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# ON CONTRACTION PRINCIPLE APPLIED TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH DERIVATIVES OF ORDER $\alpha \in (0, 1)$

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**Abstract.** One-term and multi-term fractional differential equations with a basic derivative of order $\alpha \in (0, 1)$ are solved. The existence and uniqueness of the solution is proved by using the fixed point theorem and the equivalent norms designed for a given value of parameters and function space. The explicit form of the solution obeying the set of initial conditions is given.

**1. Introduction.** Fractional differential equations are an important tool in the mathematical modelling of many systems and processes in mechanics, physics, chemistry, biochemistry, control theory, economics, engineering and bioengineering. Investigations on fractional differential equations include solving methods, the existence and uniqueness of solutions and studies of the properties of solutions as well as their applications. During last decades they yielded many essential results and the theory of fractional differential equations became an important part of pure and applied mathematics (compare monographs and review papers [5, 6, 7, 9, 10, 13, 15, 16, 17, 18, 25] and the references therein). As mentioned in the monographs [7] and [17], fractional differential equations of higher order include a class of sequential fractional differential equations. Such equations are applied for instance in hydrodynamics [4, 21, 22] and in theory of viscoelasticity [23]. Our aim is to study a variation of sequential fractional differential equations with composed differential operator including derivatives of a given order and a variable coefficient.

In the paper we consider nonlinear multi-term fractional differential equations dependent on the basic fractional derivative of arbitrary real noninteger order $\alpha \in (0, 1)$. The other types of sequential fractional differential equations were also studied in [7, 8, 19, 24].

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To prove the existence and uniqueness of the solutions in an arbitrary finite interval we follow the fixed point method and apply the Banach theorem. Our aim is to propose an efficient method of proof. A crucial point in the proof is the application of a newly-introduced one-parameter equivalent norms (and respective metrics) in the space of continuous or continuous weighted functions.

The paper is organized as follows. In the next section we recall all the necessary definitions and properties of fractional operators. We also construct one-parameter families of equivalent norms and respective metrics in the space of continuous and weighted continuous functions in a finite interval. Then we prove that certain fractional integral operators are bounded in these spaces endowed with a corresponding norm from the proposed class. The basic integral operator is generalized to a mapping, which appears to be a contraction under the respective assumptions on a parameter defining the norm and metric on the function space. Section 3 contains the main results — theorems on the existence and uniqueness of the solution to a certain nonlinear one-term and multi-term fractional differential equations. The paper is closed with a short discussion of the results and their prospective extension to further types of fractional differential equations.

## 2. Preliminaries.

In the paper we shall consider solutions of a certain class of fractional differential equations in the space of functions continuous in a finite interval $[0,b]$. The supremum norm on the space $C[0,b]$ and the respective induced metric are given by

$$\|f\| := \sup_{t\in[0,b]} |f(t)|, \quad d(f,g) := \|f-g\|. \tag{1}$$

The norm $\|\cdot\|_\gamma$ and the generated metric are active in the space of weighted continuous functions, when $\mathrm{Re}(\gamma) \in (0,1)$:

$$\|f\|_\gamma := \sup_{t\in[0,b]} |t^\gamma f(t)|, \quad d'(f,g) := \|f-g\|_\gamma. \tag{2}$$

The space $C_\gamma[0,b]$ is then given as

$$C_\gamma[0,b] := \{f \in C(0,b] : \|f\|_\gamma \leq \infty\}. \tag{3}$$

Remark. Both metric function spaces $\langle C[0,b], d\rangle$ and $\langle C_\gamma[0,b], d'\rangle$ are complete.

Now, we recall definitions of left-sided fractional operators. In our paper we shall study fractional differential equations containing Riemann–Liouville or Caputo derivatives. Both, the integral and derivatives, are defined as follows [7, 20].

Definition 2.1. If $\mathrm{Re}(\alpha) > 0$, then the *left-sided Riemann–Liouville integral* of order $\alpha$ is given by the formula

$$(I^\alpha_{0+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0, \tag{4}$$

where $\Gamma$ denotes the Euler gamma function.

If $\mathrm{Re}(\alpha) \in (n-1,n)$, then the *left-sided Riemann–Liouville derivative* is defined as

$$(D^\alpha_{0+} f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{0+} f)(t), \qquad t > 0, \tag{5}$$

and the *left-sided Caputo derivative* is defined as

$$({}^cD^\alpha_{0+}f)(t) = D^\alpha_{0+}\Big[f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)t^k}{k!}\Big], \qquad t > 0. \tag{6}$$

REMARK. In the paper we shall solve equations dependent on fractional derivatives of real order $\alpha \in (0,1)$. In this case fractional derivatives are defined by the formulas below:

$$(D^\alpha_{0+}f)(t) = \frac{d}{dt}(I^{1-\alpha}_{0+}f)(t) \tag{7}$$

$$({}^cD^\alpha_{0+}f)(t) = D^\alpha_{0+}[f(t) - f(0)]. \tag{8}$$

An important and characteristic feature of the above fractional operators is their composition rule [7, 20]. It will be applied in the transformation of the investigated equations into their equivalent integral form as well as in the derivation of the corresponding initial conditions.

