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MARCINKIEWICZ SPACES, COMMUTATORS AND NON-COMMUTATIVE GEOMETRY

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Nigel J. Kalton was one of the most eminent guests participating in the Józef Marcinkiewicz Centenary Conference. His contribution to the scientific aspect of the meeting was very essential. Nigel was going to prepare a paper based on his plenary lecture. The editors are completely sure that the paper would be a real ornament of the Proceedings. Unfortunately, Nigel's sudden death totally destroyed editors' hopes and plans. Every mathematician knows how unique were Nigel's mathematical achievements. Moreover the community of mathematicians in Poznań is very proud of Nigel's friendship demonstrated many times during his visits at the Adam Mickiewicz University. For these reasons, to commemorate Professor Kalton, the editors of the Proceedings decided to print copies of the slides that he used during his plenary talk on June 29, 2010.

The editors would like to thank very much Mrs Jennifer Kalton for her kind permission to publish Nigel's presentation. The editors are deeply grateful for such a wonderful gesture.

1. Sequence spaces and ideals. Let $\xi \in c_0$. Then its decreasing rearrangement ξ^* is given by

$$\xi_n^* = \inf\big\{\lambda > 0 : \big|\{k : |\xi_k| > \lambda\}\big| < n\big\}.$$

A symmetric sequence space E is a vector subspace of c_0 such that $\xi \in E$, $\eta \in c_0$ with $\eta^* \leqslant \xi^* \Rightarrow \eta \in E$.

If E is a symmetric sequence space then $\mathcal{S}_E$ is the ideal of compact operators T on a separable Hilbert space $\mathcal{H}$ whose singular values satisfy

$$\{s_n(T)\}_{n=1}^{\infty} \in E$$

$E \to \mathcal{S}_E$ defines a correspondence between symmetric sequence spaces and ideals of compact operators. For example ℓ^p corresponds to the Schatten class $\mathcal{S}_p$.

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2. Traces and symmetric functionals. If $\mathcal{S}_E$ is an ideal of compact operators then a *trace* on $\mathcal{S}_E$ is *any* linear functional $\tau : \mathcal{S}_E \to \mathbb{C}$ such that

$$\tau(AB) = \tau(BA), \quad A \in \mathcal{S}_E, \quad B \in \mathcal{B}(\mathcal{H}).$$

τ is called positive if $\tau(P) \geqslant 0$ for all positive $P \in \mathcal{S}_E$. If τ is a trace then we can define a linear functional on E by

$$\varphi(\xi) = \tau(\operatorname{diag}(\xi_1, \xi_2, \dots)).$$

φ is then a *symmetric* functional, i.e. $\varphi(\xi) = \varphi(\eta)$ if η is a permutation of ξ.

(Figiel) There is a correspondence between traces on $\mathcal{S}_E$ and symmetric functionals on E; precisely if φ is a symmetric functional on E there is a unique trace τ on $\mathcal{S}_E$ such that

$$\varphi\big(\{s_n(P)\}_{n=1}^\infty\big) = \tau(P), \quad P \geqslant 0, \quad P \in \mathcal{S}_E.$$

3. Nonstandard positive traces. The first construction of a nonstandard positive trace was by Dixmier. He takes E to be a Marcinkiewicz space M_ψ where ψ is a concave function with $\psi(0) = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$. M_ψ consists of all sequences such that

$$\|\xi\|_{M_\psi} = \sup_n \frac{1}{\psi(n)} \sum_{k=1}^n \xi_k^* < \infty.$$

We are particularly interested in the case $\psi(x) = \log(1 + x)$ when we write $M_\psi = M_{\log}$.

We will write $\mathcal{M}_\psi$ for the ideal $\mathcal{S}_{M_\psi}$. $\mathcal{M}_{\log}$ is the dual of the Matsaev ideal.

