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HOMOTOPY THEORY OF THE MASTER EQUATION PACKAGE APPLIED TO ALGEBRA AND GEOMETRY: A SKETCH OF TWO INTERLOCKING PROGRAMS

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Abstract. Using the algebraic theory of homotopies between maps of dga's we obtain a homotopy theory for algebraic structures defined by collections of multiplications and comultiplications. This is done by expressing these structures and resolved versions of them in terms of dga maps. This same homotopy theory of dga maps applies to extract invariants beyond homological periods from systems of moduli spaces that determine systems of chains that satisfy master equations like $dX + X * X = 0$. Minimal models of these objects resemble Postnikov decompositions in the homotopy theory of spaces and maps.

Introduction and sketch. We interpret the words "master equation and solution of master equation up to equivalence" in mathematical terms as triples (T, C, M) where T is a a free triangular dga (see below) over an operad O, C is any dga over O and $M : T \to C$ is any dga map over O and the triple is taken up to homotopy equivalence (definition 3.5) of dgOa maps. The dgOa map from the free triangular object T to the general dgOa C is called the master equation package. The nontrivial part of the story is the notion of homotopy of dgOa maps which leads directly to the lemmas mentioned below. Actually to develop all the geometric and algebraic applications, we envisage only certain specific operads that need be considered. These correspond to either composing multilinear maps or to gluing certain geometric objects together like graphs or Riemann surfaces. We refer to these special operads needed to describe gluing or composition as combination operads.

We sketch two general applications:

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Firstly, to the theory of the definition and homotopy theory of infinity or resolved versions of general algebraic structures. This is new to the author for structures such as noncompact Frobenius algebras and Lie bialgebras. Here the target C would be the total Hom complex between various tensor products of another chain complex B, $C = HomB$, O describes combinations of operations like composition and tensor product sufficient to describe the algebraic structure and one says that B has the algebraic structure in question. The master equation package is a description of the resolved or infinity form of the structure. T is generated by the labels of the multilinear operations defining the resolved structure. The differential in T describes relations in the structure that say certain combinations of operations are chain homotopic to zero. M refers to the solution of the master equation, i.e., the presentation of T, in C which is then interpreted as a dgOa map from T to C.

Secondly, to geometric systems of moduli spaces up to deformation like the moduli of J holomorphic curves in a symplectic manifold provided with an almost complex structure or the moduli space of instantons in the theory of bundles with connections over a four manifold. Here C is some geometric chain complex containing the fundamental top chains of the pseudo manifold moduli spaces of the geometric problem. It is assumed that C is provided with certain geometric operations of combination corresponding to gluing geometric objects together. These are the operations required to describe the boundary of the fundamental chains on the moduli spaces, and these are organized algebraically by the combination operad O. The common feature of these problems is that the boundary of these fundamental top chains can be described by combining together moduli spaces of simpler complexity in a master equation formalism.

We also discuss analogues of homotopy groups for dgOa's and Postnikov systems for dgOa's and for maps and impediments to using them related to linear terms in the master equation. These linear terms are called anomalies as in the physics literature because they can prevent certain evaluations related to the fundamental chains from yielding information independent of certain arbitrary choices made in the evaluation. In effect analogues of anomalies that appear there are here given a dga homotopy interpretation.

Certain operads which organize concrete composition operations or specific gluings of geometric objects and which we refer to here as combination operads arise in the algebraic study of multilinear structures and the geometric study of systems of moduli spaces. Below we treat general operads but we are really only thinking of these specific composition or gluing operads. For any operad O one may define differential graded algebras over O. Let us call them dgOa's. Fixing O they form an obvious category where the maps are dgOa maps. We will make use of a derived homotopy category based on free resolutions of dgOa's and a notion of homotopy between dgOa maps. Resolutions give a procedure to replace any dgOa by a nilpotent version of a free dgOa. There are two similar classes of examples relevant here where the combination operad O describes compositions or tensor products of multilinear operations in the first algebraic application and where the combination operad O describes gluing or union of geometric chains in the second geometric application.