PROPERTY 2.2. *Let* $\mathrm{Re}(\delta) > \mathrm{Re}(\alpha) > 0$. *Then the equalities*

$$D^\alpha_{0+}I^\delta_{0+}f(t) = I^{\delta-\alpha}_{0+}f(t) \tag{9}$$

$${}^cD^\alpha_{0+}I^\delta_{0+}f(t) = I^{\delta-\alpha}_{0+}f(t) \tag{10}$$

*hold at any point* $t \in [0,b]$ *if* $f \in C[0,b]$. *If* $f \in C_\gamma[0,b]$, *then the above composition rules hold at any point* $t \in (0,b]$.

When the orders of the fractional derivative and integral coincide, we obtain the composition rules analogous to the second main theorem of integral calculus.

PROPERTY 2.3. *Let* $\mathrm{Re}(\alpha) > 0$. *Then the equalities*

$$D^\alpha_{0+}I^\alpha_{0+}f(t) = f(t) \tag{11}$$

$${}^cD^\alpha_{0+}I^\alpha_{0+}f(t) = f(t) \tag{12}$$

*hold at any point* $t \in [0,b]$ *if* $f \in C[0,b]$. *If* $f \in C_\gamma[0,b]$, *then the above composition rules hold at any point* $t \in (0,b]$.

In the procedure of transforming the fractional differential equation under consideration into an equivalent integral equation we shall apply the stationary functions of Riemann–Liouville or Caputo derivative. They are analogues of polynomial functions from classical calculus and differential equations theory. For $\alpha \in (0,1)$ the continuous stationary function of the Caputo derivative is an arbitrary constant:

$${}^cD^\alpha_{0+}\phi_0(t) = 0 \quad \forall t \in [0,b] \qquad \Longleftrightarrow \qquad \phi_0(t) = c.$$

Respectively, the only continuous weighted stationary functions of the Riemann–Liouville derivative are proportional to the power function:

$$D^\alpha_{0+}\phi_0(t) = 0 \quad \forall t \in (0,b] \qquad \Longleftrightarrow \qquad \phi_0(t) = ct^{\alpha-1}$$

and belong to the space $C_{1-\alpha}[0,b]$.

In what follows we shall modify norms (1), (2) to the equivalent ones on the spaces $C[0,b]$ and $C_{1-\alpha}[0,b]$ respectively. To this aim we apply functions constructed using the

Mittag–Leffler function [7]. We define auxiliary functions dependent on parameters $\alpha, \beta$ and on free parameter $\kappa$, assuming $\alpha \in (0,1)$, $\alpha - \beta > 0$, $\kappa \in R_+$ and $t \in (0,b]$:

$$e^{\alpha,\beta,\kappa}(t) := \Gamma(\alpha-\beta)E_{\alpha,\alpha-\beta}(\kappa t^{\alpha}) \tag{13}$$

$$\mathcal{E}^{\alpha,\beta,\kappa}(t) := E_{\alpha-\beta,\alpha-\beta}(\kappa t^{\alpha-\beta}) \tag{14}$$

$$E^{\alpha,\beta,\kappa}(t) := \Gamma(1-\beta)E_{\alpha,1-\beta}(\kappa t^{\alpha}) \tag{15}$$

$$\mathbf{E}^{\alpha,\beta,\kappa}(t) := E_{\alpha-\beta,1-\beta}(\kappa t^{\alpha-\beta}). \tag{16}$$

In the above formulas $E_{A,B}$ is the two-parameter Mittag–Leffler function ($\mathrm{Re}(A) > 0$):

$$E_{A,B}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(Ak+B)}$$

determined (in general) on the complex plane $\mathbb{C}$ for $\mathrm{Re}(A) > 0$.

The following proposition describes integration properties of the introduced functions.

