mathbb{Z}[\pi]$-module, we obtain the cochain complex of abelian groups which can be used for the definition of the numbers $\widehat{S}^i_{(2)}(W^n)$ and $\mathbb{D}^i(W^n)$. It is proved in [11] that the chain complexes constructed from Morse functions on the manifold W^n via cellular decomposition of W^n have the same homotopy type. This means that the values of the numbers $\widehat{S}^i_{(2)}(W^n)$ and $\mathbb{D}^i(W^n)$ do not depend on the method of constructing a chain complex.

Definition 5.2. The i-th *Morse number* $\mathcal{M}_i(W^n)$ of a manifold W^n is the minimal number of critical points of index i taken over all Morse functions on W^n.

It is known [2, 10, 19] that for closed smooth manifolds of dimension greater than 6 the i-th Morse numbers are invariants of the homotopy type. In [20] the following theorem is proved.

THEOREM 5.3. *Let W^n $(n \geq 8)$ be a smooth closed manifold with $\pi = \pi_1(W^n)$. The following equality holds for the i-th Morse number $4 \leq i \leq n-4$:*

$$\mathcal{M}_i(W^n) = \mathbb{D}^i(W^n) + \widehat{S}^i_{(2)}(W^n) + \widehat{S}^{i+1}_{(2)}(W^n) + dim_{N(G)}(H^i_{(2)}(W^n, \mathbb{Z})).$$

DEFINITION 5.4. The Morse number $\mathcal{M}(W^n)$ of a manifold W^n is the minimum of the total number of critical points taken over all Morse functions on W^n.

THEOREM 5.5. *Let $(W^n, V_0^{n-1}, V_1^{n-1})$ $(n \geq 6)$ be a compact smooth manifold with boundary $\partial W^n = V_0^{n-1} \cup V_1^{n-1}$ and $\pi = \pi_1(W^n)$ be the fundamental group of the manifold W^n. Suppose that $\pi(V_i^{n-1}) \to \pi_1(W^n)$ is an isomorphism, $Wh(\pi) = 0$ and*

$$\widehat{\mathbb{D}}^0_r(W^n) = \widehat{\mathbb{D}}^{n-1}_l(W^n) = \mathbb{D}^0_r(W^n) = \mathbb{D}^{n-2}_l(W^n) = \mathbb{D}^i(W^n) = 0$$

for all i. Then

$$\mathcal{M}(W^n) = 2\sum_{i=2}^{n-3} \widehat{S}^i_{(2)}(W^n) + \widehat{S}^{n-2}_{(2)}(W^n) + \sum_{i=1}^{n-2} dim_{N[\pi]}(H^i_{(2)}(W^n, \mathbb{Z}))$$
$$+ 2\mu(H^{n-1}(\widetilde{W}^n, \mathbb{Z}[\pi])) - dim_{N[\pi]}(H^{n-1}_{(2)}(W^n, \mathbb{Z})).$$

Proof. Let f be an arbitrary ordered Morse function, ξ a gradient-like vector field on W^n, and

$$C_*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi}) : \widetilde{C}_0 \xleftarrow{\widetilde{d}_1} \widetilde{C}_1 \leftarrow \ldots \xleftarrow{\widetilde{d}_n} \widetilde{C}_n$$

the chain complex associated with them. Denote by

$$C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi}) : \widetilde{C}^{(0)} \xrightarrow{\widetilde{d}^{(0)}} \widetilde{C}^{(1)} \to \ldots \xrightarrow{\widetilde{d}^{(n-1)}} \widetilde{C}^{(n)}$$

the cochain complex constructed from the chain complex $C_*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})$. It is clear that if the chain complex $C_*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})$ is minimal in dimension i then the cochain complex $C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})$ is minimal in dimension i. It is known that the operation of stabilization of the homomorphisms $\widetilde{d}_i$ can be realized by changing the Morse function and gradient-like vector field on W^n. But the inverse operation, the elimination of contractible free chain complex of the form $0 \to \overline{C}_i \to \overline{C}_{i+1} \to 0$ from the chain complex $C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})$, cannot always be realized by changing the Morse function and gradient-like vector field on W^n. This is possible provided $n \geq 6$ and $Wh(\pi) = 0$ [19].

Let $(\overline{C}, \overline{d})$ be a minimal chain complex homotopy equivalent to the chain complex $C_*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi})$. In our situation it always exists. By Proposition 4.1 there exist contractible free chain complexes (D, ∂) and $(\overline{D}, \overline{\partial})$ such that the chain complexes

$$(C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi} \oplus D, d \oplus \partial)) \qquad \text{and} \qquad (\overline{C} \oplus \overline{D}, \overline{d} \oplus \overline{\partial})$$

are chain-isomorphic. The previous remark ensures the existence of a Morse function g and gradient-like vector field that realize the complex $(C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi} \oplus D, d \oplus \partial))$.

Using elimination of contractible free chain complexes of the form $0 \to \overline{C}_i \to \overline{C}_{i+1} \to 0$ and $0 \to \overline{C}_{i-1} \to \overline{C}_i \to 0$ we can obtain from $(C^*(\widetilde{W}^n, \widetilde{f}, \widetilde{\xi} \oplus D, d \oplus \partial))$ the minimal chain complex $(\widehat{C}, \widetilde{d})$. The conditions that $n \geq 6$ and $Wh(\pi) = 0$ ensure the existence of a Morse function g and gradient-like vector field η that realize the complex $(\widehat{C}, \widehat{d})$. The number of critical points of Morse function g can be computed using previous formulas presented in Theorem 4.13. ∎

The estimations for Morse numbers are investigated in [1, 5–9, 12–21]. Our approach differs from those papers. In next papers we shall give the values of Morse numbers for other classes of manifolds.

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