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SUMMATION PROCESSES VIEWED FROM THE FOURIER PROPERTIES OF CONTINUOUS UNIMODULAR FUNCTIONS ON THE CIRCLE

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Abstract. The main purpose of this article is to give a new method and new results on a very old topic: the comparison of the Riemann processes of summation (R,κ) with other summation processes. The motivation comes from the study of continuous unimodular functions on the circle, their Fourier series and their winding numbers. My oral presentation in Poznań at the JM–100 conference exposed the ways by which this study was developed since the fundamental work of Brézis and Nirenberg on the topological degree [5]. I shall shorten the historical part in the present article; it can be found in [3], [8] and [9].

1. Motivation. Continuous unimodular functions on the circle, Fourier series and winding numbers. The continuous unimodular functions on the circle $f(z)$, $|z|=1$, $|f(z)|=1$, or

$$f(e^{it}),\ t\in\mathbb{R}/2\pi\mathbb{Z},\ |f(e^{it})|=1,\ f \text{ continuous}, \tag{1}$$

have interesting properties that are not shared with bounded unimodular functions but can be extended to VMO-functions, that is, limits of continuous functions in the space BMO of functions with bounded mean oscillation; VMO means "vanishing mean oscillation". Their Fourier series

$$f(e^{it})\sim\sum_{-\infty}^{\infty}a_n e^{int} \tag{2}$$

have special features due to the fact that the mapping $z\mapsto f(z)$ has a topological degree, or winding number. When the mapping is very regular, for example analytic on the circle,

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the winding number is

$$\deg f = \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int f'(z)\overline{f(z)}\,dz = \sum_{-\infty}^{\infty} n|a_n|^2.$$

This formula, with its extensions and interpretations, is the starting point of the story.

The most general result on the Fourier series (2) under the assumption (1) was established recently by Jean Bourgain and myself [1]. It reads as an implication: given $s > 0$,

$$\sum_{0}^{\infty} n^{2s}|a_n|^2 < \infty \Longrightarrow \sum_{-\infty}^{\infty} |n|^{2s}|a_n|^2 < \infty. \tag{3}$$

It means that the right part of the Fourier series has an influence on the left part, what does not happen in general for bounded unimodular functions. When $s = \frac{1}{2}$ and only in that case a precision can be given: for functions f such that $\deg f = 0$, there is a constant C such that

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 < C\sum_{0}^{\infty} n|a_n|^2\,;$$

actually $C = 32$ works.

The case $s = \frac{1}{2}$ was the origin of the study. It answers a question asked by Haïm Brézis [4]: is it true that (1) and (2) imply

$$\sum_{-\infty}^{\infty} |n|\,|a_n|^2 \le |\deg f| + \sum_{0}^{\infty} n|a_n|^2\ ? \tag{4}$$

It was known already that the answer is positive when the first member is finite, meaning $f \in H^{1/2}$ (Sobolev class), therefore $f \in VMO$. Actually one of the results of Brézis and Nirenberg in [5] is that

$$\deg f = \sum_{-\infty}^{\infty} n|a_n|^2 \tag{5}$$

whenever the series converges absolutely (this works even when f is not continuous, if $\deg f$ is defined as in [5]). Since (4) is obvious when the second member is infinite, (3) proves that it is true in any case.

The relation with summation processes appears as a question in [5]. Is it possible to extend the validity of (5) when the series does not converge absolutely, by means of a convenient process of summation?

A strongly negative answer is given by a very difficult construction of Jean Bourgain and Gady Kozma [2]: there exist two functions that satisfy (1) such that their topological degrees are different and the second members of (5) are the same. When f is a signal, its energy is given by the $|a_n|^2$, therefore a pleasant way to express the result is that one cannot hear the winding number.

Previously J. Korevaar had given examples of functions f such that the partial sums

$$\sum_{-N}^{N} n|a_n|^2 \qquad (N \longrightarrow \infty), \tag{6}$$

resp. the Abel–Poisson sums

$$\sum_{-\infty}^{\infty} r^{|n|} n|a_n|^2 \qquad (r \uparrow 1), \tag{7}$$

either diverge, or converge to a value different from $\deg f$ [10].