PROPOSITION 2.4. *If $\alpha \in (0,1)$, $\alpha - \beta > 0$ and $\kappa \in \mathbb{R}_+$, then the following formulas hold in any interval* $(0,b]$

$$I^{\alpha}_{0+} t^{\alpha-\beta-1} e^{\alpha,\beta,\kappa}(t) = \frac{t^{\alpha-\beta-1}}{\kappa}\left[e^{\alpha,\beta,\kappa}(t) - 1\right] \tag{17}$$

$$I^{\alpha}_{0+} t^{\alpha-\beta-1} \mathcal{E}^{\alpha,\beta,\kappa}(t) = \frac{t^{\alpha-1}}{\kappa}\left[E_{\alpha-\beta,\alpha}(\kappa t^{\alpha-\beta}) - \frac{1}{\Gamma(\alpha)}\right] \tag{18}$$

$$I^{\alpha}_{0+} t^{-\beta} E^{\alpha,\beta,\kappa}(t) = \frac{t^{-\beta}}{\kappa}\left[E^{\alpha,\beta,\kappa}(t) - 1\right] \tag{19}$$

$$I^{\alpha}_{0+} t^{-\beta} \mathbf{E}^{\alpha,\beta,\kappa}(t) = \frac{1}{\kappa}\left[E_{\alpha-\beta,1}(\kappa t^{\alpha-\beta}) - 1\right]. \tag{20}$$

Solving the fractional differential equations we shall follow the methods from differential equations theory. We start by transforming the fractional differential equation into its equivalent integral form and then into a fixed point condition for a mapping determined on the respective function space. To prove that this mapping is contractive we extend the Bielecki method of equivalent norms [2]. A similar modification of norms was also developed in the theory of fractional differential equations [1, 3]. Lakshmikantham et al. [13, 14] used in the scaling procedure a one-parameter Mittag–Leffler function and proved the existence-uniqueness result for one-term fractional differential equation with Caputo derivative. We propose to apply in modification of norms the auxiliary functions (13)–(16). Mappings which are not contractive with respect to standard norms (1), (2) become contractions after changing the norm and metric to the equivalent ones, dependent on the free parameter $\kappa \in \mathbb{R}_+$, provided $\kappa$ is large enough.

DEFINITION 2.5. In the space $C_{1-\alpha}[0,b]$ we define a new norm and metric, provided $\kappa \in \mathbb{R}_+$ and $\alpha - \beta > 0$:

$$\|f\|_{1-\alpha,\kappa} := \sup_{t\in[0,b]} \frac{|t^{1-\alpha}f(t)|}{e^{\alpha,\beta,\kappa}(t)}, \qquad d'_{\kappa}(f,g) := \|f-g\|_{1-\alpha,\kappa} \tag{21}$$

$$\|f\|'_{1-\alpha,\kappa} := \sup_{t\in[0,b]} \frac{|t^{1-\alpha}f(t)|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)}, \qquad d''_{\kappa}(f,g) := \|f-g\|'_{1-\alpha,\kappa}. \tag{22}$$

In the space $C[0,b]$ we define a new norm and metric assuming $\kappa \in \mathbb{R}_+$ and $\alpha - \beta > 0$ (see also [13, 14] for $\beta = 0$)

$$\|f\|_\kappa := \sup_{t\in[0,b]} \frac{|f(t)|}{E^{\alpha,\beta,\kappa}(t)}, \qquad d_\kappa(f,g) := \|f-g\|_\kappa \tag{23}$$

$$\|f\|'_\kappa := \sup_{t\in[0,b]} \frac{|f(t)|}{\mathbf{E}^{\alpha,\beta,\kappa}(t)}, \qquad d^1_\kappa(f,g) := \|f-g\|'_\kappa. \tag{24}$$

Proposition 2.6. *The metrics $d'_\kappa$ and $d''_\kappa$ are equivalent to the metric $d'$ in the space $C_{1-\alpha}[0,b]$. The metrics $d_\kappa$ and $d^1_\kappa$ are equivalent to the metric $d$ in the space $C[0,b]$.*

*Proof.* The equivalence is implied by the set of inequalities fulfilled by the respective norms for arbitrary function $f \in C_{1-\alpha}[0,b]$

$$\frac{\|f\|_{1-\alpha}}{e^{\alpha,\beta,\kappa}(b)} \le \|f\|_{1-\alpha,\kappa} \le \|f\|_{1-\alpha}$$

$$\frac{\|f\|_{1-\alpha}}{\mathcal{E}^{\alpha,\beta,\kappa}(b)} \le \|f\|'_{1-\alpha,\kappa} \le \Gamma(\alpha-\beta)\|f\|_{1-\alpha}$$

and for arbitrary $f \in C[0,b]$

$$\frac{\|f\|}{E^{\alpha,\beta,\kappa}(b)} \le \|f\|_\kappa \le \|f\|$$

$$\frac{\|f\|}{\mathbf{E}^{\alpha,\beta,\kappa}(b)} \le \|f\|'_\kappa \le \Gamma(1-\beta)\|f\|. \qquad \blacksquare$$

Having defined the new norms, we shall study the properties of the integral operator $I^\alpha_{0+}t^{-\beta}$ on the spaces of continuous and weighted continuous functions with norms (23), (24) or (21), (22), respectively. Analyzing the formulas enclosed in the proposition below we note that this operator is bounded in all the cases considered and that the constant on the right-hand side is inversely proportional to the parameter $\kappa$.

