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A GEOMETRICAL/COMBINATORICAL QUESTION WITH IMPLICATIONS FOR THE JOHN–NIRENBERG INEQUALITY FOR BMO FUNCTIONS

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Abstract. The first and last sections of this paper are intended for a general mathematical audience. In addition to some very brief remarks of a somewhat historical nature, we pose a rather simply formulated question in the realm of (discrete) geometry. This question has arisen in connection with a recently developed approach for studying various versions of the function space BMO. We describe that approach and the results that it gives. Special cases of one of our results give alternative proofs of the celebrated John–Nirenberg inequality and of related inequalities due to John and to Wik. One of our main results is that an affirmative answer to the above question would lead to a version of the John–Nirenberg inequality with "dimension free" constants.

The purpose of this note is to present some material which was discussed in the lecture given by one of us at the Józef Marcinkiewicz Centenary Conference, in Poznań. Thus we will first describe a question, one of whose formulations does not require any

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specialized knowledge beyond the most basic notions of geometry or even merely discrete geometry. Then we will explain the implications of an answer to that question for the important space of functions of Bounded Mean Oscillation. These implications fit into a collection of other results about that space. Finally we will offer some very brief remarks of a somewhat historical nature about Józef Marcinkiewicz and his teachers. Additional details about all these matters can be found at or via the website [3].

1. An invitation to consider a geometrical/combinatorical question. We invite the reader, irrespective of his or her usual research interests, to consider and hopefully even answer a geometrical question whose quite simple formulation will be given in a moment. We promise that an affirmative answer to it will be greatly appreciated by a considerable number of analysts, for reasons which will be given presently.

QUESTION A. *Do there exist two constants τ and s which have the following properties?*

(i) $0 < \tau < 1/2$ *and* $s > 0$.

(ii) *For every positive integer d, whenever E_+ and E_- are two disjoint measurable subsets of the unit cube $Q = [0,1]^d$ in $\mathbb{R}^d$ whose d-dimensional Lebesgue measures satisfy*

$$\min\{\lambda(E_+), \lambda(E_-)\} > \tau\lambda(Q \setminus E_+ \setminus E_-),$$

then there exists some cube W contained in Q for which

$$\min\{\lambda(W \cap E_+), \lambda(W \cap E_-)\} \geq s\lambda(W).$$

Here, as is frequently done, we are adopting the convention, that "cube" in fact means a cartesian product of d intervals of equal length, i.e., all our cubes have sides parallel to the coordinate axes. Obviously, there is nothing special about the cube $[0,1]^d$, and, if in Question A we choose Q to be some other arbitrary fixed cube instead of $[0,1]^d$ we obtain another equivalent formulation of exactly the same question.

Of course any Lebesgue measurable subset of $[0,1]^d$ can be approximated in appropriate ways by rather simpler sets, and, using this fact, it can be shown that Question A is equivalent to a variant in which it is only necessary to consider disjoint subsets E_+ and E_- of the unit cube which are each unions of finite collections of dyadic cubes. Furthermore, the existence of a cube W with the properties sought above is clearly equivalent to the existence of such a cube which also happens to be a finite union of dyadic cubes (but possibly smaller ones than those whose unions constitute E_+ and E_-, and possibly for a slightly smaller value of the constant s). It is because of these equivalent versions of Question A, that we can consider our problem to be within the realms of combinatorics or high-dimensional discrete geometry. Indeed the above considerations lead us to the option of expressing Question A in terms of suitable finite subsets of $\mathbb{Z}^d$ with the role of Lebesgue measure replaced by the cardinality of such sets.

We can partially answer Question A affirmatively, where "partially" means that our constants τ and s depend on the dimension d. We can also affirmatively answer an analogue of this question where cubes are replaced by the special rectangles in $\mathbb{R}^d$ which Wik [8] calls "false cubes". In this latter case our constants are independent of d. (The details are in Theorem 1 below.)

2. The space BMO and its connections with Question A. We can now begin describing the setting which gives rise to the question posed in the previous section. Our main motivation comes from considering the important space BMO of functions of "bounded mean oscillation", whose definition will be recalled in a moment. In fact our results apply to a somewhat more general version of BMO than the "classical" space introduced by John and Nirenberg in [5]. We will define it with respect to some suitable collection $\mathcal{E}$ of Lebesgue measurable subsets of some fixed subset D of $\mathbb{R}^d$. We require each set E in $\mathcal{E}$ to be *admissible*, by which we mean that its Lebesgue measure must satisfy $0 < \lambda(E) < \infty$.