We have to face up to another question: for which functions f and summation processes is (5) meaningful? Of course, we have to try beyond $f \in H^{1/2}$, and we always assume (1).

The first example was given by Korevaar [10]: if $f \in BV$ (functions with bounded variation), the partial sums (6) converge to $\deg f$. Since the Abel–Poisson summation process is stronger than the process by partial sums, (7) converge to $\deg f$ also.

Here is a result of mine [7, 9]: the Abel–Poisson sums (7) converge to $\deg f$ whenever f belongs to the Zygmund class $\lambda^3_{1/3}$ [15, p. 45], meaning

$$\int_0^{2\pi} |f(e^{i(t+s)}) - f(e^{is})|^3 = o(t) \qquad (t \downarrow 0),$$

in particular when f satisfies a Hölder condition of order > 3, but it is not true for some $f \in \Lambda_{1/3}$, meaning Hölderian of order $\frac{1}{3}$.

Both positive and negative results are obtained through another process of summation. Actually

$$\deg f = \lim_{t \to 0} \sum_{-\infty}^{\infty} |a_n|^2 \frac{\sin nt}{t}$$

whenever $f \in \lambda^3_{1/3}$. For the proof it is enough to consider the case $\deg f = 0$, that is, $f(e^{is}) = e^{i\varphi(s)}$, and then to use the formula

$$\sum_{-\infty}^{\infty} |a_n|^2 \sin nt = \frac{1}{2\pi} \int_0^{2\pi} \sin(\varphi(t+s) - \varphi(s))\, ds.$$

On the other hand, convenient lacunary trigonometric series provide examples of functions $\varphi \in \Lambda_{1/3}$ (hence $f \in \Lambda_{1/3}$) such that the sums

$$\sum_{-\infty}^{\infty} |a_n|^2 \frac{\sin nt}{t} \qquad (t \to 0) \tag{8}$$

either diverge, or tend to a limit $\neq \deg f$.

If we write

$$u_m = m(|a_m|^2 - |a_{-m}|^2),$$

the sums (8) have the form

$$S(t) = \sum_{1}^{\infty} u_m \frac{\sin mt}{m}\,. \tag{9}$$

They are bounded and may tend to a limit or not when $t \to 0$. When they tend to a limit, either $\deg f$ or another value, the same holds for all processes of summation that are stronger (this property defines the term "stronger"), in particular for the Abel–Poisson process.

Most summation processes that can be found in the literature are regular, that is, stronger than the process by partial sums

$$s_n = \sum_1^n u_m \qquad (n \to \infty).$$

The process defined by (9) $(t \to 0)$ is not regular. However it has a name: $(R, 1)$, the Riemann summation process of order 1 [6, p. 88]. The Riemann process of order κ looks for the limits of

$$S_k(t) = \sum_1^\infty u_m \Big(\frac{\sin mt}{m}\Big)^\kappa \qquad (t \to 0) \tag{10}$$

$(\kappa = 1, 2, 3, \dots)$, and it is denoted by (R, κ). Riemann had introduced $(R, 2)$ in his study of convergent trigonometric series (double formal integration, then symmetric differentiation of order 2). He proved and used the fact that $(R, 2)$ is regular. There are variations about $(R, 2)$ studied by Marcinkiewicz: a process close to $(R, 2)$ such that neither is stronger than the other [12].

The second part of this article is devoted to the (R, κ) processes and tries to characterize the processes that are stronger than a given (R, κ).

2. The summation processes (R, κ) and stronger than (R, κ)

2.1. A general scheme for summation processes. Given a numerical series $\sum_0^\infty u_m$ we consider functions of the form

$$\sum_0^\infty u_m k_m(t) \qquad (t > 0) \tag{11}$$

and look for their limits as $t \to 0$. That defines the process

$$(k) = (k_m(\cdot))_{m \in \mathbb{N}}\,.$$

When the limit exists, (11) is bounded in a neighbourhood of 0. We shall reduce the interval of definition of t, $(0, t_0)$, so that (11) is bounded on this interval.