Proposition 2.7. *Let $\alpha \in (0,1)$.*

(1) *If $\beta \le 0$, then for any function $f \in C_{1-\alpha}[0,b]$*

$$\left\|(I^\alpha_{0+}t^{-\beta})^j f\right\|_{1-\alpha,\kappa} \le \left(\frac{b^{-\beta}}{\kappa}\right)^j \|f\|_{1-\alpha,\kappa} \quad (j \in \mathbb{N}). \tag{25}$$

(2) *If $\beta > 0$ and $\alpha - \beta > 0$, then there exists a sequence $(\kappa_m)$ such that $\lim_{m\to\infty}\kappa_m = \infty$ and for any function $f \in C_{1-\alpha}[0,b]$*

$$\left\|(I^\alpha_{0+}t^{-\beta})^j f\right\|'_{1-\alpha,\kappa_m} \le \left(\frac{A_L}{\kappa_m}\right)^j \|f\|'_{1-\alpha,\kappa_m} \quad (j \in \mathbb{N}), \tag{26}$$

*where the constant $L$ is determined by the condition*

$$(\alpha-\beta)L < \gamma_{\min} \le (\alpha-\beta)(L+1)$$

*and*

$$A_L = \sum_{k=1}^{L-1} \frac{\Gamma(\alpha-\beta)}{\Gamma(k(\alpha-\beta)+\alpha)} + 1.$$

(3) *If $\beta \le 0$, then for any function $f \in C[0,b]$*

$$\left\|(I^{\alpha}_{0+}t^{-\beta})^j f\right\|_{\kappa} \le \left(\frac{b^{-\beta}}{\kappa}\right)^j \|f\|_{\kappa} \quad (j \in \mathbb{N}). \tag{27}$$

(4) *If $\beta > 0$ and $\alpha-\beta > 0$, then there exists a sequence $(\kappa_m)$ such that $\lim_{m\to\infty}\kappa_m = \infty$ and for any function $f \in C[0,b]$*

$$\left\|(I^{\alpha}_{0+}t^{-\beta})^j f\right\|'_{\kappa_m} \le \left(\frac{B_L}{\kappa_m}\right)^j \|f\|'_{\kappa_m} \quad (j \in \mathbb{N}), \tag{28}$$

*where the constant $L$ is determined by the condition*

$$(\alpha-\beta)(L-1) < \gamma_{\min} + \beta - 1 \le (\alpha-\beta)L$$

*and*

$$B_L = \sum_{k=1}^{L-1} \frac{\Gamma(1-\beta)}{\Gamma(k(\alpha-\beta)+1)} + 1.$$

*Proof.* We begin with part (1) and observe that the fractional integral $I^{\alpha}_{0+}$ is bounded in the space $C_{1-\alpha}[0,b]$. Thus, it is enough to prove the case $j=1$ as $j>1$ follows from the mathematical induction principle. Applying integration property (17) from Proposition 2.4 we obtain for an arbitrary function $f \in C_{1-\alpha}[0,b]$:

$$\begin{aligned}
\left\|(I^{\alpha}_{0+}t^{-\beta})f\right\|_{1-\alpha,\kappa} &= \sup_{t\in[0,b]} \frac{\left|t^{1-\alpha}I^{\alpha}_{0+}t^{-\beta}f(t)\right|}{e^{\alpha,\beta,\kappa}(t)} \\
&= \sup_{t\in[0,b]} \frac{\left|t^{1-\alpha}I^{\alpha}_{0+}t^{-\beta}t^{\alpha-1}e^{\alpha,\beta,\kappa}(t)\frac{t^{1-\alpha}f(t)}{e^{\alpha,\beta,\kappa}(t)}\right|}{e^{\alpha,\beta,\kappa}(t)} \\
&\le \|f\|_{1-\alpha,\kappa} \sup_{t\in[0,b]} \frac{\left|t^{1-\alpha}I^{\alpha}_{0+}t^{-\beta}t^{\alpha-1}e^{\alpha,\beta,\kappa}(t)\right|}{e^{\alpha,\beta,\kappa}(t)} \\
&= \|f\|_{1-\alpha,\kappa} \sup_{t\in[0,b]} \frac{t^{-\beta}}{\kappa}\left[1 - \frac{1}{e^{\alpha,\beta,\kappa}(t)}\right] \le \frac{b^{-\beta}}{\kappa} \cdot \|f\|_{1-\alpha,\kappa}.
\end{aligned}$$

