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NORM CONVERGENCE OF FEJÉR MEANS OF TWO-DIMENSIONAL WALSH–FOURIER SERIES

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Abstract. The main aim of this paper is to prove that there exists a martingale $f \in H_{1/2}$ such that the Fejér means of the two-dimensional Walsh–Fourier series of f is not uniformly bounded in the space weak-$L_{1/2}$.

1. Introduction. The first result with respect to the a.e. convergence of the Walsh–Fejér means $\sigma_n f$ is due to Fine [1]. Later, Schipp [5] showed that the maximal operator $\sigma^* f := \sup_n |\sigma_n f|$ is of weak type $(1, 1)$, from which the a.e. convergence follows by a standard argument. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1. Fujii's theorem was extended by Weisz [8]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh–Fourier series is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [6] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [11] proved that σ^* is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$. In [3] the author proved that the maximal operator σ^* is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$. By interpolation it follows that σ^* is not bounded from the Hardy space H_p to the space weak-L_p for any $0 < p < 1/2$.

For the two-dimensional Walsh–Fourier series Weisz [9, 10] proved that the following is true.

THEOREM W1. *Let $p > 1/2$. Then the maximal operator σ^* is bounded from the Hardy space H_p to the space L_p.*

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The author [4] proved that in Theorem W1, for the maximal operator σ^*, the assumption $p > 1/2$ is essential. Moreover, we prove that the following is true.

THEOREM G. *The maximal operator σ^* is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.*

Weisz [9, 10] considered the norm convergence of Fejér means of the two-dimensional Walsh–Fourier series. In particular, the following is true.

THEOREM W2. *Let $p > 1/2$. Then*

$$\|\sigma_{n,m} f\|_{H_p} \leq c_p \|f\|_{H_p} \quad (f \in H_p).$$

In [9] Weisz conjectured that for the uniformly boundedness of the operator $\sigma_{n,m}$ from the Hardy space $H_p(G \times G)$ to the space $H_p(G \times G)$ the assumption $p > 1/2$ is essential. We give an answer to the question, moreover, we prove that the operator $\sigma_{n,n}$ is not uniformly bounded from the Hardy space $H_{1/2}(G \times G)$ to the space weak-$L_{1/2}(G \times G)$. In particular, the following is true.

THEOREM 1.1. *There exists a martingale $f \in H_{1/2}(G \times G)$ such that*

$$\sup_n \|\sigma_{n,n} f\|_{\text{weak-}L_{1/2}} = +\infty.$$

2. Dyadic Hardy spaces. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G,\ I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$
$$(x \in G,\ n \in \mathbf{N}).$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G, $I_n := I_n(0)$ $(n \in \mathbf{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbf{N})$.

For $k \in \mathbf{N}$ and $x \in G$ denote by

$$r_k(x) := (-1)^{x_k}$$

the k-th Rademacher function.

The dyadic rectangles are of the form

$$I_{n,m}(x, y) := I_n(x) \times I_m(y).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,m}(x, y) : (x, y) \in G \times G\}$ is denoted by $F_{n,m}$.

The norm (or quasinorm) of the space $L_p(G \times G)$ is defined by

$$\|f\|_p := \Big(\int_{G\times G} |f(x,y)|^p \, d\mu(x,y) \Big)^{1/p} \quad (0 < p < +\infty).$$

The space weak-$L_p(G \times G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p(G\times G)} := \sup_{\lambda>0} \lambda \mu(|f| > \lambda)^{1/p} < +\infty.$$

Let us denote by $f = (f^{(n,m)}, n, m \in N)$ a two parameter martingale with respect to $(F_{n,m}, n, m \in \mathbf{N})$ (for details see, e.g. [7, 10]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n,m\in N} \big| f^{(n,m)} \big|.$$

If $f \in L_1(G \times G)$, the maximal function can also be given by

$$f^*(x,y) = \sup_{n,m\in\mathbf{N}} \frac{1}{\mu(I_{n,m}(x,y))} \Big| \int_{I_{n,m}(x,y)} f(u,v) \, d\mu(u,v) \Big|, \quad (x,y) \in G \times G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G \times G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

3. Walsh system and Fejér means. Let $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0,1\}$ $(i \in \mathbf{N})$, i.e. n is expressed in the number system of base 2. Let $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (x \in G, \ n \in \mathbf{P}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in G \setminus I_n. \end{cases} \tag{1}$$

The Fejér kernel of order n of the Walsh–Fourier series is defined by

$$K_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).$$

The rectangular partial sums of the double Walsh–Fourier series are defined as follows:

$$S_{M,N} f(x,y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j) w_i(x) w_j(y),$$

where the number

$$\widehat{f}(i,j) = \int_{G\times G} f(x,y) w_i(x) w_j(y) \, d\mu(x,y)$$

is said to be the (i,j)-th Walsh–Fourier coefficient of the function f.

