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COMMON EXTENSIONS FOR LINEAR OPERATORS

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Abstract. The main meaning of the *common extension* for two linear operators is the following: given two vector subspaces G_1 and G_2 in a vector space (respectively an ordered vector space) E, a Dedekind complete ordered vector space F and two (positive) linear operators $T_1 : G_1 \to F$, $T_2 : G_2 \to F$, when does a (positive) linear common extension L of T_1, T_2 exist?

First, L will be defined on $\mathrm{span}(G_1 \cup G_2)$. In other results, formulated in the line of the Hahn–Banach extension theorem, the common extension L will be defined on the whole space E, by requiring the majorization of T_1, T_2 by a (monotone) sublinear operator. Note that our first Hahn–Banach common extension results were proved by using two results formulated in the line of the Mazur–Orlicz theorem. Actually, for the first of these last mentioned results, we extend the name *common extension* to the case when E is without order structure, instead of G_1, G_2 there are some arbitrary nonempty sets, instead of T_1, T_2 there are two arbitrary maps f_1, f_2, and, in addition, we are given two more maps $g_1 : G_1 \to E$, $g_2 : G_2 \to E$ and a sublinear operator $S : E \to F$. In this case we ask: *When is it possible to obtain a linear operator $L : E \to F$, dominated by S and related to the maps f_1, f_2, g_1, g_2 by some inequalities?*

To extend positive linear operators between ordered vector spaces, some authors (Z. Lipecki, R. Cristescu and myself) have used a procedure which includes the introduction of an additional set and a corresponding map. Inspired by this technique, in this paper we also solve some common positive extensions problems by using an additional set.

1. Preliminaries. In this paper the terminology, the notation and some mentioned results are classical for the theory of the ordered vector spaces and linear operators (see, for example [1], [2] and [11]); X_0 and X will be real vector spaces, E_0 and E will be ordered vector spaces and, generally, F will be a Dedekind complete ordered vector space

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(that is, every nonempty ordered bounded set in F has a supremum or, equivalently, an infimum).

For the main meaning of the *common extension problem* we consider two vector subspaces (or sets) G_1, G_2 in E_0, $E = \operatorname{span}(G_1 \cup G_2)$ and two linear operators (or arbitrary maps) $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ and we are interested to give (necessary and) sufficient conditions for the existence of a (positive) linear operator $L : E \longrightarrow F$ such that L extends T_1 and T_2, that is $L(v_1) = T_1(v_1)$ and $L(v_2) = T_2(v_2)$ for all $v_1 \in G_1$ and $v_2 \in G_2$. Obviously, a necessary condition for this is that the operators T_1 and T_2 are *consistent* (in the terminology introduced in [9]) that is, $T_1 = T_2$ on $G_1 \cap G_2$.

Such results, for the case of linear functionals, appeared in [12] and [9]. The importance of this problem appears, for example, in [9], [14], [15], [16] and [13].

The primary result in this sense is the following:

THEOREM 1.1. *Let X_0 and Y be two vector spaces, G_1 and G_2 two vector subspaces of X_0, $X = \operatorname{span}(G_1 \cup G_2)$ and $T_j : G_j \to Y$, $j \in \{1, 2\}$, two linear operators. Then, the following are equivalent*:

(i) *There exists $L : X \to Y$, a common linear extension of T_1, T_2.*
(ii) *If $v_1 + v_2 = 0$, with $v_1 \in G_1$, $v_2 \in G_2$, then $T_1(v_1) + T_2(v_2) = 0$.*
(iii) *$T_1 = T_2$ on $G_1 \cap G_2$.*

Note that, for the proof of (ii) $\Rightarrow$ (i), we define $L : X \to Y$ by $L(v_1 + v_2) = T_1(v_1) + T_2(v_2)$ for all $v_1 \in G_1$ and $v_2 \in G_2$ and, according to (ii), it follows that L is well-defined.

For a finite family $(T_j)_{j\in\{1,...,n\}}$ of linear operators, Theorem 1.1 becomes:

THEOREM 1.2. *Let X_0 and Y be two vector spaces, $(G_j)_{j\in\{1,...,n\}}$ a family of vector subspaces of X_0 and $T_j : G_j \to Y$, $j \in \{1, \ldots, n\}$ a family of linear operators. Then, the following are equivalent*:

(i) *There exists $L : \operatorname{span}(G_1 \cup \ldots \cup G_n) \to Y$, a common linear extension of $T_1, \ldots, T_n$.*
(ii) *If $v_1 + v_2 + \ldots + v_n = 0$, then $T_1(v_1) + T_2(v_2) + \ldots + T_n(v_n) = 0$, where $v_j \in G_j$ for each $j \in \{1, \ldots, n\}$.*
(iii) *For each two sets N_1, N_2 so that $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = \{1, \ldots, n\}$, $\sum\limits_{k\in N_1} T_k(v_k) = \sum\limits_{j\in N_2} T_j(v_j)$ if $\sum\limits_{k\in N_1} v_k = \sum\limits_{j\in N_2} v_j$, where $v_i \in G_i$ for $i \in \{1, \ldots, n\}$.*

It is easy to prove that (iii) from Theorem 1.2 is equivalent to the following condition:

(iii′) For any $k \in \{2, 3, \ldots, n\}$, $T_k = T_1 + T_2 + \ldots + T_{k-1}$ on $G_k \cap \operatorname{span}(G_1 \cup \ldots \cup G_{k-1})$, that is $T_k(v_k) = T_1(v_1) + T_2(v_2) + \ldots + T_{k-1}(v_{k-1})$ for any $v_k = v_1 + v_2 + \ldots + v_{k-1}$, where $v_j \in G_j$, $j \in \{1, \ldots, k\}$.

The following result is a version of Theorem 1.1 in the ordered vector spaces setting, all the linear operators which appear being positive.

THEOREM 1.3. *Let E_0 be an ordered vector space and let F be a Dedekind complete ordered vector space. Let also G_1, G_2 be two vector subspaces of E_0 and let $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ be two positive linear operators. Let us consider the following statements, where $E = \operatorname{span}(G_1 \cup G_2)$*:

(i) *There exists $L : E \to F$, a positive common linear extension of T_1 and T_2*;

(ii) *If* $v_1 + v_2 \le 0$, *where* $v_1 \in G_1, v_2 \in G_2$, *then* $T_1(v_1) + T_2(v_2) \le 0$;
(iii) *If* $v_1 + v_2 \ge 0$, *where* $v_1 \in G_1, v_2 \in G_2$, *then* $T_1(v_1) + T_2(v_2) \ge 0$;
(iv) *If* $v_1 + v_2 = 0$, *where* $v_1 \in G_1, v_2 \in G_2$, *then* $T_1(v_1) + T_2(v_2) = 0$;
(v) $T_1 = T_2$ *on* $G_1 \cap G_2$.

Then, we have: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v).

The proof of Theorem 1.3 is immediate. Also, the corresponding result which generalizes this theorem for a family $(T_j)_{j\in\{1,\dots,n\}}$ of positive linear operators can easily be formulated.

2. Common extensions in the line of Mazur–Orlicz and Hahn–Banach theorems. In the following result having as a consequence the Mazur–Orlicz theorem (see Corollary 2.3 below), we meet another meaning for the *common extension problem.* We will consider two nonempty sets A_1, A_2, four maps $g_1 : A_1 \to X$, $g_2 : A_2 \to X$, $f_1 : A_1 \to F$, $f_2 : A_2 \to F$ and a sublinear operator $S : X \to F$ such that all these maps satisfy an inequality which implies that $f_1 \le S \circ g_1$ and $f_2 \le S \circ g_2$. Then we can extend simultaneously these inequalities, obtaining the existence of a linear operator $L : E \to F$ dominated by S and such that $f_1 \le L \circ g_1$ and $f_2 \le L \circ g_2$.