In the proof of part (2) we use formula (18) from Proposition 2.4 and the monotonicity property of the Euler gamma function, namely for a real argument greater than $\gamma_{\min} \cong 1.46163$ this function is strictly increasing. We again prove the case $j=1$ as formulas for $j>1$ are a straightforward corollary. Let us observe that for any given $\alpha$ and $\beta$ fulfilling condition $\alpha-\beta>0$, there exists an integer number $L \in \mathbb{N}$ such that

$$(\alpha-\beta)L < \gamma_{\min} \le (\alpha-\beta)(L+1). \tag{29}$$

Let

$$P_L(t) = \sum_{k=1}^{L-1} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k+\alpha)}, \qquad R_L(t) = \sum_{k=L}^{\infty} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k+\alpha)},$$

and

$$\mathbf{P}_L(t) = \sum_{k=1}^{L-1} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k+\alpha-\beta)}, \qquad \mathbf{R}_L(t) = \sum_{k=L}^{\infty} \frac{(\kappa t^{\alpha-\beta})^k}{\Gamma((\alpha-\beta)k+\alpha-\beta)}.$$

It is easy to check that

$$\mathcal{E}^{\alpha,\beta,\kappa}(t) = \frac{1}{\Gamma(\alpha-\beta)} + \mathbf{P}_L(t) + \mathbf{R}_L(t).$$

We obtain the following inequalities for the norm $\|\cdot\|'_{1-\alpha,\kappa}$ of the integral $I^{\alpha}_{0+}t^{-\beta}f$:

$$\begin{aligned}
\big\|(I^{\alpha}_{0+}t^{-\beta})f\big\|'_{1-\alpha,\kappa} &= \sup_{t\in[0,b]} \frac{\big|t^{1-\alpha}I^{\alpha}_{0+}t^{-\beta}f(t)\big|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)} \\
&= \sup_{t\in[0,b]} \frac{\big|t^{1-\alpha}I^{\alpha}_{0+}t^{-\beta}t^{\alpha-1}\mathcal{E}^{\alpha,\beta,\kappa}(t)\frac{t^{1-\alpha}f(t)}{\mathcal{E}^{\alpha,\beta,\kappa}(t)}\big|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)} \\
&\le \|f\|'_{1-\alpha,\kappa} \sup_{t\in[0,b]} \frac{\big|t^{1-\alpha}I^{\alpha}_{0+}t^{-\beta}t^{\alpha-1}\mathcal{E}^{\alpha,\beta,\kappa}(t)\big|}{\mathcal{E}^{\alpha,\beta,\kappa}(t)} \\
&= \|f\|'_{1-\alpha,\kappa} \sup_{t\in[0,b]} \frac{t^{1-\alpha}t^{\alpha-1}\big[E_{\alpha-\beta,\alpha}(\kappa t^{\alpha-\beta}) - 1/\Gamma(\alpha)\big]}{\kappa\cdot\mathcal{E}^{\alpha,\beta,\kappa}(t)} \\
&= \frac{1}{\kappa}\cdot\|f\|'_{1-\alpha,\kappa} \sup_{t\in[0,b]} \frac{P_L(t)+R_L(t)}{1/\Gamma(\alpha-\beta)+\mathbf{P}_L(t)+\mathbf{R}_L(t)}.
\end{aligned}$$

Thanks to the assumption $(\alpha-\beta)(L+1)\ge\gamma_{\min}$ we can rewrite the above inequality as follows:

$$\big\|(I^{\alpha}_{0+}t^{-\beta})f\big\|'_{1-\alpha,\kappa} \le \frac{1}{\kappa}\cdot\|f\|'_{1-\alpha,\kappa} \sup_{t\in[0,b]}\Big(\frac{P_L(t)}{1/\Gamma(\alpha-\beta)+\mathbf{P}_L(t)}+1\Big).$$

In the above formula the norm $\|(I^{\alpha}_{0+}t^{-\beta})f\|'_{1-\alpha,\kappa}$ is estimated with a coefficient given as the supremum over interval $[0,b]$ of a certain rational function with positive denominator. As all these functions are continuous in $[0,b]$, the supremum is in fact a maximum value of the function at a certain $t_\kappa\in[0,b]$:

$$\big\|(I^{\alpha}_{0+}t^{-\beta})f\big\|'_{1-\alpha,\kappa} \le \frac{1}{\kappa}\cdot\|f\|'_{1-\alpha,\kappa}\Big(\frac{P_L(t_\kappa)}{1/\Gamma(\alpha-\beta)+\mathbf{P}_L(t_\kappa)}+1\Big).$$