Among natural and interesting choices for $\mathcal{E}$, we can consider the collection of all subsets of D which are cubes (this choice corresponding to the theory developed in [5]), or, alternatively, all those which are dyadic cubes (see e.g. [1]), or all those which are "false cubes" (see [8]), or euclidean balls, etc. etc.

Suppose that f is a measurable real valued function whose domain of definition contains some admissible set $E \in \mathcal{E}$ on which f is integrable. Then we define the mean oscillation of f on E to be the quantity

$$\mathbf{O}(f,E) := \inf_{c\in\mathbb{R}} \frac{1}{\lambda(E)} \int_E |f-c|\, d\lambda. \tag{1}$$

It is well known that the infimum in (1) is always attained, in fact whenever c is a median of f on E. Furthermore, choosing c to be the average $f_E := \frac{1}{\lambda(E)} \int_E f\, d\lambda$ of f on E gives an estimate for the mean oscillation to within a factor of 2, i.e.,

$$\frac{1}{\lambda(E)} \int_E |f-f_E|\, d\lambda \le 2\mathbf{O}(f,E).$$

Suppose that D is a measurable subset of $\mathbb{R}^d$ which contains all sets of our chosen collection $\mathcal{E}$. Then the space $BMO(D,\mathcal{E})$ is defined to consist of all (equivalence classes of) measurable functions $f : D \to \mathbb{R}$ whose restrictions to each of the sets $E \in \mathcal{E}$ are integrable, and for which the seminorm

$$\|f\|_{BMO(D,\mathcal{E})} := \sup_{E\in\mathcal{E}} \mathbf{O}(f,E) \tag{2}$$

is finite. If D is a cube, and $\mathcal{E}$ is the collection $\mathcal{Q}(D)$ of all cubes contained in D, then $BMO(D,\mathcal{Q}(D))$ is exactly the space introduced by John and Nirenberg in [5].

Here we can only briefly hint at the surprisingly wide range of topics in analysis in which various forms of the space BMO appear in very pertinent ways. The first application of BMO, which motivated its introduction, was related to the theory of elasticity. There immediately followed an application to generalizing Harnack's inequalities. Subsequently BMO came to play significant roles in many more contexts, among them harmonic analysis, the theory of interpolation of operators, and the study of quasiconformal mappings.

Let D be a cube in $\mathbb{R}^d$ and let $f : D \to \mathbb{R}$ be a function belonging to the space $BMO(D,\mathcal{Q}(D))$. Then

$$\lambda\big(\{x\in D : |f(x)-f_D| > \alpha\}\big) \le B\lambda(D)\exp\Big(-\frac{b\alpha}{\|f\|_{BMO(D,\mathcal{Q}(D))}}\Big) \tag{3}$$

for every $\alpha > 0$, where B and b are constants which depend only on the dimension d. This is the celebrated John–Nirenberg inequality, originally discovered and proved in [5]. It is one of the fundamental properties of $BMO(D, \mathcal{Q}(D))$, and lies behind many of its applications. Thus there are good reasons to deepen our understanding of (3). One natural problem here, which is still open, is to determine whether the constants B and b really depend on the dimension d. This is where Question A turns out to be relevant, surprisingly perhaps, since, as far as we can see, it is not related to any known proofs of (3).

For the continuation of this discussion, it is convenient to introduce and use the following terminology:

Let $\mathcal{E}$ be some collection of admissible subsets of some set D in $\mathbb{R}^d$. The pair of numbers (τ, s) is a *John–Strömberg pair for the collection* $\mathcal{E}$ if it has the properties (i) and (ii) described above in "Question A", except that now the sets Q and W of property (ii) have to be in $\mathcal{E}$. More explicitly, for this definition, property (ii) must be replaced by:

(ii)$'$ *For each $Q \in \mathcal{E}$, whenever E_+ and E_- are two disjoint measurable subsets of Q whose d-dimensional Lebesgue measures satisfy*

$$\min\{\lambda(E_+), \lambda(E_-)\} > \tau\lambda(Q \setminus E_+ \setminus E_-),$$

then there exists some set $W \in \mathcal{E}$ contained in Q for which

$$\min\{\lambda(W \cap E_+), \lambda(W \cap E_-)\} \geq s\lambda(W).$$

The research being described here has two parts or "prongs":

(A) We identify John–Strömberg pairs for various natural collections $\mathcal{E}$ of admissible subsets of $\mathbb{R}^d$.