4. Dixmier traces

THEOREM 1 (Dixmier 1966). *If*

$$\lim_{n\to\infty} \frac{\psi(2n)}{\psi(n)} = 1$$

then there is a positive trace τ on $\mathcal{M}_\psi$ defined by

$$\tau(P) = \omega\Big(\frac{1}{\psi(2^n)} \sum_{k=1}^{2^n} s_k(P)\Big), \quad P \in \mathcal{M}_\psi, \quad P \geqslant 0.$$

Here ω is a (*translation invariant*) *Banach limit.*

Traces of this type are called *Dixmier traces.*

If $\psi(x) = \log(1 + x)$ then $\tau\big(\operatorname{diag}\{1/n\}_{n=1}^\infty\big) = 1$ for every Dixmier trace.

5. Positive traces II

THEOREM 2 (Dodds, de Pagter, Semenov and Sukochev 1988). *The condition*

$$\liminf_{n\to\infty} \frac{\psi(2n)}{\psi(n)} = 1$$

is necessary and sufficient for the existence of a positive trace on $\mathcal{M}_\psi$.

Dixmier traces play a significant role in *noncommutative geometry* via the *Connes trace theorem.* Let us explain (in a very simplified case) how this goes.

6. The Connes trace theorem. Let X be a compact Riemannian manifold of dimension d. (For example $X = \mathbb{T}^d$.) Let Δ denote the negative Laplacian on X, and let dx denote the standard volume measure on X. For $f \in L_\infty(X)$ we define

$$T_f(g) = fg \quad \text{for} \quad g \in L_2(X).$$

We now state “toy” version of the Connes trace formula.

THEOREM 3 (Connes 1988). *Then for any $f \in C^\infty(X)$, and any Dixmier trace on $\mathcal{M}_{\log}$ we have $T_f(1+\Delta)^{-d/2} \in \mathcal{M}_{\log}$ and*

$$\tau\big(T_f(1+\Delta)^{-d/2}\big) = \frac{\Omega_d}{d(2\pi)^d}\int_X f(x)\,dx.$$

Here Ω_d is the surface area of the $(d-1)$-dimensional sphere.

The correct statement is for *classical pseudo-differential operators* and considers integration over the *cosphere bundle.*

7. General traces. We now consider general (not necessarily positive or continuous) traces. If E is any symmetric sequence space we define the commutator subspace of $\mathcal{S}_E$, $\operatorname{Com}\mathcal{S}_E$, to be the linear span of all commutators $[A,B] = AB - BA$ for $A \in \mathcal{S}_E$ and $B \in \mathcal{B}(\mathcal{H})$. A linear functional τ is a trace if and only if it annihilates $\operatorname{Com}\mathcal{S}_E$.

The problem of existence of non-trivial traces on $\mathcal{S}_p$ was first considered by Pearcy and Topping in 1971 who showed that $\operatorname{Com}\mathcal{S}_p = \mathcal{S}_p$ when $p > 1$, so that the only trace on these ideals is the zero trace. If $p < 1$, Anderson (1986) showed that

$$\operatorname{Com}\mathcal{S}_p = \{T \in \mathcal{S}_p : \operatorname{tr} T = d\}$$

so that the only traces are multiples of the standard trace.

The case $p = 1$ is more tricky. Weiss (1980) showed that there are discontinuous traces so that $\operatorname{Com}\mathcal{S}_1 \neq \{T \in \mathcal{S}_1 : \operatorname{tr} T = 0\}$. Kalton (1989) gave a complete characterization of $\operatorname{Com}\mathcal{S}_1$.

8. General traces II. In fact in the middle of the 1990’s there appeared a complete characterization of $\operatorname{Com}\mathcal{J}$ for any ideal $\mathcal{J}$. This was not published till 2004.

From now if T is a compact operator we write $\{\lambda_n(T)\}_{n=1}^\infty$ for the eigenvalues of T, repeated according to algebraic multiplicity and arranged in (some) order of decreasing absolute value so that

$$|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \dots .$$

If there are only finitely many eigenvalues the list is completed with zeros.

THEOREM 4 (Dykema, Figiel, Weiss, Wodzicki 1995, 2004). *Let $\mathcal{J} = \mathcal{S}_E$. Let $T \in \mathcal{J}$ be a normal operator. Then $T \in \operatorname{Com}\mathcal{J}$ if and only if*

$$\Big\{\frac{\lambda_1 + \dots + \lambda_n}{n}\Big\}_{n=1}^\infty \in E$$

where $\lambda_j = \lambda_j(T)$.