If $f \in L_1(G\times G)$ then it is easy to show that the sequence $(S_{2^n,2^m}(f) : n,m \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(n,m)} : n,m \in \mathbf{N})$ then the Walsh–Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i,j) = \lim_{\min(k,l)\to\infty} \int_{G\times G} f^{(k,l)}(x,y) w_i(x) w_j(y)\, d\mu(x,y). \tag{2}$$

The Walsh–Fourier coefficients of $f \in L_1(G \times G)$ are the same as the ones of the martingale $(S_{2^n,2^m}(f) : n,m \in \mathbf{N})$ obtained from f.

For $n,m \in \mathbf{P}$ and a martingale f the Fejér mean of order (n,m) of the double Walsh–Fourier series of the martingale f is given by

$$\sigma_{n,m} f(x,y) = \frac{1}{nm} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} S_{i,j} f(x,y).$$

For the martingale f the maximal operator is defined by

$$\sigma^* f(x,y) = \sup_{n,m} |\sigma_{n,m} f(x,y)|.$$

A function $a \in L_2$ is called a rectangle p-atom if there exists a dyadic rectangle R such that

$$\begin{cases} \operatorname{supp}(a) \subset R, \\ \|a\|_2 \le |R|^{1/2-1/p} \\ \int_G a(x,y)\, d\mu(x) = \int_G a(x,y)\, d\mu(y) = 0 \quad \text{for all } x,y \in G. \end{cases}$$

The basic result of atomic decomposition is

THEOREM W3. *A martingale $f = (f^{(n,m)} : n,m \in \mathbf{N})$ is in H_p $(0 < p \le 1)$ if there exists a sequence $(a_k, k \in \mathbf{N})$ of rectangle p-atoms and a sequence $(\mu_k, k \in \mathbf{N})$ of real numbers such that for every $n,m \in \mathbf{N}$,*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n,2^m} a_k = f^{(n,m)}, \qquad \sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \le \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}.$$

In this paper the constant C are absolute constants and may denote different constants in different contexts.

4. Auxiliary result. In order to prove the theorem we need the following lemma.

LEMMA 4.1 ([4]). *Let $2 < A \in \mathbf{P}$ and $q_A := 2^{2A} + 2^{2A-2} + \ldots + 2^2 + 2^0$. Then*

$$q_{A-1} |K_{q_{A-1}}(x)| \ge 2^{2m+2s-3}$$

for $x \in I_{2A}^{m,s} := I_{2A}(0,\ldots,0,\underset{2m}{1},0,\ldots,0,\underset{2s}{1},x_{2s+1},\ldots,x_{2A-1})$, $m = 0,1,\ldots,A-3$, $s = m+2, m+3, \ldots, A-1$.

5. Proof of the main result. *Proof of Theorem 1.1.* Since $2^m/m \uparrow \infty$ it is easy to show that there exists an increasing sequence of positive integers $\{m_k : k \in \mathbf{P}\}$ such that

$$\sum_{k=1}^{\infty} \frac{1}{m_k^{1/2}} < \infty, \tag{3}$$

$$\sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} < \frac{2^{8m_k}}{m_k}, \tag{4}$$

$$\frac{2^{8m_{k-1}}}{m_{k-1}} < \frac{2^{m_k}}{km_k}. \tag{5}$$

Let

$$f^{(A,B)}(x,y) := \sum_{\{l:2m_l<\min(A,B)\}} \lambda_l a_l(x,y),$$

where $\lambda_l := \frac{1}{m_l}$ and

$$a_l(x,y) := 2^{4m_l}\left(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\right)\left(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)\right).$$