Actually, this result will be applied to obtain a *common extension* (for two positive linear operators) in the main meaning considered in this paper and in the line of the Hahn–Banach theorem.

THEOREM 2.1. *Let* X *be a vector space,* F *a Dedekind complete ordered vector space,* A_1 *and* A_2 *two nonempty arbitrary sets,* $S : X \to F$ *a sublinear operator, and* $g_j : A_j \to X$ *and* $f_j : A_j \to F$, $j \in \{1,2\}$, *four maps. Then, the following are equivalent*:

(i) *There exists* $L : X \to F$ *a linear operator such that*
   a) $L \le S$ *on* X, *and*
   b) $f_1 \le L \circ g_1$ *on* A_1 *and* $f_2 \le L \circ g_2$ *on* A_2.
(ii) *The inequality*

$$\sum_{i=1}^{n} \lambda_i f_1(a_{1i}) + \sum_{j=1}^{m} \mu_j f_2(a_{2j}) \le S\Big(\sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{j=1}^{m} \mu_j g_2(a_{2j})\Big) \tag{2.1}$$

holds for all n, $m \in \mathbb{N}^*$, $\{a_{11}, \dots, a_{1n}\} \subset A_1$, $\lambda_1 \ge 0, \dots, \lambda_n \ge 0$, $\{a_{21}, \dots, a_{2m}\} \subset A_2$, $\mu_1 \ge 0, \dots, \mu_m \ge 0$.

Proof. First, we remark that we can suppose that $m = n$, taking $\lambda_{n+1} = \dots = \lambda_m = 0$, if $n < m$, respectively $\mu_{m+1} = \dots = \mu_n = 0$, if $m < n$.

Obviously, (i) $\Rightarrow$ (ii). Indeed, using successively (i) b), the linearity of L from (i) and (i) a), we obtain

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i f_1(a_{1i}) + \sum_{j=1}^{n} \mu_j f_2(a_{2j}) &\le \sum_{i=1}^{n} \lambda_i (L\circ g_1)(a_{1i}) + \sum_{j=1}^{n} \mu_j (L\circ g_2)(a_{2j}) \\ &= L\Big(\sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{j=1}^{n} \mu_j g_2(a_{2j})\Big) \le S\Big(\sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{j=1}^{n} \mu_j g_2(a_{2j})\Big).\end{aligned}$$

To prove that (ii) implies (i), we use the *technique of the auxiliary sublinear operator*, and apply the existence form of the Hahn–Banach theorem ("*For every sublinear operator $S_1 : X \to F$ there exists a linear operator $L_1 : X \to F$ such that $L_1 \le S_1$ on X.*"). For every $x \in X$, put $S_1(x)$ the infimum of the set

$$\Big\{S\Big(x + \sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{i=1}^{n} \mu_i g_2(a_{2i})\Big) - \sum_{i=1}^{n} \lambda_i f_1(a_{1i}) - \sum_{i=1}^{n} \mu_i f_2(a_{2i})\Big\},$$

where the infimum is taken over all finite subsets $\{a_{11}, \dots, a_{1n}\} \subset A_1$, $\{a_{21}, \dots, a_{2n}\} \subset A_2$, $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}_+$, $\{\mu_1, \dots, \mu_n\} \subset \mathbb{R}_+$ and $n \in \mathbb{N}^*$. Note that $S_1(x)$ exists because, using condition (ii) and the sublinearity of S, we have

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i f_1(a_{1i}) + \sum_{i=1}^{n} \mu_i f_2(a_{2i}) &\le S\Big(\sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{i=1}^{n} \mu_i g_2(a_{2i})\Big)\\ &\le S\Big(x + \sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{i=1}^{n} \mu_i g_2(a_{2i})\Big) + S(-x).\end{aligned}$$

Hence

$$-S(-x) \le S\Big(x + \sum_{i=1}^{n} \lambda_i g_1(a_{1i}) + \sum_{i=1}^{n} \mu_i g_2(a_{2i})\Big) - \sum_{i=1}^{n} \lambda_i f_1(a_{1i}) - \sum_{i=1}^{n} \mu_i f_2(a_{2i}).$$

This inequality holds in the Dedekind complete ordered vector space F.

It is straightforward to prove that S_1 is a sublinear operator. Then, using the existence form of the Hahn–Banach theorem ([11], p. 44), there exists a linear operator $L : X \to F$ such that

$$L(x) \le S_1(x), \quad x \in X. \tag{2.2}$$

Using the definition of S_1 we remark that

$$S_1(x) \le S(x), \quad x \in X. \tag{2.3}$$

(2.2) and (2.3) imply (i) a), that is $L(x) \le S(x)$ for all $x \in X$.

Now we prove (i) b), that is, for example, that

$$f_1 \le L \circ g_1 \text{ on } A_1. \tag{2.4}$$

But, for every $a_1 \in A_1$, we have

$$L(-g_1(a_1)) \le S_1(-g_1(a_1)) \le S(-g_1(a_1) + g_1(a_1)) - f_1(a_1) = -f_1(a_1)$$

and by using the linearity of L, we obtain (2.4). ∎

Remark 2.2. We can easily extend Theorem 2.1 for any p sets $A_1, \dots, A_p$ and $2p$ maps $g_i : A_i \to X$, $f_i : A_i \to F$, $i \in \{1, \dots, p\}$, instead of A_1, A_2 and g_1, g_2, f_1, f_2.

Corollary 2.3 (The vectorial form of the Mazur–Orlicz theorem [10]). *Let X be a vector space, F a Dedekind complete ordered vector space and $S : X \to F$ a sublinear operator. Let A be an arbitrary nonempty set, and $f : A \to F$ and $g : A \to X$ two maps. The following conditions are equivalent*:

(i) *There exists a linear operator $L : E \longrightarrow F$ with the properties*

a) *$L \le S$ on X, and* b) *$f \le L \circ g$ on A.*

(ii) *The inequality*

$$\sum_{i=1}^{n} \lambda_i f(a_i) \le S\Big(\sum_{i=1}^{n} \lambda_i g(a_i)\Big)$$

holds for all finite subsets $\{a_1, \ldots, a_n\} \subset A$ *and* $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$.

Proof. Put in Theorem 1.2, $A_1 = A$, $A_2 = \{0\} \subset X$, $g_1 = g$, $f_1 = f$, $g_2 = 0$, $f_2 = 0$. ∎

The following result is the version of Theorem 2.1 for ordered vector spaces.

THEOREM 2.4. *Let E be an ordered vector space, F a Dedekind complete ordered vector space, and K_1, K_2 two nonempty convex sets, and $S : E \to F$ a monotone sublinear operator. For each $i \in \{1, 2\}$, let $P_i : K_i \to E$ be a convex operator and $Q_i : K_i \to F$ a concave operator. Then, the following conditions are equivalent*:

(i) *There exists a positive linear operator $L : E \to F$ such that*

a) $L \le S$ *on* E, *and*

b) $Q_1 \le L \circ P_1$ *on* K_1 *and* $Q_2 \le L \circ P_2$ *on* K_2.

(ii) *The inequality*

$$\lambda Q_1(a_1) + \mu Q_2(a_2) \le S(\lambda P_1(a_1) + \mu P_2(a_2)) \tag{2.5}$$

holds for all $a_1 \in K_1$, $a_2 \in K_2$, $\lambda \ge 0$ *and* $\mu \ge 0$.