Let us choose a sequence $(\kappa_m)$ such that

$$\lim_{m\to\infty}\kappa_m=\infty$$

and the limit $\lim_{m\to\infty}\kappa_m t_{\kappa_m}^{\alpha-\beta}$ (finite or infinite) exists. For this sequence of parameters we obtain the following sequence of inequalities:

$$\begin{aligned}
\big\|(I^{\alpha}_{0+}t^{-\beta})f\big\|'_{1-\alpha,\kappa_m} &\le \frac{1}{\kappa_m}\cdot\|f\|'_{1-\alpha,\kappa_m}\Big(\frac{P_L(t_{\kappa_m})}{1/\Gamma(\alpha-\beta)+\mathbf{P}_L(t_{\kappa_m})}+1\Big) \\
&\le \frac{1}{\kappa_m}\cdot\|f\|'_{1-\alpha,\kappa_m}\left(\frac{\sum_{k=1}^{L-1}1/\Gamma(k(\alpha-\beta)+\alpha)}{1/\Gamma(\alpha-\beta)}+1\right) = \frac{A_L}{\kappa_m}\cdot\|f\|'_{1-\alpha,\kappa_m}.
\end{aligned}$$

In the above calculations we used the property of the Euler gamma function: the inequalities

$$\Gamma(k(\alpha-\beta)+\alpha)\ge\Gamma(k(\alpha-\beta)+\alpha-\beta)$$

are valid for $k\ge L$, $k\in\mathbb{N}$ when $(\alpha-\beta)(L+1)\ge\gamma_{\min}$.

We omit the proof of parts (3) and (4) as it is analogous to the calculations presented above in detail. ∎

**3. Main results.** In this section we shall apply the norms and metrics constructed in Definition 2.5 to prove the existence and uniqueness of the solution for certain fractional differential equations. We begin by studying two one-term equations in the form of

$$t^{\beta} D^{\alpha}_{0+} f(t) = \Psi(t, f(t)) \tag{30}$$

$$t^{\beta} \cdot {}^{c}D^{\alpha}_{0+} f(t) = \Psi(t, f(t)) \tag{31}$$

and we shall show that there exists a unique solution for each of the above equations in the respective function space and for arbitrarily long interval $[0, b]$. Thanks to the composition rules from Property 2.3 we can rewrite (30), (31) as the equivalent fractional integral equations ($c \in \mathbb{R}$ arbitrary)

$$f(t) = I^{\alpha}_{0+} t^{-\beta} \Psi(t, f(t)) + ct^{\alpha-1} \tag{32}$$

$$f(t) = I^{\alpha}_{0+} t^{-\beta} \Psi(t, f(t)) + c. \tag{33}$$

Finally, denoting the mapping on the right-hand-side by $T_{\phi_0}$ in both cases and assuming $T_{\phi_0}\phi_0 \neq \phi_0$, we obtain equations (30), (31) reformulated as the fixed point conditions on $C_{1-\alpha}[0, b]$ and $C[0, b]$, respectively. Thus, we can solve equations (30), (31) applying the Banach fixed point theorem, provided the constructed mapping is a contraction on the corresponding function space. The following propositions and corollaries describe the solution for the considered equations.

Proposition 3.1. *Let $\alpha \in (0, 1)$ and $\alpha - \beta > 0$. If the function $\Psi \in C([0, b] \times \mathbb{R})$ fulfils the Lipschitz condition*

$$|\Psi(t, x) - \Psi(t, y)| \leq M \cdot |x - y| \qquad \forall t \in [0, b] \quad \forall x, y \in \mathbb{R}, \tag{34}$$

*then each stationary function $\phi_0$ of the operator $D^{\alpha}_{0+}$ generates a unique solution $f \in C_{1-\alpha}[0, b]$ of the equation*

$$t^{\beta} D^{\alpha}_{0+} f(t) = \Psi(t, f(t)).$$

*The solution $f$ is the limit of iterations of the mapping $T_{\phi_0}$:*

$$f(t) = \lim_{k \to \infty} (T_{\phi_0})^k \psi(t),$$

*where the function $\psi \in C_{1-\alpha}[0, b]$ is arbitrary.*

*Proof.* Let functions $g, h \in C_{1-\alpha}[0, b]$ be arbitrary and $\beta \leq 0$. Their images $T_{\phi_0} g$ and $T_{\phi_0} h$ look as follows