(B) We obtain "analytic" conditions, which imply or are implied by the existence of John–Strömberg pairs. These conditions are inequalities expressed in terms of BMO "seminorms" of certain functions f and their rearrangements f^*. They are of the form

$$\|f^*\|^{(\mathbf{J},\sigma)}_{BMO(I,\mathcal{Q}(I))} \leq \|f\|^{(\mathbf{J},s)}_{BMO(Q,\mathcal{E}(Q))} \quad \text{for } I = (0, \lambda(Q)). \tag{4}$$

(The notation will be explained below.)

The results alluded to in part (B) enable us to obtain various abstract variants of the the fundamental inequality (3), some cases of which are already known, because they enable us to reduce these proofs to the easy case where $d = 1$ and $\mathcal{E}$ consists of all subintervals of a given interval, and the relevant function is non-increasing. When combined with results of part (A) these also give proofs of various concrete versions of (3).

In particular, if (τ, s) is a John–Strömberg pair for the collection of all cubes in $\mathbb{R}^d$ then we can show that

$$\lambda\left(\{x \in Q : |f(x) - m_{f,Q}| \geq \alpha\}\right) \leq \max\left\{\frac{1}{2\tau}, 2\sqrt{\frac{1}{2\tau}}\right\} \cdot \lambda(Q) \cdot \exp\left(-\frac{\alpha s \log \frac{1}{2\tau}}{8\|f\|_{BMO}}\right) \tag{5}$$

for every $\alpha \geq 0$. (Some slightly stronger inequalities also hold.) Here $m_{f,Q}$ is any median of the measurable function f on the cube Q in $\mathbb{R}^d$, and $\|f\|_{BMO}$ is the seminorm which was essentially introduced (with different terminology) in [5] and which is a special case of the general BMO seminorm $\|f\|_{BMO(D,\mathcal{E})}$ defined above in (2).

This means that an affirmative answer to Question A, with τ and s independent of d, would give a "dimension free" version of the original John–Nirenberg inequality.

We have written "seminorms" in quotation marks above, because, in addition to using the usual seminorms for BMO, we often use the suprema of a certain special functional which is not a seminorm. This functional was introduced by John [4] and further studied by Strömberg [7]. We use the notation $\mathbf{J}(f,Q,s)$ to denote it. It is defined, for each admissible set Q, for each measurable function $f : Q \to \mathbb{R}$, and each $s > 0$, by

$$\mathbf{J}(f,Q,s) = \inf_{c\in\mathbb{R}} \bigl(\inf\{\alpha \geq 0 : \lambda(\{x \in Q : |f(x) - c| > \alpha\}) < s\lambda(Q)\}\bigr).$$

One of our tools is the following apparently quite useful alternative formula which we obtain for $\mathbf{J}(f,E,s)$ in terms of non-increasing rearrangements, for all non-negative measurable functions f which are defined on some set containing Q:

$$\mathbf{J}(f,Q,s) = \frac{1}{2} \inf\bigl\{(f\chi_Q)^*(u) - (f\chi_Q)^*(u + (1-s)\lambda(Q)) : 0 < u < s\lambda(Q)\bigr\}. \tag{6}$$

We will indicate the proof of this formula in Section 3.

We can now explain the notation used above in (4). If $\mathcal{E}$ is a collection of admissible sets contained in some $D \subset \mathbb{R}^d$, and if s is some number in $(0,1/2)$, then, for each measurable $f : D \to \mathbb{R}$, we set

$$\|f\|^{(\mathbf{J},s)}_{BMO(D,\mathcal{E})} := \sup_{E\in\mathcal{E}} \mathbf{J}(f,E,s)\,.$$

(This is a natural generalization of the functional of John and Strömberg which is denoted by $\|f\|_{BMO_{0,s}}$ in [7].)