Proof. First we remark that inequality (2.5) is equivalent to inequality (2.1) from Theorem 2.1. Indeed, it is obvious that (2.1) implies (2.5), if we put in (2.1) $m = n = 2$, and $A_i = K_i$, $g_i = P_i$ and $f_i = Q_i$, $i \in \{1, 2\}$. To prove the converse, if $a_{11}, \ldots, a_{1n} \in K_1$, $a_{21}, \ldots, a_{2n} \in K_2$, $\lambda_1 \ge 0$, $\ldots$, $\lambda_n \ge 0$, $\mu_1 \ge 0$, $\ldots$, $\mu_n \ge 0$, we can suppose that $\lambda := \lambda_1 + \ldots + \lambda_n > 0$ and $\mu := \mu_1 + \ldots + \mu_n > 0$. Let $\alpha_i = \frac{\lambda_i}{\lambda}$ and $\beta_i = \frac{\mu_i}{\mu}$, for each $i \in \{1, \ldots, n\}$. It follows that $\alpha_1 + \ldots + \alpha_n = 1$, $\beta_1 + \ldots + \beta_n = 1$ and hence, using that P_1, P_2 are convex operators and Q_1, Q_2 are concave operators, we obtain:

$$P_1\Big(\sum_{i=1}^{n} \alpha_i a_{1i}\Big) \le \sum_{i=1}^{n} \alpha_i P_1(a_{1i}), \quad P_2\Big(\sum_{i=1}^{n} \beta_i a_{2i}\Big) \le \sum_{i=1}^{n} \beta_i P_2(a_{2i}),$$

and

$$Q_1\Big(\sum_{i=1}^{n} \alpha_i a_{1i}\Big) \ge \sum_{i=1}^{n} \alpha_i Q_1(a_{1i}), \quad Q_2\Big(\sum_{i=1}^{n} \beta_i a_{2i}\Big) \ge \sum_{i=1}^{n} \beta_i Q_2(a_{2i}).$$

Then, using (2.5) and the condition that S is a monotone operator we have:

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i Q_1(a_{1i}) + \sum_{i=1}^{n} \mu_i Q_2(a_{2i}) &= \lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda}\, Q_1(a_{1i}) + \mu \sum_{i=1}^{n} \frac{\mu_i}{\mu}\, Q_2(a_{2i}) \\
= \lambda \sum_{i=1}^{n} \alpha_i Q_1(a_{1i}) + \mu \sum_{i=1}^{n} \beta_i Q_2(a_{2i}) &\le \lambda Q_1\Big(\sum_{i=1}^{n} \alpha_i a_{1i}\Big) + \mu Q_2\Big(\sum_{i=1}^{n} \beta_i a_{2i}\Big) \\
&\le S\Big(\lambda P_1\Big(\sum_{i=1}^{n} \alpha_i a_{1i}\Big) + \mu P_2\Big(\sum_{i=1}^{n} \beta_i a_{2i}\Big)\Big) \\
&\le S\Big(\lambda\Big(\sum_{i=1}^{n} \alpha_i P_1(a_{1i})\Big) + \mu\Big(\sum_{i=1}^{n} \beta_i P_2(a_{2i})\Big)\Big).
\end{aligned}$$

Moreover, to prove (ii) $\Rightarrow$ (i), we use that any linear operator $L : E \to F$ dominated by a monotone and positive homogeneous operator $S : E \to F$ is a positive operator (see, for example [4], Remark 2.3). ∎

Corollary 2.5 (Mazur–Orlicz theorem for ordered vector spaces, see [4], Theorem 2.4). *Let E be an ordered vector space, F a Dedekind complete ordered vector space and $S : E \to F$ a monotone sublinear operator. Let K be a nonempty convex set, $P : K \to E$ a convex operator, and $Q : K \to F$ a concave operator. Then the following conditions are equivalent*:

(i) *There exists a positive linear operator $L : E \longrightarrow F$ with the properties*:

a) $L \leq S$ *on* E, *and* b) $Q \leq L \circ P$ *on* K.

(ii) *The inequality $Q \leq S \circ P$ holds on K.*

Now we remember two vectorial forms of the Hahn–Banach extension theorem, for cases in which the domain space is an arbitrary vector space, respectively an ordered vector space.

Theorem 2.6. *Let X be a vector space, F a Dedekind complete ordered vector space, and $S : X \to F$ a sublinear operator. Let G be a vector subspace of X and $T : G \to F$ a linear operator. The following conditions are equivalent*:

(i) *There exists a linear operator $L : X \longrightarrow F$ with the properties*

a) $L \leq S$ *on* X, *and* b) $L = T$ *on* G.

(ii) $T \leq S$ *on* G.

Theorem 2.7. *Let E be an ordered vector space, F a Dedekind complete ordered vector space and $S : E \to F$ a monotone sublinear operator. Let G be a vector subspace of E and $T : G \to F$ a positive linear operator. Then, the following are equivalent*:

(i) *There exists a positive linear operator $L : E \to F$ such that*

a) $L \leq S$ *on* E, *and* b) $L = T$ *on* G.

(ii) $T \leq S$ *on* G.

Remark that Corollary 2.3 (the Mazur–Orlicz theorem) is a generalization of Theorem 2.6 (the vectorial form of the Hahn–Banach extension theorem).

The following common extension result will be formulated in the line of the Hahn–Banach extension theorem with a vector space as the domain space (see Theorem 2.6).

Theorem 2.8. *Let X be a vector space, F a Dedekind complete ordered vector space, and $S : X \to F$ a sublinear operator. Let G_1 and G_2 be two vector subspaces of X and $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ two linear operators. The following conditions are equivalent*:

(i) *There exists a linear operator $L : X \to F$ with the properties*:

a) $L \leq S$ *on* X, *and*

b) $L = T_1$ *on* G_1, $L = T_2$ *on* G_2.

(ii) *The following inequality holds for all* $v_1 \in G_1$ *and* $v_2 \in G_2$,

$$T_1(v_1) + T_2(v_2) \leq S(v_1 + v_2). \tag{2.6}$$

Proof. Obviously, (i) implies (ii). To prove the converse we can apply Theorem 2.1 for $A_i = G_i$, $f_i = T_i$ and $g_i =$ the inclusion of G_i in X, for each $i \in \{1, 2\}$. We obtain a linear operator $L : X \to F$ such that $L \leq S$ on X and $T_i \leq L$ on G_i, for $i \in \{1, 2\}$. Actually, we have even $T_i = L$ on G_i, that is L is an extension of T_i, because $T_i \leq L$ on G_i, and T_i and L are linear. (Indeed, if, for example $v_1 \in G_1$, we have: $T_1(-v_1) \leq L(-v_1)$ and hence $-T_1(v_1) \leq -L(v_1)$. It follows that $L(v_1) \leq T_1(v_1) \leq L(v_1)$). Therefore L is a common extension of T_1, T_2. ■

Note that inequality (2.6) implies that

1) $T_1 \leq S$ on G_1 and $T_2 \leq S$ on G_2.
2) $T_1 = T_2$ on $G_1 \cap G_2$.

Indeed, to prove 2), let $v \in G_1 \cap G_2$ and put in (2.6) $v_1 = v$ and $v_2 = -v$. Then $T_1(v) + T_2(-v) \leq S(0) = 0$ and hence $T_1(v) \leq T_2(v)$; similarly, $T_2(v) \leq T_1(v)$ and therefore $T_1(v) = T_2(v)$.