1. The setup of the algebraic application. Consider collections of multilinear j to k operations for various j and k positive that define examples of algebraic structures like traceless Frobenius algebra or Lie bialgebra. Such examples of algebraic structures, because multiple outputs appear, cannot themselves be described as algebras over operads. However the concepts that might be used in terms of dioperads, properads or props to describe these structures will be replaced here by the notion of the master equation package. This will be possible because any prop, properad, dioperad respectively can be described as an algebra over a specific combination operad, namely, over the combination operad describing the combinations of operations required in the definition of that prop, properad, dioperad, respectively.

There are a few choices for which combinations of operations are allowed varying from the most parsimonious to the most generous.

For example consider the definition of a Lie bialgebra involving a two-to-one product and a one-to-two coproduct. These are both skew commutative and satisfy three quadratic relations requiring one variable substitution. There are four combination operads that could be employed here. 1) All the operations freely generated by composing these two in all possible ways with only one output inserted in to an input; 2) same as 1) but with multiple outputs inserted into multiple inputs; 3) same as is 1) with tensor products thrown in; 4) same as 2) with tensor products thrown in.

Since the generating operations are both graded skew symmetric the combination operads in choices 1) 2) 3) 4) can be described as follows: 1) by gluing trivalent trees with labeled inputs and outputs; 2) by gluing all connected directed trivalent graphs with labeled non empty sets of input and output vertices and no directed cycles: 3) as in 1) but not necessarily connected; 4) same as in 2) but not necessarily connected. On the other hand consider the definition of a noncommutative Frobenius algebra (without a trace). This is defined by an associative product and a coassociative coproduct satisfying two quadratic compatibility relations. The four corresponding combination operads will be described by the graphs as above but where the half edges at a vertex have the additional structure of a cyclic order. We will see that resolved or infinity versions of these algebraic structures will use higher valence graphs as well.

2. The setup of the geometric application. Consider gluing operations describing compactifications in a system of moduli spaces coming from some geometric problem involving, for example, Riemann surfaces, connections on G bundles or configuration spaces of manifolds. The main point is a hereditary property of compactifications of these moduli spaces. The homotopy theory of the master equation described here becomes relevant if the points added in the compactifications of these moduli spaces can be described in terms of other moduli spaces of the same system. This description uses gluing operations on moduli spaces or their fundamental chains and these operations in turn are described abstractly by a relevant gluing or combination operad.

The category of dgOa's will be used in the two settings above via its associated homotopy theory.

3. Basic facts of dgOa homotopy theory. 1) One knows that there are free dgOa algebras associated to any system of generating vector spaces with zero differential. For example in the associative case the free algebra is the tensor algebra on the generating spaces without the unit or ground ring term, namely the augmentation ideal of the tensor algebra with unit.

One says a dgOa is free if it is free in this sense after suppressing the differential. A free dgOa is called triangular if there is a partial ordering on the generators, with all descending chains finite, so that the differential of any generator is a sum of O-operations applied to strictly smaller generators for the partial order.

A free triangular dgOa is the analog of nilpotent space or nilpotent differential Lie algebra in usual homotopy theory.

LEMMA 3.1. *If A is any dgOa there is a dgOa map $T \to A$ from a free triangular dgOa T to A inducing an isomorphism on homology. Such maps are called resolutions.*

This follows from a staightforward induction.

The first step of one induction, which is not the most efficient, is to choose a generating set for the homology of A, form the free algebra on these over O, define the differential to be zero there and map the generators to cycles representing the named homology classes. The second step of this induction is to add generators to the domain whose differentials put in a spanning set of homology relations among the cycles in A chosen in the first step. These second stage generators are mapped to elements in A which exist because these homology relations are satisfied. The third step puts in relations that kill the kernel of this dgOa map etc. There are many constructions of resolutions.

In the homotopy theory of CW complexes the analog of resolution in this sense is the Quillen plus construction. It is unique up to homotopy.

2) One also knows how to regain uniqueness of free resolutions in the case of modules over a ring using chain equivalences and chain homotopies. Chain homotopies between chain maps of free chain complexes which are bounded from below say can be defined inductively by solving linear equations like $dx = y$, with y determined inductively and x unknown. There is an analogous but more nontrivial notion of homotopy between dgOa maps from a free triangular dgOa T into A an arbitrary dgOa. The theory follows the same line as developed in [8] for dga's over the graded commutative operad. Now one is inductively solving a triangular system of equations $dx =$ sum of O-operations of y's with x unknown and the right hand side determined inductively. This theory of homotopies is described in detail for associative algebras over Novikov rings in [1]. This notion of homotopy is also used in papers by Markl [5, 6, 7].