For a finite sum $\sum u_m$, we want that (k) works and gives the sum. The condition is always assumed, that is

$$\lim_{t \to 0} k_m(t) = 1 \qquad (m \in \mathbb{N}). \tag{12}$$

We say that the process (k) entails the process $(\ell) = (\ell_m(\cdot))_{m \in \mathbb{N}}$ and we write

$$(k) \longrightarrow (\ell) \tag{13}$$

(Hardy's notation [6]) if

$$\lim_{t \to 0} \sum_0^\infty u_m \ell_m(t) = \lim_{t \to 0} \sum_0^\infty u_m k_m(t)\,,$$

meaning that the first member exists whenever the second exists and they are equal. Due to (12) that is equivalent to

$$\sum_0^\infty u_m k_m(t) = o(1) \Longrightarrow \sum_0^\infty u_m \ell_m(t) = o(1) \quad (t \to 0).$$

REMARK. Whatever $a > 0$, the processes $(k_m(\cdot))_{m\in\mathbb{N}}$ and $(k_m(a\cdot))_{m\in\mathbb{N}}$ are equivalent:

$$(k_m(\cdot))_{m\in\mathbb{N}} \longleftrightarrow (k_m(a\cdot))_{m\in\mathbb{N}}\,;$$

only the interval $(0, t_0)$ where (11) is bounded may change.

Another way to express (13) is that (ℓ) is stronger than (k). If the reader disagree with the expression "(k) is stronger than (k)", he can forget about it.

EXAMPLE. Let μ be a probability measure carried by the interval $(0, 1)$, and

$$\ell_m(t) = \int k_m(at)\,\mu(da).$$

Then (k) entails (ℓ), with $t_0(\ell) = t_0(k)$. If $t_0(k) = \infty$, this holds when μ is carried by $(0, \infty)$.

The processes (R, κ) correspond to

$$k_m(t, \kappa) = \Big(\frac{\sin mt}{mt}\Big)^{\kappa}$$

on their images through a linear change of variable $t \to at$. Writing

$$2\int_0^1 \frac{\sin mat}{mat}\,a\,da = 4\frac{\sin^2 t/2}{m^2t^2} = k_m\Big(\frac{t}{2}, 2\Big)$$

we see that

$$(R, 1) \longrightarrow (R, 2).$$

2.2. $(R, 1) \longrightarrow (k)$? $(R, 1)$ deals with the sums

$$S(t) = S_1(t) = \sum_1^\infty u_m \frac{\sin mt}{mt} \tag{14}$$

already met in (9) and (10). The $(R, 1)$-sum of the series $\sum_1^\infty u_m$ is $\lim_{t\to 0} S(t)$ whenever it exists. Then $S(t)$ is bounded on some interval $(0, t_0)$, $t_0 > 0$. In the cases we met before, we can take $t_0 = \pi$.

If we assume $t_0 = \pi$, that is

$$\sup_t |S(t)| < \infty,$$

$(R, 1)$ is replaced by a slightly weaker process, which I denote by $(R_w, 1)$. Thus

$$(R_w, 1) \longrightarrow (R, 1)$$

but the reverse is not true. Since $(R_w, 1)$ is easier to handle with than $(R, 1)$, I shall begin with $(R_w, 1)$.

2.2.1. *A sufficient condition for $(R_w, 1) \longrightarrow (k)$.* Assume that $\sum_1^\infty u_m$ is $(R_w, 1)$-summable to 0. It means that the series in (14) converges to $S(t)$ for all t, and that $S(t)$ is bounded and tends to 0 as $t \to 0$. Formally

$$(tS(t))' = \sum_1^\infty u_m \cos mt \tag{15}$$

and

$$\sum_1^\infty u_m k_m(s) = \frac{1}{\pi}\int_{-\pi}^{\pi} (tS(t))' K_s(t)\,dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} tS(t)\,dK_s(t) \tag{16}$$

with

$$K_s(t) = k_0 + \sum_1^\infty k_m(s)\cos mt. \tag{17}$$

Let us assume that $t\,dK_s(t)$ is a bounded measure on $(-\pi,\pi)$:

$$\int_{-\pi}^{\pi} |t\,dK_s(t)| < \infty. \tag{18}$$

Then the first and last members of (16) are equal by the Lebesgue convergence theorem. Thus $\sum_1^\infty u_m$ is (k)-summable to 0 if and only if

$$\lim_{s\to 0}\int_0^\pi tS(t)\,dK_s(t) = 0. \tag{19}$$

In all cases we consider, $K_s(\cdot)$ is absolutely continuous on every interval (ε,π) $(0<\varepsilon<\pi)$ when s is small enough, $s<s(\varepsilon)$. From now on we make this assumption.