The following common extension result will be formulated in the line of the Hahn–Banach extension theorem with an ordered vector space as the domain space (see Theorem 2.7).

THEOREM 2.9. *Let E be an ordered vector space, F a Dedekind complete ordered vector space and $S : E \to F$ a monotone sublinear operator. Let G_1 and G_2 be two vector subspaces of X and $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ two positive linear operators. Then, the following are equivalent*:

(i) *There exists a positive linear operator* $L : E \to F$ *such that*

a) $L \leq S$ *on* E,

b) $L = T_1$ *on* G_1, $L = T_2$ *on* G_2.

(ii) $T_1(v_1) + T_2(v_2) \leq S(v_1 + v_2)$, *for all* $v_1 \in G_1$ *and* $v_2 \in G_2$.

Proof. We apply Theorem 2.4. ■

The following result is a consequence of Theorem 2.9.

COROLLARY 2.10. *Let E, F, G_1, G_2 and T_1, T_2 be like in the previous theorem. Then, the following are equivalent*:

(i) *There exists a positive linear operator* $L : E \to F$ *such that* $L = T_1$ *on* G_1 *and* $L = T_2$ *on* G_2.

(ii) *There exists a monotone sublinear operator* $S : E \to F$ *such that*

$$T_1(v_1) + T_2(v_2) \leq S(v_1 + v_2)$$

for all $v_1 \in G_1$ *and* $v_2 \in G_2$.

In the following result, which is a consequence of Corollary 2.10, the condition that the sublinear operator S is monotone is dropped.

THEOREM 2.11. *Let E be an ordered vector space, F a Dedekind complete ordered vector space and G_1, G_2 two vector subspaces of E. Let also $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ be two positive linear operators. Then, the following are equivalent*:

(i) *There exists a positive linear operator $L : E \to F$ such that $L = T_1$ on G_1 and $L = T_2$ on G_2.*

(ii) *There exists $S : E \to F$ a sublinear operator such that*

$$v_1 + v_2 \leq v \Rightarrow T_1(v_1) + T_2(v_2) \leq S(v) \tag{2.7}$$

where $v_1 \in G_1$, $v_2 \in G_2$ and $v \in E$.

Proof. (i) $\Rightarrow$ (ii). We put $S = L$ and use that L is a positive linear common extension of T_1 and T_2. We have $v_1 + v_2 \leq v \Rightarrow L(v_1) + L(v_2) \leq L(v) \Rightarrow T_1(v_1) + T_2(v_2) \leq S(v)$.

(ii) $\Rightarrow$ (i). Conversely, let $S : E \to F$ be a sublinear operator satisfying (2.7). We apply the *technique of the auxiliary sublinear operator*, defining $S_1 : E \to F$ by the formula

$$S_1(v) = \inf\{S(w) \mid w \in E, w \geq v\}, \text{ for each } v \in E.$$

This infimum exists in F, because the set $\{S(w) \mid w \in E,\ w \geq v\}$ is minorized in F by $-S(-v)$. Indeed, we have for $v_1 = v_2 = 0$, and $u \geq 0$: $0 = T_1(0) + T_2(0) \leq S(u) = S(v+u-v) \leq S(v+u)+S(-v)$, hence $-S(-v) \leq S(v+u)$, for all $u \geq 0$, or, equivalently, $-S(-v) \leq S(w)$, for all $w \in E$, $w \geq v$.

Obviously $S_1 \leq S$ on E. In addition the operator S_1 has the following properties:

1) S_1 is sublinear,
2) S_1 is monotone,
3) $T_1(v_1) + T_2(v_2) \leq S_1(v_1 + v_2)$ for all $v_1 \in G_1$, $v_2 \in G_2$.

Now, we can apply Corollary 2.10, (ii) $\Rightarrow$ (i), for S_1 instead of S, obtaining a positive common linear extension of T_1 and T_2. ∎

REMARK 2.12. Many results of this paper, including Theorem 2.9, can easily be generalized in the line of the Maharam theorem (1972).

THEOREM 2.13 (Maharam theorem). *Let E be a vector lattice with an order unit $e \in E_+$ and $(G_\delta)_{\delta\in\Delta}$ a family of subspaces of E such that $e \in \operatorname{span}\big(\bigcup_{\delta\in\Delta} G_\delta\big)$. Let also F be a Dedekind complete ordered vector space and let $\{T_\delta : G_\delta \to F \mid \delta \in \Delta\}$ be a family of positive linear operators. Then, the following conditions are equivalent*:

(i) *There exists $T : E \to F$ a positive linear extension of the family $(T_\delta)_{\delta\in\Delta}$ (that is, $T(x) = T_\delta(x)$ for all $\delta \in \Delta$ and $x \in G_\delta$).*

(ii) *The inequality $0 \leq T_\delta(v_\delta)$ holds for every family $(v_\delta)_{\delta\in\Delta} \in \Phi((G_\delta))$, satisfying $0 \leq \sum_{\delta\in\Delta} v_\delta$, where $\Phi((G_\delta)_{\delta\in\Delta})$ is the collection of all families $\{v_\delta \in G_\delta \mid \delta \in \Delta\}$ such that $v_\delta \neq 0$ for at most finitely many $\delta \in \Delta$.*

This theorem was originally proved by D. Maharam in [9] (see also [13], Theorem 6.3).

The following result (see [4], Theorem 5.4) is an easy generalization of Theorem 2.13, because if the ordered vector space E has an order unit $e > 0$ and $G \subseteq E$ is a vector subspace so that $e \in G$, then G is a majorizing subspace of E.

THEOREM 2.14. *Let E be an ordered vector space and let $(G_\delta)_{\delta\in\Delta}$ be a family of subspaces of E, such that there exists at least one which is majorizing, say G_{δ_0}. Let F be a Dedekind complete ordered vector space and let $\{T_\delta : G_\delta \to F \mid \delta \in \Delta\}$ be a family of positive linear operators. Then the following conditions are equivalent*:

(i) *The family $\{T_\delta : G_\delta \to F \mid \delta \in \Delta\}$ has a positive common linear extension $T : E \to F$.*

(ii) *The implication $\sum_{\delta\in\Delta} v_\delta \geq 0 \Rightarrow \sum_{\delta\in\Delta} T_\delta(v_\delta) \geq 0$ holds for every family $(v_\delta)_{\delta\in\Delta} \in \Phi((G_\delta)_{\delta\in\Delta})$.*

REMARK 2.15. If we generalize Corollary 2.10 in the line of the Maharam theorem, we obtain Theorem 2.14, and hence Theorem 2.13 too, as consequences. To prove this it suffices to prove that Corollary 2.10 implies the version of Theorem 2.14 for $\Delta = \{1, 2\}$. For this aim it is necessary to prove that (ii′) ⇒ (ii) if at least one of the subspaces G_1, G_2, say G_1, is majorizing, where (ii) and (ii′) are the following statements:

(ii) *There exists a monotone sublinear operator S such that*

$$T_1(v_1) + T_2(v_2) \leq S(v_1 + v_2)$$

for all $v_1 \in G_1$ and $v_2 \in G_2$.

(ii′) *If $v_1 + v_2 \leq 0$, then $T_1(v_1) + T_2(v_2) \leq 0$ for all $v_1 \in G_1$ and $v_2 \in G_2$.*

Suppose that (ii′) is valid. Let us define $T : \operatorname{span}(G_1 \cup G_2) \to F$ by the equality

$$T(v_1 + v_2) = T_1(v_1) + T_2(v_2)$$

for all $v_1 \in G_1$ and $v_2 \in G_2$.