First, we prove that the martingale $f := (f^{(A,B)} : A, B \in \mathbf{N})$ belongs to the Hardy space $H_{1/2}(G \times G)$. Indeed, since $\|a_l\|_2 \le c2^{6m_l}$ and

$$S_{2^A,2^B} a_k(x,y) = \begin{cases} 0, & \text{if } \min(A,B) \le 2m_k, \\ a_k(x,y), & \text{if } \min(A,B) > 2m_k, \end{cases}$$

we can write

$$f^{(A,B)}(x,y) := \sum_{\{l:2m_l<\min(A,B)\}} \lambda_l a_l(x,y) = \sum_{k=0}^{\infty} \lambda_k S_{2^A,2^B} a_k(x,y).$$

From (3) and Theorem W3 we conclude that $f \in H_{1/2}(G \times G)$.

Now, we investigate the Fourier coefficients. Since

$$\int_{G\times G} f^{(A,B)}(x,y) w_i(x) w_j(y)\, d\mu(x,y)$$
$$= \begin{cases} 0, & (i,j) \notin \bigcup_{k=0}^{\infty}\{2^{2m_k},\dots,2^{2m_k+1}-1\} \times \{2^{2m_k},\dots,2^{2m_k+1}-1\}, \\ 0, & (i,j) \in \{2^{2m_k},\dots,2^{2m_k+1}-1\}\times\{2^{2m_k},\dots,2^{2m_k+1}-1\}, \min(A,B) \le 2m_k, \\ \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k},\dots,2^{2m_k+1}-1\}\times\{2^{2m_k},\dots,2^{2m_k+1}-1\}, \min(A,B) > 2m_k, \end{cases}$$

we can write (see (2))

$$\widehat{f}(i,j) = \begin{cases} \frac{2^{4m_k}}{m_k}, & (i,j) \in \{2^{2m_k},\dots,2^{2m_k+1}-1\}\times\{2^{2m_k},\dots,2^{2m_k+1}-1\},\ k \in \mathbf{P}, \\ 0, & (i,j) \notin \bigcup_{k=1}^{\infty}\{2^{2m_k},\dots,2^{2m_k+1}-1\} \times \{2^{2m_k},\dots,2^{2m_k+1}-1\}. \end{cases} \tag{6}$$

Let $q_{m_k} := 2^{2m_k} + 2^{2m_k-2} + \ldots + 2^2 + 2^0$. Then we can write

$$\begin{aligned}\sigma_{q_{m_k},q_{m_k}} f(x,y) &= \frac{1}{q_{m_k}^2} \sum_{i=0}^{q_{m_k}-1} \sum_{j=0}^{q_{m_k}-1} S_{i,j} f(x,y) \\ &= \frac{1}{q_{m_k}^2} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=0}^{2^{2m_k}-1} S_{i,j} f(x,y) + \frac{1}{q_{m_k}^2} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=0}^{2^{2m_k}-1} S_{i,j} f(x,y) \\ &+ \frac{1}{q_{m_k}^2} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S_{i,j} f(x,y) + \frac{1}{q_{m_k}^2} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} S_{i,j} f(x,y) \\ &= I + II + III + IV.\end{aligned} \tag{7}$$

Let $(i,j) \in \{2^{2m_k}, \ldots, q_{m_k}-1\} \times \{2^{2m_k}, \ldots, q_{m_k}-1\}$. Then from (6) we have

$$\begin{aligned}&S_{i,j} f(x,y) = \sum_{v=0}^{i-1} \sum_{\mu=0}^{j-1} \widehat{f}(\nu,\mu) w_\nu(x) w_\mu(y) \\ &= \sum_{l=1}^{k-1} \sum_{\nu=2^{m_l}}^{2^{m_l+1}-1} \sum_{\mu=2^{m_l}}^{2^{m_l+1}-1} \widehat{f}(\nu,\mu) w_\nu(x) w_\mu(y) + \sum_{\nu=2^{2m_k}}^{i-1} \sum_{\mu=2^{2m_k}}^{j-1} \widehat{f}(\nu,\mu) w_\nu(x) w_\mu(y) \\ &= \sum_{l=1}^{k-1} \frac{2^{4m_l}}{m_l} \big(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\big)\big(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)\big) \\ &\quad + \frac{2^{4m_k}}{m_k} \big(D_i(x) - D_{2^{2m_k}}(x)\big)\big(D_j(y) - D_{2^{2m_k}}(y)\big).\end{aligned} \tag{8}$$