Furthermore, for each $Q \in \mathcal{E}$, we let $\mathcal{E}(Q)$ denote the collection of all those sets in $\mathcal{E}$ which are contained in Q. In particular $\mathcal{Q}(I)$ denotes the collection of all subintervals of the interval $I = (0, \lambda(Q))$.

Here are some of our main results.

The two parts of this first theorem are special cases of a rather more general but somewhat technical result (see Theorem 5 below).

Theorem 1.

(i) *The pair* $(\tau,s) = \bigl(\sqrt{2}-1, \frac{3-2\sqrt{2}}{2^d}\bigr)$ *is a John–Strömberg pair for the collection of all cubes in* $\mathbb{R}^d$, *and also for the collection of all dyadic cubes in* $\mathbb{R}^d$.

(ii) *The pair* $(\tau,s) = \bigl(\sqrt{2}-1, \frac{3-2\sqrt{2}}{2}\bigr)$ *is a John–Strömberg pair for the collection of all "false cubes" in* $\mathbb{R}^d$.

(For the reader's convenience, we recall Wik's definition [8]: A "false cube" is a d-dimensional rectangle in $\mathbb{R}^d$ whose sides are parallel to the axes and, for some r, have side lengths either r or $2r$.)

Theorem 2. *Let* $\mathcal{E}$ *be a collection of admissible subsets of* $\mathbb{R}^d$. *Let* Q *be a set in* $\mathcal{E}$ *which contains all other sets of* $\mathcal{E}$. *Let* τ *and* s *be numbers in* $(0,1/2)$ *for which* (τ,s) *is a John–Strömberg pair for* $\mathcal{E}$. *Then, for every constant* r *in the range* $1 \leq r \leq 1/2\tau$, *the inequalities*

$$\lambda\bigl(\{x \in Q : |f(x) - m| \geq \alpha\}\bigr) \leq \max\bigl\{r, 2\sqrt{r}\bigr\} \cdot \lambda(Q) \cdot \exp\Biggl(-\frac{\alpha \log r}{8\,\|f\|^{(\mathbf{J},s)}_{BMO(Q,\mathcal{E})}}\Biggr)$$

and

$$\lambda\bigl(\{x \in Q : |f(x) - m| \geq \alpha\}\bigr) \leq \max\left\{r, 2\sqrt{r}\right\} \cdot \lambda(Q) \cdot \exp\left(-\frac{\alpha s \log r}{8\,\|f\|_{BMO(Q,\mathcal{E})}}\right)$$

hold for every $\alpha \geq 0$, every measurable $f : Q \to \mathbb{R}$, and every median m of f on Q.

One of our main tools for proving the previous theorem is this next one.

THEOREM 3. *Let $\mathcal{E}$ be a collection of admissible subsets of $\mathbb{R}^d$. Let Q be a set in $\mathcal{E}$ and let $\mathcal{E}(Q)$ be the collection of all sets in $\mathcal{E}$ which are contained in Q. Let τ and s be numbers in $(0, 1/2)$ for which (τ, s) is a John–Strömberg pair for $\mathcal{E}$. Suppose that the function $f : Q \to \mathbb{N} \cup \{0\}$ is measurable and satisfies $\|f\|^{(\mathbf{J},s)}_{BMO(Q,\mathcal{E}(Q))} \leq 1/2$. Let σ be a number in the range*

$$\frac{2\tau}{1+2\tau} < \sigma \leq \frac{1}{2}\,.$$

Then the function $f^* : (0, \lambda(Q)) \to [0, \infty)$, i.e., the non-increasing rearrangement of f restricted to the interval $I := (0, \lambda(Q))$, satisfies

$$\|f^*\|^{(\mathbf{J},\sigma)}_{BMO(I,\mathcal{Q}(I))} \leq \|f\|^{(\mathbf{J},s)}_{BMO(Q,\mathcal{E}(Q))}\,.$$

Here is a sort of "converse" to Theorem 3:

THEOREM 4. *Let $\mathcal{E}$ be a collection of admissible subsets of $\mathbb{R}^d$. Let s and σ be two given numbers in $(0, 1/2)$ and let τ be any number in $(0, 1)$ satisfying*

$$0 < \sigma < \frac{\tau}{1+2\tau}\,.$$

Suppose that, for each $Q \in \mathcal{E}$, the inequality

$$\|f^*\|^{(\mathbf{J},\sigma)}_{BMO(I,\mathcal{Q}(I))} \leq \|f\|^{(\mathbf{J},s)}_{BMO(Q,\mathcal{E}(Q))}$$

holds for the interval $I = (0, \lambda(Q))$ and for every measurable function $f : Q \to \mathbb{R}$ which assumes only the three values 0, 1 and 2. Here $\mathcal{E}(Q)$ denotes the collection of all those sets in $\mathcal{E}$ which are contained in Q.