9. General traces III. The DFWW theorem implies that $\operatorname{Com}\mathcal{J} = \mathcal{J}$ (or, equivalently, $\mathcal{J}$ admits no nonzero traces) if $\mathcal{J} = \mathcal{S}_E$ where $\xi \in E$ implies that $\{\frac{1}{n}(\xi_1^* + \ldots + \xi_n^*)\}_{n=1}^\infty \in E$.

In the same paper the authors show that $T \in \operatorname{Com}\mathcal{J}$ if and only if T can be expressed as the sum of at most 3 commutators $[A_j, B_j]$ where $A_j \in \mathcal{J}$ and $B_j \in \mathcal{B}(\mathcal{H})$.

10. Formulas for nonstandard traces. Suppose τ is a trace on $\mathcal{J} = \mathcal{S}_E$. The DFWW theorem implies that to calculate $\tau(T)$ we should write $T = H + iK$ where H, K are hermitian and then

$$\tau(T) = \tau\big(\operatorname{diag}\{\lambda_n(H) + i\lambda_n(K)\}_{n=1}^\infty\big).$$

Are there other natural formulas?

THEOREM 5 (Kalton 1998). *Suppose E is a Banach or quasi-Banach sequence space. Then*

$$\tau(T) = \tau\big(\operatorname{diag}\{\lambda_n(T)\}_{n=1}^\infty\big).$$

This theorem fails for arbitrary ideals (Dykema–Kalton 1998). This can be expressed by saying that if E is quasi-Banach then a form of Lidskii's theorem holds, i.e. all traces are *spectral*.

11. Taking the diagonal. When can we compute τ just from the diagonal of the operator? Let $(e_n)_{n=1}^\infty$ be an orthonormal basis of $\mathcal{H}$; can we compute

$$\tau(T) = \tau\big(\operatorname{diag}\{(Te_n, e_n)\}_{n=1}^\infty\big)?$$

Not in general of course.

THEOREM 6 (Kalton, Lord, Potapov, Sukochev 2010). *Let A be a positive compact operator on $\mathcal{H}$. Let $(e_n)_{n=1}^\infty$ be an orthonormal basis of eigenvectors for $\mathcal{H}$. Let τ be any trace on $\mathcal{J}_A$ (the smallest ideal containing A). Then for any $T \in \mathcal{B}(\mathcal{H})$ we have*

$$\tau(TA) = \tau(AT) = \tau\big(\operatorname{diag}\{(TAe_n, e_n)\}_{n=1}^\infty\big).$$

12. Idea of the proof. We can take T to be hermitian. Let $(f_n)_{n=1}^\infty$ be an orthonormal basis of eigenvectors for $H = \frac{1}{2}(TA + AT)$ arranged so that $Hf_n = \mu_n f_n$ with $|\mu_n|$ decreasing. We similarly suppose that $Ae_n = \lambda_n e_n$ with λ_n decreasing.

The key is to estimate the difference

$$\sum_{k=1}^n (Hf_k, f_k) - \sum_{k=1}^n (He_k, e_k).$$

To do this we let E_n be the span of $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ and P_n be the orthogonal projection on E_n. Then

$$\Big|\sum_{k=1}^n (Hf_k, f_k) - \operatorname{tr}(P_n H)\Big|, \Big|\sum_{k=1}^n (He_k, e_k) - \operatorname{tr}(P_n H)\Big| \leqslant n\lambda_n \|T\|.$$

These estimates (and a little more) and the DFWW theorem allow us to show that $\operatorname{diag}\{(Hf_k, f_k) - (He_k, e_k)\}_{k=1}^\infty \in \operatorname{Com}\mathcal{J}_A$.

13. Back to the Connes trace formula. Let X be a compact Riemannian manifold of dimension d. (For example $X = \mathbb{T}^d$). Let Δ denote the Laplacian on X, and let dx denote the standard volume measure on X. For $f \in L_\infty(X)$ we define $T_f g = fg$ for $g \in L_2(X)$.