Using natural obstruction theory arguments one can show two lemmas:

LEMMA 3.2. *A dgOa map from T, a free triangular dgOa, into any dgOa A can be lifted up to homotopy into B for any dgOa map $B \to A$ which induces an isomorphism on homology. The lift is unique up to homotopy.*

COROLLARY 3.3. *A map between free triangular dgOa's $T \to T'$ inducing isomorphisms on homology is a homotopy equivalence in the usual sense: there is a map $T' \to T$ so that each composition is homotopic to the identity.*

LEMMA 3.4. *Given two resolutions $T \to A$ and $T' \to A$ there is a homotopy equivalence between T and T' which is unique up to homotopy such that the diagram into A commutes up to homotopy.*

DEFINITION 3.5. Two maps $T \to A$ and $T' \to A'$ are said to be *homotopy equivalent* if there are homotopy equivalences f between T and T' and g between the resolutions of A and A' so that the lifted maps from T and T' to these resolutions together with f and g form a commutative square up to homotopy.

4. First application: general algebraic structures, resolved infinity versions thereof and their homotopy theory. Any algebraic structure described by j to k multilinear operations, for various k and j positive, on a chain complex C satisfying various identities can be viewed as a dgOa map over some composition operad O. The domain of this "structure map" is a dgOa whose presentation in terms of generators and relations defines the algebraic structure in question where the minimal combination operad O is determined by just those kind of combinations required to express the relations defining the algebraic structure. For example in the Lie bialgebra case above there were two generators and three quadratic relations which in the most parsimonious description required a composition operad defined by gluing together at single external edges directed trivalent trees. Then O may be enlarged if desired by adding further combinations as illustrated in the examples above. The range of the structure map is the total *hom* complex, denoted $Hom(C)$ between the various tensor powers of C endowed with the composition and tensor product operations labeled by the operad chosen O.

DEFINITION 4.1. An *algebraic structure* on B is defined to be a dgOa map of any dgOa into $HomB$ regarded as a dgOa where O is the combination operad describing the operations of composition and tensor product considered as part of the structure. If the domain of the dgOa map is a free triangular dgOa the structure is called a *resolved* or *infinity algebraic structure*. Any algebraic structure has a resolved or infinity version obtained by replacing the domain of the structure map by a resolution of the domain. Forming composition with the resolution map associates with one particular instance of an algebraic structure a particular instance of a resolved or infinity algebraic structure.

Remark: familiar examples of resolved or infinity versions of algebraic structures include associative infinity, Lie infinity and commutative infinity. Other useful examples are Gerstenhaber infinity and BV infinity. The above theory now yields Frobenius infinity (noncompact case) and Lie bialgebra infinity.

Together with a notion of homotopy equivalence (see below) all of these correspond to free resolutions of the appropiate algebra over the appropiate composition operad defining the structure. The corresponding free triangular dgOa in each case is by definition formal, which by definition means it is homotopy equivalent to its own homology algebra as a dgOa.

Note however we have extended the terminology of resolved or infinity algebra beyond this formal case to include those structures defined by any free triangular dgOa. We can think of these examples as being resolved or infinity versions of their own homotopy type which may not be formal. Their theory however has the same form as the formal examples. A case in point that appears in string topology and in symplectic topology is a structure dubbed a quantum Lie bialgebra. Here there is an operation with k inputs and j outputs for each genus $g = 0, 1, 2, \dots$ with k and j both positive integers. The differential applied to the generators are quadratic expressions obtained summing over all binary gluings of g-decorated corollae with inputs and outputs labeled.

The genus zero part of the structure defines an infinity Lie bialgebra. There is a genus one relation, called the involutive relation, holding in the homology, which may not be completely resolved by the rest of the structure and this homotopy type may not be formal. Nevertheless it is a natural resolved structure that appears independently in three contexts: string topology, symplectic topology and in algebra via bar and cobar constructions.