Theorem 1. *If $K_1(\cdot)$, defined in* (17)*, satisfies* (18) *and the above condition, and moreover*

$$\overline{\lim_{s\to 0}}\int_0^\varepsilon t\,|dK_s(t)| < \infty \tag{20}$$

for some $\varepsilon>0$, and

$$\lim_{s\to 0}\int_\varepsilon^\pi \varphi(t)\,dK_s(t) = 0 \tag{21}$$

for every $\varepsilon>0$ and $\varphi\in L^\infty(\varepsilon,\pi)$, then

$$(R_w,1) \longrightarrow (k) \tag{22}$$

Proof. (20) and (21) imply (19). ■

Remarks.

1. Theorem 1 applies when $k_m(s) = \frac{\sin ms}{ms}$. Then $K_s(\cdot)$ is a step function and dK_s is carried by the points s and $-s$. We shall see other examples later.
2. In (20) usually $\varepsilon=\pi$ works. The role of ε will appear in the reciprocal statement (Theorem 3).

2.2.2. *A sufficient condition for $(R,1)\longrightarrow(k)$.* Let us move to $(R,1)$ instead of $(R_w,1)$. Then $|S(t)|$ can be arbitrarily large on the interval (t_0,π). The control we have is

$$u_m = o(m) \qquad (m\longrightarrow\infty).$$

Let me recall Cantor's proof: assuming the contrary, there exist $c>0$ and a sequence of integers m_j such that $|u_{m_j}|>cm_j$ and $m_{j+1}>4m_j$, then there exists a $t\in(0,t_0)$ such that $(\operatorname{sign} u_{m_j})\sin m_j t>\frac12$ for j large enough, hence $\frac{u_{m_j}}{m_j}\sin m_j t>\frac{c}{2}$, contrary to the definition of $S(t)$.

We shall replace (21) by a stronger condition, namely

$$\begin{cases} \forall m \displaystyle\int_\varepsilon^\pi \sin mt\,dK_s(t) = o(1) \quad (s\to 0)\\ \displaystyle\sum_m \sup_\varepsilon\Big|\int_\varepsilon^\pi \sin mt\,dK_s(t)\Big| < \infty. \end{cases} \tag{23}$$

When (23) holds,

$$\int_\varepsilon^\pi \sum \frac{u_m}{m} \sin mt \, dK_s(t)$$

makes sense and tends to 0 as $s \to 0$. Hence:

THEOREM 2. *Same statement as Theorem 1, replacing (21) by (23) and the conclusion (22) by*

$$(R, 1) \longrightarrow (k).$$

2.2.3. *A sufficient condition for* $(R_w, 1) \nrightarrow (k)$. Let us turn to the reverse: define a class of (k)s that are not entailed by $(R, 1)$, nor even by $(R_w, 1)$.

THEOREM 3. *Suppose that for arbitrarily small* ε

$$\overline{\lim_{s\to 0}} \int_0^\varepsilon t \, |dK_s(t)| = \infty \tag{24}$$

and

$$\int_\varepsilon^\pi \varphi(t) \, dK_s(t) = O(1) \qquad (s \to 0) \tag{25}$$

for every $\varphi \in L^\infty(\varepsilon, \pi)$. *Then*

$$(R_w, 1) \nrightarrow (k). \tag{26}$$

Proof. Let us assume (24) and (25) and write

$$B(\varepsilon, s) = \int_0^\varepsilon t \, |dK_s(t)|.$$

There exists a C^∞-function $\varphi_{\varepsilon,s}(t)$ carried by $[0, \varepsilon]$, with supremum norm 1, such that

$$\int_0^\varepsilon t\varphi_{\varepsilon,s}(t) \, dK_s(t) > \frac{1}{2} B(\varepsilon, s). \tag{27}$$