Then (τ, s) is a John–Strömberg pair for $\mathcal{E}$.

It may seem at first surprising, and perhaps even amusing, that in this theorem we only have to deal with functions taking only three (consecutive integer) values here. But also, in an important part of the proof of Theorem 3, we find ourselves having to deal only with such kinds of functions.

We conclude this summary of our main results by stating the theorem mentioned above, which implies that certain pairs (τ, s) are John–Strömberg pairs for various collections $\mathcal{E}$ of subsets of $\mathbb{R}^d$. We will not explain all the terminology which appears in the formulation of this theorem. For that we refer you to our forthcoming detailed paper, whose preprint is available via [3].

THEOREM 5. *Let $\mathcal{E}$ be an M-multidecomposable collection of admissible subsets of $\mathbb{R}^d$ for some $M \geq 2$. Let τ be a number satisfying $0 < \tau < 1$. Let δ be a bi-density constant for $\mathcal{E}$.*

(i) *Suppose that Q is a set in $\mathcal{E}$ and there exist three pairwise disjoint measurable sets E_+, E_- and G which satisfy*

$$Q = E_+ \cup E_- \cup G$$

and

$$\min\{\lambda(E_+), \lambda(E_-)\} > \tau\lambda(G)\,.$$

Then there exists a subset W of Q such that $W \in \mathcal{E}$ and

$$\min\{\lambda(E_+ \cap W), \lambda(E_- \cap W)\} \geq s\lambda(W) \tag{7}$$

where

$$s = \min\Big\{\frac{\tau - \tau^2}{M(1+\tau)}, \delta\Big\}.$$

(ii) *In particular, if the above-mentioned sets E_+, E_- and G satisfy*

$$\min\{\lambda(E_+), \lambda(E_-)\} > (\sqrt{2} - 1)\lambda(G),$$

then the above-mentioned set W can be chosen so that it satisfies (7) *where s has the value $s = \min\{\frac{3-2\sqrt{2}}{M}, \delta\}$.*

3. How to prove the formula (6). Restated informally, the formula (6) tells us that $2\mathbf{J}(f, Q, s)$ is the "minimum" amount that $(f\chi_Q)^*$ can decrease on any closed subinterval of $(0, \lambda(Q))$ of length exactly $(1-s)\lambda(Q)$. The main step of the proof of (6) is to consider the special case where $d = 1$ and Q is an interval $Q = (0, q)$ and $f : (0, q) \to [0, \infty)$ is non-increasing. The general case can then be deduced from this special case via applications of the fact that f and f^* are equimeasurable.

Let us then deal with this special case. I.e., we have to show that the formula

$$\mathbf{J}(f, (0, q), s) = \frac{1}{2}\inf\big\{f^*(u) - f^*(u + (1-s)q) : 0 < u < sq\big\} \tag{8}$$

holds for each $s \in (0, 1)$. To avoid some messy technical details, while still conveying the main ideas behind the general proof, we will allow ourselves the extra assumptions here that the non-increasing function $f : (0, q) \to [0, \infty)$ is also uniformly continuous and strictly decreasing. In this case we have $f = f^*$, and that, furthermore, f has a unique extension to a continuous function on $[0, q]$ which we will also denote by f. For each pair of numbers $c \in \mathbb{R}$ and $\alpha \geq 0$, let

$$E(c, \alpha) := \{t \in [0, q] : |f(t) - c| \leq \alpha\} = \{t \in [0, q] : c - \alpha \leq f(t) \leq c + \alpha\}\,.$$