The operator T has the following properties: 1) T is well-defined, according to (ii′); 2) T is linear; 3) T is positive.

Because we supposed that G_1 is a majorizing subspace, it follows that the subspace $G = \operatorname{span}(G_1 \cup G_2)$ is majorizing, too. Define $S : E \to F$, $S(x) = \overline{T}(x)$, for all $x \in E$, (that is $S(x) = \inf\{T(z) \mid z \in G,\ z \geq x\}$). It is known that S is a monotone sublinear operator and $T \leq S$ on E. We have: $T_1(v_1) + T_2(v_2) = T(v_1 + v_2) \leq S(v_1 + v_2)$ for all $v_1 \in G_1$ and $v_2 \in G_2$, that is, (ii) is valid.

3. Common positive extensions using an additional set. In the following result we will give a sufficient condition for the existence of a positive linear operator L satisfying the converse inequalities of Theorem 2.1(i) b). This condition is an implication between two inequalities and next we will simplify the form of the left and respectively of the right member of these inequalities. Note that, instead of majorization of L by a sublinear operator S, we will assume the existence of an additional set M and of two maps $h : M \to E$ and $r : M \to F$, obtaining that $L \circ h \leq r$ on M.

THEOREM 3.1. *Let E_0 be an ordered vector space, F a Dedekind complete ordered vector space, and let A_1, A_2 and M be arbitrary nonempty sets. Let also $g_j : A_j \to E_0$, $f_j : A_j \to F$, $j \in \{1, 2\}$ and $h : M \to (E_0)_+$, $r : M \to F$ be arbitrary maps, and $E =$*

$\operatorname{span}(g_1(A_1) \cup g_2(A_2) \cup h(M)) \subseteq E_0$. *Suppose that*

$$\sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) \leq \sum_{i=1}^{n} h(z_i)$$
$$\Rightarrow \sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) \leq \sum_{i=1}^{n} r(z_i), \quad (3.1)$$

where $n \in \mathbb{N}^*$, *and* $a_{1i} \in A_1$, $a_{2i} \in A_2$, $z_i \in M$, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$, *for each* $i \in \{1, \ldots, n\}$.

Then, there exists a positive linear operator $L : E \to F$ *such that*

a) $L \circ g_1 \leq f_1$ *on* A_1, $L \circ g_2 \leq f_2$ *on* A_2,
b) $L \circ h \leq r$ *on* M.

Proof. Step 1. Remark that condition (3.1) is equivalent to the following condition:

$$\sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) \leq \sum_{i=1}^{n} \lambda_i h(z_i)$$
$$\Rightarrow \sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) \leq \sum_{i=1}^{n} \lambda_i r(z_i), \quad (3.2)$$

where $n \in \mathbb{N}^*$, and $a_{1i} \in A_1$, $a_{2i} \in A_2$, $z_i \in M$, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$, $\lambda_i \geq 0$, for each $i \in \{1, \ldots, n\}$.

Obviously, (3.2) $\Rightarrow$ (3.1). To prove that (3.1) $\Rightarrow$ (3.2), we analyze three cases:

Case 1. Suppose that $\lambda_1 \in \mathbb{N}^*, \ldots, \lambda_n \in \mathbb{N}^*$. We define the elements $(y_i)_{i=1}^{\lambda_1+\ldots+\lambda_n} \in M$ as follows:

$$y_1 = \ldots = y_{\lambda_1} = z_1$$
$$y_{\lambda_1+1} = \ldots = y_{\lambda_1+\lambda_2} = z_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_{\lambda_1+\ldots+\lambda_{n-1}+1} = \ldots = y_{\lambda_1+\ldots+\lambda_n} = z_n.$$

We set $m = \lambda_1 + \ldots + \lambda_n \in \mathbb{N}^* \Rightarrow m \geq n$ because $\lambda_i \geq 1$ for all $i \in \{1, \ldots, n\}$. Now, we have:

$$\sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) \leq \sum_{i=1}^{m} h(y_i)$$
$$\overset{(3.1)}{\Rightarrow} \sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) \leq \sum_{i=1}^{m} r(y_i) = \sum_{i=1}^{n} \lambda_i r(z_i).$$

Case 2. Assume that $\lambda_i \in \mathbb{Q}_+$, for all $i \in \{1, \ldots, n\}$. Let us suppose that $\lambda_i = \frac{p_i}{q_i}$, where $p_i \in \mathbb{N}$ and $q_i \in \mathbb{N}^*$ for all $i \in \{1, \ldots, n\}$. Denote by q the least common multiple of $q_1, \ldots, q_n$. It follows that for all $i \in \{1, \ldots, n\}$ there exist $k_i \in \mathbb{N}$ such that $q = k_i q_i$. If

$$\sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) \leq \sum_{i=1}^{n} \frac{p_i}{q_i} h(z_i) = \sum_{i=1}^{n} \frac{p_i k_i}{q} h(z_i),$$

then

$$\sum_{i=1}^{n} q\alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} q\beta_i g_2(a_{2i}) \le \sum_{i=1}^{n} p_i k_i h(z_i)$$

$$\overset{\text{Case 1}}{\Rightarrow} \sum_{i=1}^{n} q\alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} q\beta_i f_2(a_{2i}) \le \sum_{i=1}^{n} p_i k_i r(z_i)$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) \le \sum_{i=1}^{n} \frac{p_i k_i}{q} r(z_i) = \sum_{i=1}^{n} \frac{p_i}{q_i} r(z_i) = \sum_{i=1}^{n} \lambda_i r(z_i).$$

Case 3. Suppose that $\lambda_i \in \mathbb{R}_+$, for all $i \in \{1, \dots, n\}$. We apply *Case 2* and use that F is Archimedean.

Step 2. We will prove that there exists a monotone sublinear operator $S : E \to F$ such that

$$S \circ g_1 \le f_1 \text{ on } A_1,\ S \circ g_2 \le f_2 \text{ on } A_2, \text{ and } S \circ h \le r \text{ on } M.$$

Define $S : E \to F$ by the formula

$$S(x) = \inf\Big\{\sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) + \sum_{i=1}^{n} \lambda_i r(z_i) \;\Big|$$
$$x \le \sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) + \sum_{i=1}^{n} \lambda_i h(z_i),\ n \in \mathbb{N}^*, \text{ and } a_{1i} \in A_1,\ a_{2i} \in A_2,$$
$$z_i \in M,\ \alpha_i \in \mathbb{R},\ \beta_i \in \mathbb{R},\ \lambda_i \ge 0 \text{ for all } i \in \{1, \dots, n\}\Big\}$$

for each $x \in E$. (Remember that $E = \operatorname{span}(g_1(A_1) \cup g_2(A_2) \cup h(M)) \subseteq E_0$.)