We define by induction two sequences (ε_j) and (s_j) positive, decreasing and tending to 0, such that

1) $B(\varepsilon_j, s_j) > 2^j$ (use (24)),
2) (27) holds with $\varepsilon = \varepsilon_j$, $s = s_j$, and $\varphi_{\varepsilon,s}$ carried by $[\varepsilon_{j+1}, \varepsilon_j]$,
3) $\displaystyle\int \sum_{k<j} \frac{1}{k^2} \varphi_{\varepsilon_k, s_k}(t) t \, dK_{s_j}(t) = O(1)$ $(j \to \infty)$ (use (25)),
4) $\displaystyle\int \sum_{k>j} \frac{1}{k^2} \varphi_{\varepsilon_k, s_k}(t) t \, dK_{s_j}(t) = O(1)$ $(j \to \infty)$.

Then we choose

$$S(t) = \sum \frac{1}{j^2} (\varphi_{\varepsilon_j, s_j}(t) + \varphi_{\varepsilon_j, s_j}(-t))$$

$$\frac{u_m}{m} = \frac{1}{\pi} \int_{-\pi}^\pi tS(t) \sin mt \, dt \qquad (m = 1, 2, \dots).$$

Since $tS(t)$ is odd and C^∞ out of 0, its Fourier series on $(-\pi, \pi)$ converges everywhere, that is,

$$tS(t) = \sum_1^\infty \frac{u_m}{m} \sin mt \qquad (-\pi < t < \pi)$$

pointwise. Moreover $S(t)$ is bounded on $(-\pi,\pi)$ and $\lim_{t\to 0} S(t)=0$, therefore $\sum u_m$ is $(R_w,1)$-summable to 0. On the other hand, using the above conditions 1) to 4),

$$\overline{\lim_{s\to 0}}\int_{-\pi}^{\pi} tS(t)\,dK_s(t)=\infty,$$

therefore (see (16)) $\sum u_m$ is not (k)-summable to 0. Hence (26).

Remarks.

1. If (26) works, then a fortiori $(R,1)\not\twoheadrightarrow(k)$.
2. In many examples s is not a continuous but a discrete parameter, for instance $s=\frac{1}{N}$, N integer. All statements remain valid.
3. The general meaning of Theorems 1, 2, 3 is that, under a condition that guarantees a good behaviour of $K_s(\cdot)$ on (ε,π) when $s<s(\varepsilon)$, ε being arbitrarily small, the necessary and sufficient condition for $(R_w,1)\to(k)$, and also for $(R,1)\to(k)$, is
$$\exists\varepsilon>0:\int_0^{\varepsilon} t\,|dK_s(t)|=O(1)\qquad(s\to 0).\tag{28}$$
4. In many cases $k_m(s)=k(ms)$, so that (17) reads
$$K_s(t)=k_0+\sum_1^{\infty}k(ms)\cos mt.\tag{29}$$
Then it may be appropriate to consider $K(\cdot)$, the Fourier transform of $k(\cdot)$, and to compare $K_s(t)$ with $\frac{1}{s}K\big(\frac{t}{s}\big)$. Under a good condition on the behaviour of $K(\cdot)$ out of a neighbourhood of 0,
$$\int_0^1 t\,|dK(t)|<\infty$$
will replace (28) as a necessary and sufficient condition for $(R_w,1)\to(k)$. The "good condition" expresses that $dK(t)$ integrates bounded periodic functions. That may derive from an oscillatory character of $K(t)$, as we shall see in the examples. A different "good condition" is required for $(R,1)\to(k)$: $dK(t)$ should integrate formally a certain type of periodic Schwartz distributions (namely, locally derivatives of Fourier transforms of bounded functions). Again, an oscillatory character of $K(t)$ will work.

2.2.4. *Examples.* In all examples we shall consider Remark 4 applies: $k_m(s)=k(ms)$ and $K_s(t)$ has the form (29).