Substituting (8) in IV, we obtain

$$\begin{aligned}IV &= \frac{1}{q_{m_k}^2} (q_{m_k} - 2^{2m_k})^2 \sum_{l=1}^{k-1} \frac{2^{4m_l}}{m_l} \big(D_{2^{2m_l+1}}(x) - D_{2^{2m_l}}(x)\big)\big(D_{2^{2m_l+1}}(y) - D_{2^{2m_l}}(y)\big) \\ &+ \frac{1}{q_{m_k}^2} \frac{2^{4m_k}}{m_k} \sum_{i=2^{2m_k}}^{q_{m_k}-1} \sum_{j=2^{2m_k}}^{q_{m_k}-1} \big(D_i(x) - D_{2^{2m_k}}(x)\big)\big(D_j(y) - D_{2^{2m_k}}(y)\big) \\ &= IV_1 + IV_2.\end{aligned} \tag{9}$$

Since

$$D_{j+2^{2m_k}}(x) = D_{2^{2m_k}}(x) + w_{2^{2m_k}}(x) D_j(x), \quad j = 0, 1, \ldots, 2^{2m_k} - 1,$$

we can write

$$\begin{aligned}IV_2 &= \frac{1}{q_{m_k}^2} \frac{2^{4m_k}}{m_k} w_{2^{2m_k}}(x) w_{2^{2m_k}}(y) \sum_{i=0}^{q_{m_{k-1}}-1} D_i(x) \sum_{j=0}^{q_{m_{k-1}}-1} D_j(y) \\ &= \frac{1}{q_{m_k}^2} \frac{2^{4m_k}}{m_k} w_{2^{2m_k}}(x) w_{2^{2m_k}}(y) q_{m_k-1}^2 K_{q_{m_k-1}}(x) K_{q_{m_k-1}}(y).\end{aligned} \tag{10}$$

Since

$$|D_{2^n}(x)| \le 2^n, \quad n \in N, \quad x \in G,$$

by (4) and (5) we obtain

$$|IV_1| \le C \sum_{l=1}^{k-1} \frac{2^{8m_l}}{m_l} \le C \frac{2^{m_k}}{km_k}. \tag{11}$$

Combining (9)–(11) we have

$$IV \ge \frac{Cq^2_{m_k-1}}{m_k} \left|K_{q_{m_k-1}}(x)\right| \left|K_{q_{m_k-1}}(y)\right| - \frac{C2^{m_k}}{km_k}. \tag{12}$$

Let

$$\begin{aligned}(i,j) \in &(\{2^{2m_k}, \dots, q_{m_k} - 1\} \times \{0,1,\dots,2^{2m_k}-1\}) \\ &\cup (\{0,1,\dots,2^{2m_k}-1\} \times \{2^{2m_k},\dots,q_{m_k}-1\}) \\ &\cup (\{0,1,\dots,2^{2m_k}-1\} \times \{0,1,\dots,2^{2m_k}-1\}).\end{aligned}$$

Then from (6), (4) and (5) it is easy to show that

$$|S_{i,j} f(x,y)| \le \sum_{l=0}^{k-1} \sum_{\nu=2^{2m_l}}^{2^{2m_l+1}-1} \sum_{\mu=2^{2m_l}}^{2^{2m_l+1}-1} |\widehat{f}(\nu,\mu)| \le \sum_{l=0}^{k-1} \frac{2^{8m_l}}{m_l} \le \frac{C2^{m_k}}{km_k}.$$

Consequently,

$$|I| \le \frac{1}{q^2_{m_k}} \sum_{i=0}^{2^{2m_k}-1} \sum_{j=0}^{2^{2m_k}-1} |S_{i,j} f(x,y)| \le C \frac{2^{4m_k}}{q^2_{m_k}} \frac{2^{m_k}}{km_k} \le \frac{C2^{m_k}}{km_k} \tag{13}$$

$$|II| \le \frac{2^{2m_k}(q_{m_k} - 2^{2m_k})}{q^2_{m_k}} \frac{2^{m_k}}{km_k} \le C \frac{2^{m_k}}{km_k} \tag{14}$$

$$|III| \le \frac{C2^{m_k}}{km_k}. \tag{15}$$

Combining (7), (9)–(15) we obtain

$$|\sigma_{q_{m_k},q_{m_k}} f(x,y)| \ge \frac{Cq^2_{m_k-1}}{m_k} \left|K_{q_{m_k-1}}(x)\right| \left|K_{q_{m_k-1}}(y)\right| - \frac{C2^{m_k}}{km_k}. \tag{16}$$