First we will prove that the above infimum exists in F. Let

$$x = \sum_{j=1}^{m} \alpha'_j g_1(a'_{1j}) + \sum_{j=1}^{m} \beta'_j g_2(a'_{2j}) + \sum_{j=1}^{m} \lambda'_j h(z'_j)$$
$$\le \sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) + \sum_{i=1}^{n} \lambda_i h(z_i),$$

where $a'_{1j} \in A_1$, $a'_{2j} \in A_2$, $z'_j \in M$, $\alpha'_j \in \mathbb{R}$, $\beta'_j \in \mathbb{R}$, $\lambda'_j \in \mathbb{R}$, $j \in \{1, \dots, m\}$ are fixed and $a_{1i} \in A_1$, $a_{2i} \in A_2$, $z_i \in M$, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$, $\lambda_i \ge 0$, $i \in \{1, \dots, n\}$ are arbitrary. Obviously, we can suppose that $m = n$. Then we can write:

$$\sum_{j=1}^{n} \alpha'_j g_1(a'_{1j}) + \sum_{j=1}^{n} \beta'_j g_2(a'_{2j}) - \sum_{j=1}^{n} \alpha_j g_1(a_{1j}) - \sum_{j=1}^{n} \beta_j g_2(a_{2j})$$
$$\le \sum_{j=1}^{n} \lambda_j h(z_j) - \sum_{j=1}^{n} \lambda'_j h(z'_j).$$

Since (3.2) holds, $-\lambda'_j \le |\lambda'_j|$ for each $j \in \{1, \dots, n\}$ and h takes positive values, we obtain the inequality:

$$\sum_{j=1}^{n} \alpha'_j f_1(a'_{1j}) + \sum_{j=1}^{n} \beta'_j f_2(a'_{2j}) - \sum_{j=1}^{n} \alpha_j f_1(a_{1j}) - \sum_{j=1}^{n} \beta_j f_2(a_{2j}) \le \sum_{j=1}^{n} \lambda_j r(z_j) + \sum_{j=1}^{n} |\lambda'_j| r(z'_j),$$

and hence,

$$\sum_{j=1}^{n}\alpha'_j f_1(a'_{1j})+\sum_{j=1}^{n}\beta'_j f_2(a'_{2j})-\sum_{j=1}^{n}|\lambda'_j| r(z'_j)\leq \sum_{j=1}^{n}\alpha_j f_1(a_{1j})+\sum_{j=1}^{n}\beta_j f_2(a_{2j})+\sum_{j=1}^{n}\lambda_j r(z_j).$$

So, the set appearing in the definition of $S(x)$ is minorized in F and hence there exists its infimum (denoted by $S(x)$).

It is straightforward to prove that S is sublinear and monotone. Moreover we have:

1) $S\circ g_j\leq f_j$ on A_j, for each $j\in\{1,2\}$. (Indeed, for example, for $j=1$ and $a_1\in A_1$, we have $g_1(a_1)=1\cdot g_1(a_1)+0\cdot g_2(a_2)+0\cdot h(z)$, with some $a_2\in A_2$ and $z\in M$, it follows that $S(g_1(a_1))\leq 1\cdot f_1(a_1)+0\cdot f_2(a_2)+0\cdot r(z)$.)

2) $S\circ h\leq r$ on M. (Indeed, if $z\in M$, then for some $a_1\in A_1$ and $a_2\in A_2$, we have $h(z)=0\cdot g_1(a_1)+0\cdot g_2(a_2)+1\cdot h(z)$ and hence $S(h(z))\leq 0\cdot f_1(a_1)+0\cdot f_2(a_2)+1\cdot r(z)=r(z)$.)

Step 3. Now we will prove the existence of a positive linear operator $L:E\to F$ such that

a) $L\circ g_j\leq f_j$ on A_j for each $j\in\{1,2\}$, and
b) $L\circ h\leq r$ on M.

We apply *Step 2* and the existence form of the Hahn–Banach theorem. Also, we apply the remark mentioned at the end of the proof of Theorem 2.4. ∎

Now we will simplify successively the form of the left members in the inequalities which appear in (3.1).

THEOREM 3.2. *Let E_0 be an ordered vector space, F a Dedekind complete ordered vector space, and let G_1, G_2 be two ordered vector spaces and M a nonempty set. Let also $h: M\to (E_0)_+$ and $r: M\to F$ be two maps, $P_j: G_j\to E_0$ linear operators and $T_j: G_j\to F$ positive linear operators, where $j\in\{1,2\}$. Let $E=\operatorname{span}(P_1(G_1)\cup P_2(G_2)\cup h(M))\subseteq E_0$. Then, the following conditions are equivalent*:

(i) *There exists a positive linear operator $L:E\to F$ such that*

  a) *$L\circ P_j=T_j$ on G_j for $j\in\{1,2\}$, and*

  b) *$L\circ h\leq r$ on M.*

(ii) *The following implication holds*

$$P_1(v_1)+P_2(v_2)\leq\sum_{i=1}^{n}h(z_i)\Rightarrow T_1(v_1)+T_2(v_2)\leq\sum_{i=1}^{n}r(z_i), \tag{3.3}$$

where $n\in\mathbb{N}^*$, $v_1\in G_1$, $v_2\in G_2$ and $z_i\in M$, for all $i\in\{1,\dots,n\}$.

Proof. (i) ⇒ (ii) is immediate. Indeed, if $P_1(v_1)+P_2(v_2)\leq\sum\limits_{i=1}^{n}h(z_i)$, then, because L is a positive linear operator, we have

$$L(P_1(v_1)+P_2(v_2))\leq\sum_{i=1}^{n}L(h(z_i))\overset{\text{(i)}}{\Rightarrow} T_1(v_1)+T_2(v_2)\leq\sum_{i=1}^{n}r(z_i).$$

(ii) ⇒ (i) is a consequence of Theorem 3.1. Indeed, let us prove that, for example, $L\circ P_1=T_1$ on G_1. If $v_1\in G_1$, then, because $L\circ P_1\leq T_1$ on G_1 we have $L(P_1(-v_1))\leq$

$T_1(-v_1)$ and since L, P_1 and T_1 are linear, it follows that $-L(P_1(v_1)) \leq -T_1(v_1)$, that is $L \circ P_1 \geq T_1$ on G_1. ■

We remark that the form of the left-hand side in the inequalities appearing in (3.3) can be still simplified, if G_1 and G_2 are two vector subspaces of the ordered vector space E_0.

THEOREM 3.3. *Let E_0 be an ordered vector space, F a Dedekind complete ordered vector space, and let G_1, G_2 be two ordered vector subspaces of E_0 and M an arbitrary set. Let also $h : M \to (E_0)_+$, and $r : M \to F$ be two maps, and $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ two positive linear operators. Let $E = \operatorname{span}(G_1 \cup G_2 \cup h(M)) \subseteq E_0$. Then, the following conditions are equivalent*:

(i) *There exists a common positive linear extension L of T_1, T_2 to the space E (that is $L = T_j$ on G_j, for $j \in \{1,2\}$) such that $L \circ h \leq r$ on M.*

(ii) *The following implication holds*

$$v_1 + v_2 \leq \sum_{i=1}^{n} h(z_i) \Rightarrow T_1(v_1) + T_2(v_2) \leq \sum_{i=1}^{n} r(z_i), \tag{3.4}$$

for $n \in \mathbb{N}^*$, $v_1 \in G_1$, $v_2 \in G_2$ and $z_i \in M$, for each $i \in \{1, \dots, n\}$.

Proof. Apply Theorem 3.2 for $P_j = i_j$, the inclusion of G_j in E_0 for $j \in \{1,2\}$. ■

A new step to simplify the right members of the inequalities that arise in (3.4) is to choose M an arbitrary subset of $(E_0)_+$ and to take $h = i$, the inclusion of M in E_0.