1. $k_m(s)=\frac{\sin ms}{ms}$. It is the only example I gave after Theorem 1, and fortunately Theorem 2 also applies. The measures dK_s are discrete in that case.
2. $k_m(s)=\big(\frac{\sin ms}{ms}\big)^{\kappa}$. $K_s(\cdot)$ is a convolution power of a step function, dK_s is a continuous measure carried by the interval $[-\kappa s,\kappa s]$. Again Theorems 1 and 2 apply:
$$(R_w,1)\longrightarrow(R_w,\kappa),\qquad(R,1)\longrightarrow(R,\kappa).$$
3. $k_m(s)=(1-ms)^+$, $s=\frac{1}{N}$. $K_s(t)$ is the usual Fejér kernel $\frac{\sin^2 Nt/2}{\sin^2 t/2}$, (21) and (23) (the "good conditions") are satisfied, but we have (24) instead of (20). Now Theorem 3 applies:
$$(R_w,1)\not\twoheadrightarrow(C,1),\tag{30}$$

the Cesàro summation process of order 1.

4. $k_m(s) = ((1-ms)^+)^\beta$, $\beta > 1$, $s = \frac{1}{N}$. Now $K_s(t)$ is a regularized Fejér kernel and (20) is satisfied:
$$(R,1) \longrightarrow (H,\beta) \longleftrightarrow (C,\beta). \tag{31}$$
(H,β) is the Marcel Riesz summation process of order β, equivalent to the Cesàro process of order β, (C,β) ([6, pp. 112–113].
5. $k_m(s) = e^{-ms}$. Here $K_s(t)$ is the Poisson kernel, (20), (21) and (23) are valid:
$$(R,1) \longrightarrow (\text{Abel–Poisson}),$$
a consequence of (31) by the way.

2.3. $(R,2) \longrightarrow (k)$? The method of 2.2 applies, but here
$$S(t) = S_2(t) = \sum_1^\infty u_m \Big(\frac{\sin mt}{mt}\Big)^2$$
instead of (14), and formally
$$(t^2 S(t))'' = 2\sum_1^\infty u_m \cos 2mt$$
instead of (15). $(R_w, 2)$ means $(R,2)$ with the additional assumption that $S(t)$ is bounded. As in (17) we write
$$K_s(t) = k_0 + \sum_1^\infty k_m(s)\cos mt$$
and we now assume that the formal derivative of the series is the Fourier series of a function with bounded variation, $K_s'(t)$, and
$$\int_{-\pi}^{\pi} t^2\,|dK_s'(t)| < \infty$$
for each s. Then, in case $S(\cdot)$ bounded,
$$\sum_1^\infty u_m k_m(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 S\Big(\frac{t}{2}\Big)\,dK_s'(t).$$
Hence

THEOREM 4. *Assume that*
$$\lim_{s\to 0}\int_0^\pi t^2\varphi(t)\,dK_s'(t) = 0 \tag{32}$$
for every $\varphi \in L^\infty(0,\pi)$ *such that* $\lim_{t\to 0}\varphi(t) = 0$. *Then*
$$(R_w, 2) \longrightarrow (k).$$

In order to have the analogue of Theorem 1 we have to assume that $K_s'(\cdot)$ is absolutely continuous on (ε,π) when $s < s(\varepsilon)$. Then (32) is implied, and can be replaced in Theorem 4 by
$$\varlimsup_{s\to 0}\int_0^\varepsilon t^2\,|dK_s'(t)| < \infty$$

for some $\varepsilon > 0$, and

$$\lim_{s\to 0}\int_{\varepsilon}^{\pi}\varphi(t)\,dK'_s(t) = 0$$

for every ε and $\varphi \in L^\infty(\varepsilon,\pi)$.

In order to have the analogue of Theorem 2 we shall make a strong assumption, namely that $K_s(\cdot)$ is C^∞ on (ε,π) when $s < s(\varepsilon)$, and that

$$\lim_{s\to 0}\langle\varphi, K_s\rangle = 0 \tag{33}$$

for every Schwartz distribution φ carried by $]0,\pi]$ (open on the left). The proof of Theorem 3 works the same, and (33) is a "good condition" in the sense of Remark 3 in 2.2.3. Expressing (28) again, together with the results of this section, we can state a simple and final result:

Theorem 5. *Assuming that $K_s(\cdot)$ is C^∞ on (ε,π) when $s < s(\varepsilon)$ and that* (33) *is valid for every distribution φ carried by $(0,\pi]$, the questions asked in* 2.2 *and* 2.3 *have the following answers*:

$$\exists\varepsilon > 0 : \overline{\lim_{s\to 0}}\int_0^{\varepsilon} t\,|dK_s(t)| < \infty \Longleftrightarrow \big((R_w,1)\longrightarrow(k)\big) \Longleftrightarrow \big((R,1)\longrightarrow(k)\big) \tag{34}$$

$$\exists\varepsilon > 0 : \overline{\lim_{s\to 0}}\int_0^{\varepsilon} t^2\,|dK'_s(t)| < \infty \Longleftrightarrow \big((R_w,2)\longrightarrow(k)\big) \Longleftrightarrow \big((R,2)\longrightarrow(k)\big) \tag{35}$$

Remark. (34) means that $K_s(\cdot)$ has locally bounded variation, and (35) that $K_s(\cdot)$ is a primitive of a function with bounded variation locally. If that is not the case, (k) is not entailed by $(R,1)$ resp. $(R,2)$.

Examples.

$$(R,2)\longrightarrow(R,\kappa) \qquad (\kappa = 2,3,\dots)$$

(R,κ) is an example of process (k) such that $K_s(\cdot)$ is not C^∞ on $(0,\pi)$, but is C^∞ (namely, 0) on (ε,π) for $s < s(\varepsilon)$.

$$(R,2)\to(H,\beta)\longleftrightarrow(C,\beta) \qquad (\beta > 2)$$
$$(R,2)\to(\text{Abel–Poisson})$$
$$(R,2)\not\twoheadrightarrow(H,E)\leftrightarrow(C,2). \tag{36}$$

The crucial case is (36). Then $k_m(s) = ((1-ms)^+)^2$. Taking $s = \frac{1}{N}$, N integer, and

$$C_N(t) = 1 + e^{it} + \ldots + e^{i(N-1)t} = \frac{e^{iNt}-1}{e^{it}-1}$$

we have

$$K_s(t) = \mathrm{Re}\Big(C_N(t) + \frac{2i}{N}C'_N(t) - \frac{1}{N^2}C''_N(t)\Big).$$

If we develop $K_s^{(j)}(t)$, all terms contain $\cos Nt$ or $\sin Nt$ as a factor, so that $\langle\varphi, K_s\rangle$ in (33) is the N-th Fourier coefficient of a bounded function, therefore (33) is valid. Near 0, the dominant term of $t^2K''_s(t)$ is $\frac{2}{N^2}\frac{\cos Nt}{t}$, hence the first part of (35) fails, therefore $(R,2)\not\twoheadrightarrow(H,2)$.

Going back to (30) and (31), we see that

$$(R,1) \to (C,1+\varepsilon) \quad \text{when } \varepsilon > 0 \tag{37}$$

$$(R,2) \to (C,2+\varepsilon) \quad \text{when } \varepsilon > 0 \tag{38}$$

$$(R,1) \nrightarrow (C,1) \tag{39}$$

$$(R,2) \nrightarrow (C,2) \tag{40}$$

(37) and (38) have been known for a long time [11, 13]. (39) and (40) are new.

2.4. What about $(R,3) \longrightarrow (k)$? In view of (37) and (38) one could presume that $(R,\kappa) \longrightarrow C(\kappa+\varepsilon)$ for all integers κ and $\varepsilon > 0$. But that is not true. Actually

$$(R,3) \nrightarrow (\text{Abel–Poisson}),$$

that is, $(R,3) \nrightarrow (k)$ when $k_1(m) = e^{-ms}$. That was established in the 1930s [11, 13]. Let me reproduce Verblunsky's proof.