THEOREM 7 (Connes 1988). *For any $f \in C^\infty(X)$ and any Dixmier trace on $\mathcal{M}_{\log}$ we have $T_f(1+\Delta)^{-d/2} \in \mathcal{M}_{\log}$ and*

$$\tau(T_f(1+\Delta)^{-d/2}) = \frac{\Omega_d}{d(2\pi)^d}\int_X f(x)\,dx.$$

Here Ω_d is the surface area of the $(d-1)$-dimensional sphere.

Let us take $X = \mathbb{T}^d$ and use the preceding ideas.

14. The Connes trace formula for the torus. In this case $A = (1+\Delta)^{-d/2}$ and so $\mathcal{J}_A = \mathcal{S}_{1,\infty}$ is associated with the weak-ℓ^1 sequence space $\ell_{1,\infty} = \{\xi : \xi^* = O(1/n)\}$. This is strictly contained in the Marcinkiewicz space $\mathcal{M}_{\log}$.

The basis of eigenvectors is given by $e_n(2\pi)^{-d/2}e^{i\langle n,\theta\rangle}$ for $n \in \mathbb{Z}^d$. Here $n = (n_1,\ldots,n_d) \in \mathbb{Z}^d$ and $\theta = (\theta_1,\ldots,\theta_d) \in (-\pi,\pi]^d$.

If $T = T_f$ is a multiplication operator with $f \in L_\infty(\mathbb{T}^d)$ then

$$(T_f e_n, e_n) = \frac{1}{(2\pi)^d}\int f(x)\,dx$$

Hence if $|n| = (n_1^2 + \ldots + n_{\mathrm{d}}^2)^{1/2}$,

$$(T_f(1+\Delta)^{-d/2}e_n, e_n) = \frac{1}{(2\pi)^d}(1+|n|^2)^{-d/2}\int f(x)\,dx.$$

15. The Connes trace formula for the torus II. Thus if τ is *any* trace on $\mathcal{S}_{1,\infty}$ we have

$$\tau(T_f(1+\Delta)^{-d/2}) = \frac{1}{(2\pi)^d}\,\tau\big(\{(1+|n|^2)^{-d/2}\}_{n\in\mathbb{Z}^d}\big)\int f(x)\,dx.$$

If τ is *normalized* so that $\tau(\{1/n\}_{n=1}^\infty) = 1$ (e.g. a Dixmier trace) one can evaluate

$$\tau\big(\{(1+|n|^2)^{-d/2}\}_{n\in\mathbb{Z}^d}\big) = d^{-1}\Omega_d.$$

This proves the Connes trace formula for L_∞-functions (not just C^∞-functions) and for every (perhaps discontinuous) trace on $\mathcal{S}_{1,\infty}$ (not just Dixmier traces).

For Dixmier traces the extension to L_∞-functions was proved in Lord-Potapov-Sukochev 2010.

16. Eigenvalues. Let us interpret this extension of Connes's trace formula in terms of the eigenvalues of $S = T_f(1+\Delta)^{-d/2}$. Let $\lambda_n = \lambda_n(S)$.

THEOREM 8 (Kalton, Lord, Potapov, Sukochev 2010). *There exists a constant C such that*

$$\left|\sum_{k=1}^n \lambda_k - \frac{\Omega_d \log n}{d(2\pi)^d}\int f(x)\,dx\right| \leqslant C.$$

Note that $\ell_{1,\infty}$ is quasi-Banach so all traces are spectral.

17. An improvement

THEOREM 9 (Kalton, Lord, Potapov, Sukochev 2010). *Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ and let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator such that for some constant C we have*

$$n \sum_{k=n+1}^{\infty} \|Te_k\|^2 \leqslant C.$$

Then:

- $T \in \mathcal{S}_{1,\infty}$,
- $\sup_n \left| \sum_{k=1}^{n} \lambda_k(T) - \sum_{k=1}^{n} (Te_k, e_k) \right| < \infty$,
- $\tau(T) = \tau(\{(Te_k, e_k)\}_{k=1}^{\infty})$ *for every trace τ on $\mathcal{S}_{1,\infty}$.*

18. Operators on $L_2(\mathbb{R}^d)$. We consider operators $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ of the form

$$Tf(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p_T(x,\xi) \widehat{f}(\xi)\, d\xi.$$

T is compactly supported if there is a compact set K so that $p_T(x,\xi) = 0$ for $x \notin K$ and $Tf = 0$ if $f = 0$ on K.