$$T_{\phi_0} g(t) = I^{\alpha}_{0+} t^{-\beta} \Psi(t, g(t)) + \phi_0(t)$$

$$T_{\phi_0} h(t) = I^{\alpha}_{0+} t^{-\beta} \Psi(t, h(t)) + \phi_0(t).$$

We estimate the distance $d'_{\kappa}$ of the above images

$$\begin{aligned} d'_{\kappa}(T_{\phi_0} g, T_{\phi_0} h) &= \left\| I^{\alpha}_{0+} t^{-\beta} [\Psi(t, g(t)) - \Psi(t, h(t))] \right\|_{1-\alpha,\kappa} \\ &\leq \left\| I^{\alpha}_{0+} t^{-\beta} M |g(t) - h(t)| \right\|_{1-\alpha,\kappa} \\ &\leq \frac{Mb^{-\beta}}{\kappa} \cdot \|g - h\|_{1-\alpha,\kappa} = \frac{Mb^{-\beta}}{\kappa} d'_{\kappa}(g, h). \end{aligned}$$

Thus $T_{\phi_0}$ is a contraction in $\langle C_{1-\alpha}[0,b], d'_\kappa\rangle$, provided $\kappa$ is large enough. Then, it follows from the Banach theorem that there exists a unique fixed point $f \in C_{1-\alpha}[0,b]$

$$f(t) = T_{\phi_0} f(t).$$

The function $f$ is the solution of equation (30) and can be explicitly calculated as a limit of iterations of the mapping $T_{\phi_0}$ acting on an arbitrary start function $\psi$.

Next, we consider the case $\beta > 0$, $\alpha - \beta > 0$ and a mapping $T_{\phi_0}$ acting on the space $C_{1-\alpha}[0,b]$ endowed with the metric $d''_{\kappa_m}$. Let us note that for any function $g \in C_{1-\alpha}[0,b]$ its image $T_{\phi_0} g \in C_{1-\alpha}[0,b]$. To prove this fact we apply the norm (22) equivalent to the standard norm (2):

$$\|T_{\phi_0} g\|'_{1-\alpha,\kappa_m} \le \| \, I^\alpha_{0+} t^{-\beta}\Psi(t,g(t))\|'_{1-\alpha,\kappa_m} + \|\phi_0\|'_{1-\alpha,\kappa_m}$$
$$\le \frac{A_L}{\kappa_m}\|\Psi(t,g(t))\|'_{1-\alpha,\kappa_m} + \|\phi_0\|'_{1-\alpha,\kappa_m} < \infty.$$

In the calculations we applied the fact (yielded by the Lipschitz condition) that the composed function $\Psi(t,g(t)) \in C_{1-\alpha}[0,b]$.

Now, on $C_{1-\alpha}[0,b]$, the distance $d''_{\kappa_m}$ of images $T_{\phi_0} g$ and $T_{\phi_0} h$ looks as follows

$$\begin{aligned} d''_{\kappa_m}(T_{\phi_0} g, T_{\phi_0} h) &= \left\| I^\alpha_{0+} t^{-\beta}[\Psi(t,g(t)) - \Psi(t,h(t))]\right\|'_{1-\alpha,\kappa_m} \\ &\le \left\| I^\alpha_{0+} t^{-\beta} M|g(t)-h(t)| \,\right\|'_{1-\alpha,\kappa_m} \\ &\le \frac{M\cdot A_L}{\kappa_m}\cdot \|g-h\|'_{1-\alpha,\kappa_m} = \frac{M\cdot A_L}{\kappa_m}\, d''_{\kappa_m}(g,h). \end{aligned}$$

Similar to the previous case, we conclude that for $\kappa_m$ large enough $T_{\phi_0}$ is a contraction on the space $\langle C_{1-\alpha}[0,b], d''_{\kappa_m}\rangle$. Hence the fixed point $f$ exists thanks to the Banach theorem and can be constructed in a way described in the above proposition. ∎

COROLLARY 3.2. *If the assumptions of Proposition* 3.1 *are fulfilled, then the equation*

$$t^\beta D^\alpha_{0+} f(t) = \Psi(t, f(t))$$

*has a unique solution* $f \in C_{1-\alpha}[0,b]$ *fulfilling the initial condition*

$$I^{1-\alpha}_{0+} f(0) = d.$$

*The solution* $f$ *is the limit of iterations of the mapping* $T_{\phi_0}$ *generated by the stationary function* $\phi_0 = dt^{\alpha-1}/\Gamma(\alpha)$.