Similarly there is a construction in riemannian geometry by Costello producing a structure which is like the associative analogue of what was just said for Lie and which has been dubbed a quantum cyclic A-infinity algebra. Now one has an operation for each planar g-decorated corollae with inputs and outputs interspersed around the boundary of the corollae.

The role of the involutive relation in the Lie context above which created some complexity (like possible non formality) is now played by an euler relation which creates complexity.

Definition 4.2. If C and D are two chain complexes, a *$HomO$ quasi isomorphism* from C to D is a homotopy class of homotopy equivalences between a resolution of $HomC$ and a resolution of $HomD$ as dgOa algebras.

It follows from the definitions that one may transport infinity algebraic structures up to equivalence back and forth between C and D by a $HomO$ quasi isomorphism.

To use this notion the following lemma is useful.

Lemma 4.3. *Suppose C and D are two quasi isomorphic chain complexes over the rationals. Then the dgOa algebras $HomC$ and $HomD$ have homotopy equivalent resolutions. In other words, an ordinary quasi isomorphism implies a $HomO$ quasi isomorphism.*

The idea of the proof is to prove it for the case when $D = H$ is the homology of C. In this case there is a dgOa map from $HomH$ to $HomC$ using a purely algebraic analogue of a Hodge decomposition of C. A multilinear operation on tensor products of harmonic elements can be extended to all of C by defining it to be zero on tensors with exact or coexact factors. This map induces an isomorphism on homology of the Hom complexes.

Definition 4.4. Two algebraic structures with possibly different presentations on possibly different chain complexes are called *quasi isomorphic* or *homotopy equivalent* if their associated infinity versions have homotopy equivalent structure maps (Definition 3.5) after lifting them to resolutions of the Hom complexes.

5. Second application: nonlinear homology of systems of geometric moduli spaces. 1) Various geometric problems that resonate with quantum or string discussions in theoretical physics give rise to systems of oriented pseudomanifolds with boundary where the codimension one pieces of the boundary can be described by gluing and intersection operations applied to earlier pseudomanifolds in the system. A correct formal description of the pieces in such a theory leads to a free triangular dgOa where the combination operad O describes the operations used in this description. Let X denote the tuple labeling all the formal moduli pieces of the system. Then this description usually takes the form of a "master equation" like $dX + X * X = 0$ or $dX + LX + X * X = 0$. Here $*$ denotes the binary gluing operations of the description and L the unary operations required in the description. The formal identity $dd = 0$ follows from the geometric fact that the boundary of a boundary of the formal moduli is zero. We obtain from the master equation at this formal level a presentation of a free triangular dgOa. The partial ordering may come from a dimension consideration or from an energy consideration.

Solving the PDEs defining the formal moduli yields a set of chains solving the master equation. The solutions of these equations in the geometric chain complex C defines a dgOa map from T the triangular free dgOa into C the geometric dgOa.

So we see the same dgOa formalism that applies to algebraic structures also applies to give a description of systems of moduli spaces assuming the hereditary property: in the compactifications the ideal points are described by lower (in the sense of dimension or energy) moduli spaces of the system. The difference is that now the range of the dgOa map is not necessarily homotopy equivalent to a complex of the form $C = HomB$ as it was in the case of algebraic structures.

2) Varying the choices in the geometric equations perturbing the PDEs, e.g., to create transversality, is meant to lead to a homotopy equivalence of the dgOa map associated to this moduli package.

3) The linear terms in the master equation as just described are called anomalies. They make the above discussion vulnerable to being homotopically trivial. This is analyzed by looking at the linearized homology of the free triangular dgOa which will be discussed next.

6. Postnikov systems and minimal models in the dgOa context. 1) Given a free triangular dgOa T we can of course form the usual or global homology which is an algebra over O. We can also form a linearized chain complex and its homology which is called the linearized homology. The linearized complex is the quotient of the free dgOa by the d submodule defined by the image of O operations with at least two inputs. The linearized homology behaves like the homotopy groups of T or rather their dual spaces. The natural map from the global homology of T to the linearized homology of T is analogous to the dual of the Hurewicz homomorphism in topology from homotopy to homology.