Here

$$S(t) = S_3(t) = \sum_1^\infty u_m \Big(\frac{\sin mt}{mt}\Big)^3. \tag{41}$$

We shall construct a sequence (u_m) such that $S(t)$ vanishes in a neighbourhood of 0 and

$$\overline{\lim_{s\to 0}} \sum_1^\infty u_m e^{-ms} \neq \underline{\lim_{s\to 0}} \sum_1^\infty u_m e^{-ms}.$$

In view of

$$4\sin^3 t = 3\sin t - \sin 3t$$

(41) can be written formally as

$$\begin{cases} 4t^3 S(t) = 3\Phi(t) - \Phi(3t) \\ \Phi(t) = \displaystyle\sum_1^\infty \frac{u_m}{m^3}\sin mt. \end{cases} \tag{42}$$

Let us define first

$$\varphi(t) = \sum_1^\infty \frac{u_m}{m^2}\cos mt$$

by the conditions

$$\varphi(t) = \varphi(3t) \qquad (t < t_1 \leq \pi/3)$$

and $\varphi(t)$ is bounded and not constant on $(0,t_1)$. Integrating φ we obtain Φ and S, with $\sum_1^\infty \frac{|u_m|}{m^3} < \infty$, therefore (42) is justified, and $S(t) = 0$ on $[0,t_1]$ if we choose Φ in order that $\Phi(3t) = 3\Phi(t)$. Now

$$\sum_1^\infty \frac{u_m}{m^2} e^{-ms} \ \Big(= \int \varphi P_s\Big)$$

(P_1 being a Poisson kernel) has no limit as $s \to 0$. The same is true for $\sum_1^\infty u_m e^{-ms}$ for the following reason.

Lemma 1. *If $\sum_1^\infty a_m e^{-ms} = \psi(s)$ is bounded on $(0,1)$, $\sum_1^\infty \frac{a_m}{m} e^{-ms}$ is defined on $(0,1)$ and converges to a limit as $s \downarrow 0$.*

Proof. The first series converges uniformly on any interval $(s_0, 1)$ $(s_0 > 0)$, and the second series converges for $0 < s < 1$ by the Abel transformation. If $0 < s_0 < s_1 < 1$, $\Phi(s_1) - \Phi(s_0) = \int_{s_0}^{s_1} \psi(s)\,ds = o(1)$ $(s_1 \downarrow 0)$, hence the result. ∎

This argument extends to other but not all summation processes (k). For example, one can replace (Abel–Poisson) by (Weierstrass): $k_m(s) = e^{-m^2 s}$. But it is not true in general that the convergence as $s \downarrow 0$ of $\sum u_m e^{-ms}$ implies the same for $\sum \frac{u_m}{m^2} e^{-ms}$.

However $(R,3) \nrightarrow (k)$ is a quite general fact. Using Φ instead of φ, we look for a $\Phi \in \mathcal{F}\ell^1$ such that Φ is odd,

$$3\Phi(t) = \Phi(3t) \qquad (0 < t < t_0)$$

and

$$\varlimsup_{s\to 0} \Big| \int \Phi K_s''' \Big| > 0.$$

Then $S(t) = 0$ on $(0, t_0)$ and $\sum u_m k_m(s)$ does not tend to 0 as $s \to 0$, therefore

$$(R,3) \nrightarrow (k). \tag{43}$$

In order to construct Φ we choose a triangular function ψ carried by an interval $(a-\varepsilon, a+\varepsilon)$ $(\varepsilon \ll \frac{a}{2})$ and take

$$\Phi(t) = \sum_{j=0}^{\infty} 3^{-j} \big(\psi(3^j t) - \psi(-3^j t)\big).$$

Then

$$\int \Phi K_s''' \simeq C_\varepsilon \sum 3^{-2j} K_s'''(3^j a). \tag{44}$$

If the second member does not tend to 0 as $s \to 0$, (43) takes place.

As an example, take

$$\begin{aligned} k_m(s) &= \Big(\frac{\sin ms}{ms}\Big)^4 \\ K_s &= (2s)^{-4} \chi_s * \chi_s * \chi_s * \chi_s \qquad (\chi_s = 1_{[-s,s]}) \\ K_s''' &= (2s)^{-4} \chi_s' * \chi_s' * \chi_s' * \chi_s = (3 \times 1_{(0,2)} - 1_{(2,4)})(2s)^{-4} \end{aligned}$$

If we choose $s = a3^{-k}$, the sum of the series in (44) is $C_s' s^{-2}$. Hence

$$(R,3) \nrightarrow (R,4).$$

Besides Hardy's book [6] the main useful references are [14, 11] and [13].

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