This set is clearly a closed interval $[u, u + r]$ contained in $[0, q]$, on which f attains a minimum value m (at $u + r$) and a maximum value M (at u), and these values both lie in the interval $[c - \alpha, c + \alpha]$. If we set $c' = \frac{1}{2}(M + m)$ and $\alpha' = \frac{1}{2}(M - m)$ then of course $E(c', \alpha') = E(c, \alpha)$ and $0 \leq \alpha' \leq \alpha$. Of course the length r of the interval $E(c, \alpha)$ is the same as the length of the not necessarily closed interval $\{t \in (0, q) : |f(t) - c| \leq \alpha\}$. So, in order to calculate $\mathbf{J}(f, (0, q), s)$, we have to consider all intervals $E(c, \alpha)$ which have length exceeding $(1-s)q$ and find the infimum of all values of α which they can have. If, as above, we write $E(c, \alpha)$ as $[u, u + r]$, then $M = h(u)$ and $m = h(u + r)$ and

$$\alpha' = \frac{1}{2}(M - m) = \frac{1}{2}(f(u) - f(u + r)).$$

Thus $\mathbf{J}(f, (0, q), s)$ is the infimum of the set Ω of all numbers $\frac{1}{2}(f(u) - f(u+r))$ for which $r > (1-s)q$ and $0 \leq u \leq u + r \leq q$. In view of the continuity and monotonicity of f, we can optimally choose $r = (1-s)q$, so that the above infimum is equal to the infimum of the set Ω_1 of all numbers $\frac{1}{2}(f(u) - f(u + (1-s)q))$ for which $0 \le u \le u + (1-s)q \le q$. (The infimum is of course attained for some particular $u \in [0, q]$.) Again by continuity, this infimum is also equal to the infimum of the subset Ω_2 of Ω_1

$$\Omega_2 = \left\{ \frac{1}{2}(f(u) - f(u + (1-s)q)) : 0 < u, u + (1-s)q < q \right\}.$$

This last fact is exactly what is expressed by the formula (8), since $f^* = f$. Thus our proof of (8) is complete in this special case.

4. Some very brief historical remarks. The remarkable conference, at which we presented the above results, included several moving and detailed tributes to Józef Marcinkiewicz and surely at least some of them will find expression in this volume of the proceedings of the conference. Let us paraphrase some of the brief remarks made by one of the authors of this note in his lecture at the conference. In terms of "mathematical genealogy", Marcinkiewicz was a towering figure in the immediate mathematical family of two of us. Of course no words can adequately express our sorrow and outrage that tyrannical regimes led to his murder in or near Katyń, and also, during those same dark years, to the murder of so many of our families.

Thus we could experience Marcinkiewicz's genius and his personality only via his writings and via what those who had known him could tell us. We had the extraordinary privilege of knowing Marcinkiewicz's mathematical "father" Antoni Zygmund. His heartfelt tribute to Marcinkiewicz is in the volume of Marcinkiewicz's Collected Papers [6].

It is also intriguing to look back further into Marcinkiewicz's scientific genealogy. We can of course do that via the well known "mathematical genealogical site" [2] of North Dakota State University. There we can see, among many other noteworthy things, that there is a line connecting Józef Marcinkiewicz back to another of the giants of Polish science, Mikołaj Kopernik.

Acknowledgments. Given that this paper is something of a recapitulation of things which one of us said in a lecture at the conference, we would like to once more express, as was of course also expressed by the other speakers, very warm thanks to all the organizers who put so much thought, energy and good will into the tasks of making the conference so successful and making their visitors feel so much at home in Poznań. In our particular case special thanks are due to Professor Mieczysław Mastyło for his kind hospitality on this occasion and indeed for many years of precious friendship and stimulating mathematical interaction.

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References

[1] J. B. Garnett, P. W. Jones, *BMO from dyadic BMO*, Pacific J. Math. 99 (1982), 351–371.

[2] `http://genealogy.math.ndsu.nodak.edu/`

[3] `http://www.math.technion.ac.il/~mcwikel/bmo`

[4] F. John, *Quasi isometric mappings*, in: Seminari 1962/63 di Analisi, Algebra, Geometria e Topologia, Vol. II, Ist. Naz. Alta Mat., Ediz. Cremonese, Rome, 1965, 462–473.

[5] F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415–426.

[6] J. Marcinkiewicz, *Collected Papers*, edited by A. Zygmund, Instytut Matematyczny Polskiej Akademii Nauk, Państwowe Wydawnictwo Naukowe, Warsaw, 1964.

[7] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. 28 (1979), 511–544.

[8] I. Wik, *On John and Nirenberg's theorem*, Ark. Mat. 28 (1990), 193–200.