THEOREM 3.4. *Let E_0 be an ordered vector space, F a Dedekind complete ordered vector space, and let G_1, G_2 be two ordered vector subspaces of E_0 and M an arbitrary subset of $(E_0)_+$. Let also $r : M \to F$ be a map, and $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ two positive linear operators. Denote by E the vector space $\operatorname{span}(G_1 \cup G_2 \cup M) \subseteq E_0$. Then the following statements are equivalent*:

(i) *There exists a common positive linear extension L of T_1, T_2 to the space E such that $L \leq r$ on M.*

(ii) *The following implication holds*

$$v_1 + v_2 \leq \sum_{i=1}^{n} z_i \Rightarrow T_1(v_1) + T_2(v_2) \leq \sum_{i=1}^{n} r(z_i), \tag{3.5}$$

where $n \in \mathbb{N}^*$, $v_1 \in G_1$, $v_2 \in G_2$ and $z_i \in M$, for each $i \in \{1, \dots, n\}$.

REMARK 3.5.

1) Note that this theorem generalizes a result formulated without proof in [5], and applied in [6]; for the proof, see Theorem 1, p. 63 in [7]. Also, Theorem 3.4 generalizes Theorem 6.4 in [4]. This result is the consequence of our Theorem 3.4, obtained taking $G_2 = \{0\}$ and $T_2 = 0$ (the null operator on G_2).

2) If, additionally, the cone $(E_0)_+$ in Theorem 3.4 is generating and $M = (E_0)_+$, then $E = E_0$ and thus Theorem 3.4 gives the existence of a common extension of T_1, T_2 to the whole E_0.

3) We have also $E = E_0$ if E_0 has a positive algebraic basis, chosen instead of M.

Note that we can also simplify the form of the right-hand side in the inequalities appearing in condition (ii) in all previous theorems of this section. It suffices to choose as M a nonempty set closed under addition (in an arbitrary ordered vector space E_1 for Theorem 3.1 and Theorem 3.2) and to assume that the maps $-h$ and r are subadditive. So, for example (3.1) becomes

$$\sum_{i=1}^{n} \alpha_i g_1(a_{1i}) + \sum_{i=1}^{n} \beta_i g_2(a_{2i}) \leq h(z) \Rightarrow \sum_{i=1}^{n} \alpha_i f_1(a_{1i}) + \sum_{i=1}^{n} \beta_i f_2(a_{2i}) \leq r(z),$$

for $n \in \mathbb{N}^*$, $z \in M$, and $a_{1i} \in A_1$, $a_{2i} \in A_2$, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$, for each $i \in \{1, \ldots, n\}$. Also, (3.5) becomes: $v_1 + v_2 \leq z \Rightarrow T_1(v_1) + T_2(v_2) \leq r(z)$, where $v_1 \in G_1$, $v_2 \in G_2$ and $z \in M$.

Remark 3.6. As consequences of the results included in this section, we obtain respectively Theorems 6.1, 6.2, 6.3 and 6.4 from [4].

4. Other common positive linear extensions using an additional set. The following common extension result is in the line of a result of R. Cristescu, concerning the extension of a positive linear operator. This result by R. Cristescu generalizes a result obtained by Z. Lipecki (see Corollary 4.3 below) for the extension of a positive linear operator defined on a majorizing vector subspace of an ordered vector space. Note that in the following theorem, F, the range of the operators is an ordered vector space, not necessary Dedekind complete.

Theorem 4.1. *Let E_0 and F be two ordered vector spaces, G_1, and G_2 be two vector subspaces of E_0 and $M \subseteq E_0$ a nonempty set. Let also $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ be positive linear operators and $P : E_0 \to F$ a monotone sublinear operator such that $P = T_1$ on G_1 and $P = T_2$ on G_2. Let $E = \operatorname{span}(G_1 \cup G_2 \cup M)$ and suppose that*

$$P\Big(\sum_{i=1}^{n} z_i\Big) = \sum_{i=1}^{n} P(z_i) \tag{4.1}$$

where $n \in \mathbb{N}^*$ and $z_1, \ldots, z_n \in M$.

Then, there exists a positive linear operator $L : E \to F$ such that

a) *$L = T_1$ on G_1, $L = T_2$ on G_2, and*
b) *$L = P$ on M.*

Proof. Define $L : E \to F$ by the following equality:

$$L\Big(v_1 + v_2 + \sum_{i=1}^{n} \alpha_i z_i\Big) = T_1(v_1) + T_2(v_2) + \sum_{i=1}^{n} \alpha_i P(z_i),$$

where $n \in \mathbb{N}^*$, $v_1 \in G_1$, $v_2 \in G_2$ and $z_i \in M$, $\alpha_i \in \mathbb{R}$, for all $i \in \{1, \ldots, n\}$. We intend to prove that L is well-defined. First, we will prove that (4.1) $\Rightarrow$ (4.2), where (4.2) is the following statement:

$$P\Big(\sum_{i=1}^{n} \lambda_i z_i\Big) = \sum_{i=1}^{n} \lambda_i P(z_i) \tag{4.2}$$

with $n \in \mathbb{N}^*$, $z_i \in M$, $\lambda_i \in \mathbb{R}_+$, for all $i \in \{1, \ldots, n\}$ (actually the statements (4.1) and (4.2) are equivalent). Of course, it suffices to prove the inequality "$\geq$" in (4.2). Fix

$\lambda \in \mathbb{R}_+$ with $\lambda_i \leq \lambda$, for all $i \in \{1, \dots, n\}$. Then, the subadditivity of P, the property of P to be positive homogeneous together with our assumption (4.1) yield:

$$P\Big(\sum_{i=1}^{n} \lambda_i z_i\Big) \geq P\Big(\lambda \sum_{i=1}^{n} z_i\Big) - P\Big(\sum_{i=1}^{n} (\lambda - \lambda_i) z_i\Big)$$
$$\geq \lambda \sum_{i=1}^{n} P(z_i) - \sum_{i=1}^{n} (\lambda - \lambda_i) P(z_i) = \sum_{i=1}^{n} \lambda_i P(z_i).$$

Next we show that

$$v_1 + v_2 + \sum_{i=1}^{n} \lambda_i z_i \geq 0 \Longrightarrow T_1(v_1) + T_2(v_2) + \sum_{i=1}^{n} \lambda_i P(z_i) \geq 0 \tag{4.3}$$

if $v_1 \in G_1$, $v_2 \in G_2$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, $z_1, \dots, z_n \in M$. Indeed, put $I = \{1 \leq i \leq n \mid \lambda_i \geq 0\}$, and $J = \{1 \leq j \leq n \mid \lambda_j < 0\}$. We have

$$v_1 + v_2 + \sum_{i \in I} \lambda_i z_i \geq \sum_{j \in J} (-\lambda_j) z_j,$$

and hence, by the monotonicity of P, it follows that

$$P\Big(v_1 + v_2 + \sum_{i \in I} \lambda_i z_i\Big) \geq P\Big(\sum_{j \in J} (-\lambda_j) z_j\Big).$$

Now, we will use again the subadditivity of P and the equalities $P = T_1$ on G_1, $P = T_2$ on G_2, obtaining

$$T_1(v_1) + T_2(v_2) + P\Big(\sum_{i \in I} \lambda_i z_i\Big) \geq P\Big(\sum_{j \in J} (-\lambda_j) z_j\Big).$$