Let $(x,y) \in I^{l_1,l_1+2}_{2m_k} \times I^{l_2,l_2+2}_{2m_k}$, $(l_1,l_2) \in \{0,1,\dots,m_k-3\} \times \{0,1,\dots,m_k-3\}$. Then from Lemma 4.1 we can write

$$q_{m_k-1}\left|K_{q_{m_k-1}}(x)\right| \ge C2^{4l_1} \quad \text{and} \quad q_{m_k-1}\left|K_{q_{m_k-1}}(y)\right| \ge C2^{4l_2},$$

consequently,

$$q^2_{m_k-1}\left|K_{q_{m_k-1}}(x)\right| \left|K_{q_{m_k-1}}(y)\right| \ge C2^{4l_1+4l_2},$$

$$\left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \ge \frac{C}{m_k} 2^{4l_1+4l_2} - \frac{C2^{m_k}}{km_k}. \tag{17}$$

Let

$$A(m_k) := \left\{(l_1,l_2) : 0 \le l_2 \le m_k - 3,\ 0 \le l_1 \le \frac{m_k}{4},\ l_1 + l_2 \ge \frac{m_k}{4}\right\}$$

and

$$\alpha_k := \frac{C2^{m_k}}{m_k}.$$

Since (see (17) and $(l_1,l_2) \in A(m_k)$)

$$\left|\sigma_{q_{m_k},q_{m_k}} f(x,y)\right| \ge \frac{C}{m_k} 2^{m_k} - \frac{C2^{m_k}}{km_k} \ge \frac{C2^{m_k}}{m_k} = \alpha_k \quad \text{for sufficiently large } k,$$

we have

$$\mu\{(x,y)\in G\times G:\left|\sigma_{q_{m_k},q_{m_k}}f(x,y)\right|\geq\alpha_k\}$$
$$\geq\sum_{(l_1,l_2)\in A(m_k)}\mu\{(x,y)\in I_{2m_k}^{l_1,l_1+2}\times I_{2m_k}^{l_2,l_2+2}:\left|\sigma_{q_{m_k},q_{m_k}}f(x,y)\right|\geq\alpha_k\}$$
$$\geq C\sum_{l_1=0}^{[m_k/4]}\sum_{l_2=[m_k/4]-l_1}^{m_k-3}\sum_{x_{2l_1+5}=0}^{1}\cdots\sum_{x_{2m_k-1}=0}^{1}\sum_{x_{2l_2+5}=0}^{1}\cdots\sum_{x_{2m_k-1}=0}^{1}\mu\big(I_{2m_k}^{l_1,l_1+2}\times I_{2m_k}^{l_2,l_2+2}\big)$$
$$\geq C\sum_{l_1=0}^{[m_k/4]}\sum_{l_2=[m_k/4]-l_1}^{m_k-3}\frac{1}{2^{2l_1+2l_2}}\geq\frac{Cm_k}{2^{m_k/2}}.$$

Consequently,

$$\alpha_k\Big(\mu\{(x,y):|\sigma_{q_{m_k},q_{m_k}}f(x,y)|\geq C\alpha_k\}\Big)^2\geq C\frac{2^{m_k}}{m_k}\frac{m_k^2}{2^{m_k}}=Cm_k\to\infty\quad\text{as}\quad k\to\infty,$$

$$\sup_k\left\|\sigma_{q_{m_k},q_{m_k}}f\right\|_{\text{weak-}L_{1/2}}$$
$$:=\sup_k\sup_{\lambda>0}\lambda\big(\mu\{(x,y)\in G\times G:\sigma_{q_{m_k},q_{m_k}}f(x,y)>\lambda\}\big)^2=+\infty.$$

Theorem 1.1 is proved. ∎

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