When O is the graded commutative operad the dual of linearized homology of a dgOa has the structure of a Lie algebra which is the leading part of a Lie infinity structure. In topology this corresponds to Whitehead products and higher order Whitehead products on homotopy.

This generalizes in the following way over the rationals.

2) Let H denote the linearized homology of a free triangular dgOa T.

Corollary 6.1. *There is a triangular differential in $O(H)$, the free dgOa generated by H, so that this dgOa is homotopy equivalent to T.*

The proof in this setting is direct. View the differential in T as a dgOa map from a fixed free triangular dgOa into Hom of the linearized complex. Lift this to $HomH$ using lemma 3.2 and the proof of lemma 4.3.

Note that the differential in $O(H)$ consists of maps from H to various tensor products of H with itself. This is a set of coproducts satisfying quadratic identities. In the dual picture these provide the indicated generalization of Whitehead products and higher order products to the O context. The commutative case of corollary 6.1 appeared in [8] but the L infinity interpretation was missing before the work of Hinich and Schechtman [2, 3]. Corollary 6.1 in this generality is certainly due to Markl [5, 6, 7]. See also Kadeishvili [4] and the thesis of Bruno Vallette [11].

3) Minimal models.

Definition 6.2. The $O(H)$ version of the homotopy type of T just described is called the *minimal model* of the homotopy type.

The minimal model is built up inductively by adding layers of dual homotopy groups in an algebraic way that resembles combining base and fibre to get the total space in a fibration. This is the analogue in the free triangular dgOa world of the nilpotent or untwisted Postnikov system in homotopy theory.

4) More generally there is the algebraic analogue of the Postnikov system of a map in homotopy theory where the stages follow the homotopy groups of the fibre of the map. Namely given $T \to A$ one inductively adds generators to T and extends the map to make it into a homology isomorphism i.e., a resolution of A. There are minimal versions of this process which reveal a set of invariants of the homotopy type of the original map.

7. Anomalies. If starting from the geometry one finds a master equation of the form $dX + X * X+$ higher order terms...= 0, then the domain of the dgOa map is in minimal form and each component of X represents a nontrivial dual element to the homotopy groups. Thus there are in general nontrivial invariants of such dgOa maps up to homotopy equivalence. On the other hand if there are linear terms in the master equation $dX + LX + X * X+$ higher terms...= 0, the linearized differential is the unary operator L and its homology might be zero. In various geometric contexts it is sometimes possible to use symmetry or other geometric devises to kill some of the linear terms of L by additional gluing or filling in. Reducing L increases the linearized homology and thus the fund of possible invariants. See [1, 9, 10, 12].

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References

[1] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction*, chapters 4 and 5, http://www.math.kyoto-u.ac.jp/~fukaya/fukaya.html.

[2] V. Hinich and V. Schechtman, *Deformation theory and Lie algebra homology. I*, Algebra Colloq. 4 (1997), 213–240.

[3] V. Hinich and V. Schechtman, *Deformation theory and Lie algebra homology. II*, Algebra Colloq. 4 (1997), 291–316.

[4] T. V. Kadeishvili, *The category of differential coalgebras and the category of A_∞-algebras*, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 77 (1985), 50–70 (in Russian).

[5] M. Markl, *Homotopy algebras via resolutions of operads*, in: The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999), Rend. Circ. Mat. Palermo (2) Suppl. 63 (2000), 157–164.

[6] M. Markl, *Homotopy algebras are homotopy algebras*, Forum Math. 16 (2004), 129–160.

[7] M. Markl, *Free homotopy algebras*, Homology Homotopy Appl. 7 (2005), 123–137 (electronic).

[8] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977), 269–331 (1978).

[9] D. Sullivan, *String topology: background and present state*, Current Developments in Mathematics 2005, International Press of Boston and arXiv:0710.4141.

[10] D. Sullivan, *Sigma models and string topology*, in: Graphs and Patterns in Mathematics and Theoretical Physics, Proc. Sympos. Pure Math. 73, Amer. Math. Soc., Providence, RI, 2005, 1–11.

[11] B. Vallette, *A Koszul duality for props*, arXiv:math/0411542.

[12] D. Xu, *Graph configurations and the master equation*, PhD dissertation, State Univ. of New York, Stony Brook, 2008.