According to (4.2) we have

$$T_1(v_1) + T_2(v_2) + \sum_{i \in I} \lambda_i P(z_i) \geq \sum_{j \in J} (-\lambda_j) P(z_j),$$

and hence

$$T_1(v_1) + T_2(v_2) + \sum_{i=1}^{n} \lambda_i P(z_i) \geq 0.$$

Now we will prove that L is *well-defined*. Let

$$v_1' + v_2' + \sum_{i=1}^{m} \alpha_i z_i' = v_1'' + v_2'' + \sum_{j=1}^{n} \beta_j z_j'',$$

where v_1', $v_1'' \in G_1$, v_2', $v_2'' \in G_2$, m, $n \in \mathbb{N}^*$, $z_i' \in M$, $\alpha_i \in \mathbb{R}$ for all $i \in \{1, \dots, m\}$, and $z_j'' \in M$, $\beta_j \in \mathbb{R}$ for all $j \in \{1, \dots, n\}$. Then

$$(v_1' - v_1'') + (v_2' - v_2'') + \sum_{i=1}^{m} \alpha_i z_i' + \sum_{j=1}^{n} (-\beta_j) z_j'' = 0,$$

so, according to (4.3),

$$T_1(v_1' - v_1'') + T_2(v_2' - v_2'') + \sum_{i=1}^{m} \alpha_i P(z_i') + \sum_{j=1}^{n} (-\beta_j) P(z_j'') \geq 0.$$

It follows that

$$T_1(v_1') + T_2(v_2') + \sum_{i=1}^{m} \alpha_i P(z_i') = T_1(v_1'') + T_2(v_2'') + \sum_{j=1}^{n} \beta_j P(z_j'')$$
$$\Rightarrow L\Big(v_1' + v_2' + \sum_{i=1}^{m} \alpha_i z_i'\Big) = L\Big(v_1'' + v_2'' + \sum_{j=1}^{n} \beta_j z_j''\Big),$$

that is L is well-defined. It is straightforward to prove that L is a linear operator. By (4.3) it follows that L is positive, too.

Clearly, L extends T_1 and T_2. (Indeed, for example, taking $v_1 \in G_1$, $v_2 = 0 \in G_2$ and $z \in M$ we can write $v_1 = v_1 + 0 + 0 \cdot z$ and therefore $L(v_1) = T(v_1) + T_2(0) + 0 \cdot P(z)$, that is $L = T_1$ on G_1.) Also, obviously, $L = P$ on M. ■

Remark 4.2. The conditions of Theorem 4.1 determine L uniquely. Suppose by contradiction that there exists $L_1 : \operatorname{span}(G_1 \cup G_2 \cup M) \to F$ such that: a) L_1 is positive and linear; b) $L_1 = T_1$ on G_1, $L_1 = T_2$ on G_2; c) $L_1 = P$ on M. Then we have

$$L_1\Big(v_1 + v_2 + \sum_{i=1}^{n} \alpha_i z_i\Big) = L_1(v_1) + L_1(v_2) + \sum_{i=1}^{n} \alpha_i L_1(z_i)$$
$$= T_1(v_1) + T_2(v_2) + \sum_{i=1}^{n} \alpha_i P(z_i) = L\Big(v_1 + v_2 + \sum_{i=1}^{n} \alpha_i z_i\Big),$$

and so $L_1 = L$.

Taking in Theorem 4.1 $G_1 = G$, $T_1 = T$ and $G_2 = \{0\} \subset E_0$, $T_2 : G_2 \to F$, $T_2(0) = 0$, and $E = \operatorname{span}(G \cup M)$, we obtain a result of R. Cristescu (see [3]). This result generalizes a theorem of Z. Lipecki (see [8]). Actually this Lipecki's result is a consequence of our Theorem 4.1. Remember that a vector subspace G of an ordered vector space E_0 is called a *majorizing* subspace if for each $x \in E_0$, there exists $v \in G$ such that $x \leq v$ (or, equivalently, there exists $u \in G$ such that $u \leq x$).

Also, if G is a majorizing vector subspace of E_0, F a Dedekind complete ordered vector space, and $T : G \to F$ is a positive linear operator, the operator $\overline{T} : E \to F$ (well-) defined by $\overline{T}(x) = \inf\{T(v) \mid v \in G,\ v \geq x\}$, $x \in E_0$ is *monotone* and *sublinear*. Also $T = \overline{T}$ on G, and if $L : E_0 \to F$ is a positive linear operator which extends T, then $L \leq \overline{T}$ on E_0.

Corollary 4.3 ([8]). *Let E_0 be an ordered vector space, F a Dedekind complete ordered vector space, G a majorizing vector subspace of E_0, $M \subseteq E_0$ a nonempty set and $T : G \to F$ a positive linear operator. Then, the following are equivalent*:

(i) *T extends to a* (*unique*) *positive linear operator $L : E \to F$ such that $L = \overline{T}$ on M*;
(ii) $\overline{T}\big(\sum_{i=1}^{n} z_i\big) = \sum_{i=1}^{n} \overline{T}(z_i)$, *where $n \in \mathbb{N}^*$, and $z_1, \ldots, z_n \in M$.*

Proof. (ii) $\Rightarrow$ (i) follows from Theorem 4.1.

Conversely, if $L : E \to F$ is a positive linear extension of T such that $L = \overline{T}$ on M, we have, for $n \in \mathbb{N}^*$, and $z_1, \ldots, z_n \in M$,

$$\sum_{i=1}^{n} \overline{T}(z_i) = \sum_{i=1}^{n} L(z_i) = L\Big(\sum_{i=1}^{n} z_i\Big) \leq \overline{T}\Big(\sum_{i=1}^{n} z_i\Big).$$

Hence, according to the subadditivity of $\overline{T}$, we have $\sum_{i=1}^{n} \overline{T}(z_i) = \overline{T}\big(\sum_{i=1}^{n} z_i\big)$. ∎

The following common positive linear extension result is a consequence of Theorem 4.1, formulated in the line of Corollary 4.3.

Corollary 4.4. *Let E_0 be an ordered vector space, F a Dedekind complete ordered vector space, G_1 and G_2 be two vector subspaces of E_0, one of them, say G_1, majorizing, and M a nonempty subset of E_0. Let $T_1 : G_1 \to F$ and $T_2 : G_2 \to F$ be two positive linear operators such that $\overline{T}_1 = T_2$ on G_2. Let $E = \operatorname{span}(G_1 \cup G_2 \cup M)$. Then the following statements are equivalent*:

(i) *There exists a positive linear operator $L_1 : E_1 \to F$ such that*

a) *$L = T_1$ on G_1, $L = T_2$ on G_2, and*

b) *$L = \overline{T}_1$ on M.*

(ii) $\overline{T}\big(\sum_{i=1}^{n} z_i\big) = \sum_{i=1}^{n} \overline{T}(z_i)$, *where $n \in \mathbb{N}^*$, and $z_1, \ldots, z_n \in M$.*

Proof. (ii) $\Rightarrow$ (i) is obviously, according to Theorem 4.1 applied for $P = \overline{T}_1$.

(i) $\Rightarrow$ (ii) can be proved like in Corollary 4.3, by putting T_1 instead of T. ∎

The following result is a consequence of Theorem 4.1 for the case when the set $M \subseteq E_0$ is *closed under addition.*

Corollary 4.5. *Let E_0 and F be two ordered vector spaces, G_1 and G_2 be two vector subspaces of E_0, and M a nonempty subset of E_0, closed under addition. Let $P : E_0 \to F$ be a monotone sublinear operator, and $T_1 : G_1 \to F$, $T_2 : G_2 \to F$ two positive linear operators such that $P = T_1$ on G_1 and $P = T_2$ on G_2. Let $E = \operatorname{span}(G_1 \cup G_2 \cup M)$. Then, the following are equivalent*:

(i) *There exists a positive linear operator $L : E \to F$ such that*

a) *$L = T_1$ on G_1, $L = T_2$ on G_2, and*

b) *$L = P$ on M.*

(ii) *P is additive on M.*

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