T is pseudo-differential operator of order m if p_T is C^∞ and satisfies estimates of the type

$$\left| \partial_x^\alpha \partial_\xi^\beta p_T(x,\xi) \right| \leqslant C_{\alpha,\beta} (1 + |\xi|)^{m-|\beta|}.$$

However we do not need to consider such smooth kernels ... We assume only that p_T is measurable.

19. Operators on $L_2(\mathbb{R}^d)$ II

THEOREM 10 (Kalton, Lord, Potapov, Sukochev 2010). *Suppose for some constant C*

$$\int_{\mathbb{R}^d} \int_{|\xi| \geqslant t} |p_T(x,\xi)|^2 \, d\xi \, dx \leqslant Ct^{-d}$$

and

$$Tf(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p_T(x,\xi) \widehat{f}(\xi)\, d\xi$$

is compactly supported. For example T could be a pseudo-differential operator of order $-d$. Then

- $T \in \mathcal{S}_{1,\infty}$,
- $\sup_n \left| \sum_{k=1}^{n} \lambda_k(T) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{|\xi| \leqslant n^{1/d}} p_T(x,\xi)\, d\xi \, dx \right| < \infty.$

20. The Connes trace formula again

THEOREM 11 (Kalton, Lord, Potapov, Sukochev 2010). *Let X be a compact Riemannian manifold and consider the operator $S = T_f(1+\Delta)^{-d/2}$ where $f \in L_2(X)$. Then $S \in \mathcal{S}_{1,\infty}$. If τ is a normalized trace on $\mathcal{S}_{1,\infty}$ we have*

$$\tau(S) = \frac{\Omega_d}{d(2\pi)^d} \int_X f(x)\, dx$$

or, equivalently

$$\sup_n \left| \sum_{k=1}^{n} \lambda_k(S) - \frac{\Omega_d \log n}{d(2\pi)^d} \int_X f(x)\,dx \right| < \infty.$$

Notice now we require $f \in L_2(X)$ (not $f \in L_\infty(X)$) so that T_f is potentially unbounded.

21. L_p for $p < 2$? One cannot expect $T_f(1+\Delta)^{-d/2}$ be necessarily even a bounded operator if $p < 2$. However one can consider $(1+\Delta)^{-d/4}T_f(1+\Delta)^{-d/4}$. All the previous results would have worked in this case.

THEOREM 12 (Lord, Potapov and Sukochev 2010). *If $f \in L_p$ where $p > 1$ then $(1+\Delta)^{-d/4}T_f(1+\Delta)^{-d/4} \in \mathcal{M}_{\log}$ and if τ is a Dixmier trace on $\mathcal{M}_{\log}$ one still has the trace formula.*

However for $f \in L_1$ it is not necessarily true that $(1+\Delta)^{-d/4}T_f(1+\Delta)^{-d/4}$ is even a bounded operator.

22. $p < 2$

THEOREM 13 (Kalton, Lord, Potapov, Sukochev 2010). *Suppose $X = \mathbb{S}^d$ or $X = \mathbb{T}^d$. Suppose $f \in \mathrm{L}(\log\mathrm{L})^2(\log\log\mathrm{L})(X)$, i.e.*

$$\int |f(x)|(\log_+ |f(x)|)^2 \log_+ \log_+ |f(x)|\,dx < \infty.$$

Then $(1+\Delta)^{-d/4}T_f(1+\Delta)^{-d/4} \in \mathcal{M}_{\log}$ and for every normalized trace τ on $\mathcal{M}_{\log}$, we have

$$\tau\big((1+\Delta)^{-d/4}T_f(1+\Delta)^{-d/4}\big) = \frac{\Omega_d}{d(2\pi)^d} \int f(x)\,dx.$$