*Proof.* Due to the one-to-one correspondence between the stationary function and the unique generated solution of equation (30), it is enough to show that the function $\phi_0 = dt^{\alpha-1}/\Gamma(\alpha)$ generates a solution $f$ obeying the initial condition $I^{1-\alpha}_{0+} f(0) = d$. The solution $f$ fulfils equation (30) and the equivalent integral equation

$$f(t) = I^\alpha_{0+} t^{-\beta}\Psi(t,f(t)) + \frac{dt^{\alpha-1}}{\Gamma(\alpha)}\,.$$

Integrating the above equation we arrive at the relation

$$\begin{aligned} I^{1-\alpha}_{0+} f(t) &= I^{1-\alpha}_{0+} I^\alpha_{0+} t^{-\beta}\Psi(t,f(t)) + I^{1-\alpha}_{0+}\frac{dt^{\alpha-1}}{\Gamma(\alpha)} \\ &= I^1_{0+} t^{(-\beta+1)-1}\Psi(t,f(t)) + d, \end{aligned}$$

where we applied the integration formula [7]: $I^{\gamma}_{0+}t^{\delta-1} = t^{\delta+\gamma-1}\Gamma(\delta)/\Gamma(\gamma+\delta)$ valid for $\mathrm{Re}(\delta) > 0$ and the inequality $-\beta+1 > 1-\alpha > 0$ in case $\beta > 0$.

Now for $t = 0$ we obtain

$$I^{1-\alpha}_{0+}f(0) = d$$

what ends the proof. ■

Similar results are valid for one-term equation with Caputo derivative (31). We quote Proposition 3.3 and Corollary 3.4 without proof. They both are straightforward corollaries of Property 2.3 and Proposition 2.6 as well as of Proposition 2.7 (part (3) and (4)).

PROPOSITION 3.3. *Let* $\alpha \in (0,1)$ *and* $\alpha - \beta > 0$. *If* $\Psi \in C([0,b]\times\mathbb{R})$ *fulfils the Lipschitz condition* (34), *then each stationary function* $\phi_0$ *of the operator* ${}^cD^{\alpha}_{0+}$ *generates a unique solution* $f \in C[0,b]$ *of equation*

$$t^{\beta}\cdot {}^cD^{\alpha}_{0+}f(t) = \Psi(t, f(t)).$$

*The solution* $f$ *is the limit of iterations of the mapping* $T_{\phi_0}$*:*

$$f(t) = \lim_{k\to\infty}(T_{\phi_0})^k\psi(t),$$

*where* $\psi \in C[0,b]$ *is arbitrary.*

COROLLARY 3.4. *If the assumptions of Proposition* 3.3 *are fulfilled, then the equation*

$$t^{\beta}\cdot {}^cD^{\alpha}_{0+}f(t) = \Psi(t, f(t))$$

*has a unique solution* $f \in C[0,b]$ *fulfilling the initial condition*

$$f(0) = d.$$

*The solution* $f$ *is the limit of iterations of the mapping* $T_{\phi_0}$ *generated by the stationary function* $\phi_0 = d$.

The results given in the above propositions and corollaries, valid for one-term fractional differential equations, can be extended to multi-term equations. In the present paper we shall discuss such equations with basic Riemann–Liouville derivative of order $\alpha \in (0,1)$. A similar procedure can be applied to the analogous equations with Caputo derivative and will be described in a subsequent paper.

We assume that the nonlinear part of the fractional differential equation does not depend on the derivative and consider in finite interval a class of equations in the form

$$\Big[(t^{\beta}D^{\alpha}_{0+})^m - \sum_{j=1}^{m-1} c_j (t^{\beta}D^{\alpha}_{0+})^j\Big] f(t) = \Psi(t, f(t)), \tag{35}$$

where $\alpha \in (0,1)$, $c_j \in \mathbb{R}$, $j = 1,\ldots,m-1$, $\Psi \in C([0,b]\times\mathbb{R})$ and $\alpha - \beta > 0$.

Using the composition rules from Property 2.3 we reformulate the above equation as follows

$$(t^{\beta}D^{\alpha}_{0+})^m\Big[\Big(1 - \sum_{j=1}^{m-1} c_j (I^{\alpha}_{0+}t^{-\beta})^{m-j}\Big) f(t) - (I^{\alpha}_{0+}t^{-\beta})^m \Psi(t, f(t))\Big] = 0. \tag{36}$$

Thus, we note that on the space $C_{1-\alpha}[0,b]$ equation (35) is equivalent to the fractional integral equation

$$\Big(1-\sum_{j=1}^{m-1} c_j (I_{0+}^{\alpha} t^{-\beta})^{m-j}\Big) f(t) - (I_{0+}^{\alpha} t^{-\beta})^{m} \Psi(t, f(t)) = \phi_0(t), \tag{37}$$

where the function $\phi_0 \in C_{1-\alpha}[0,b]$ belongs to the kernel of the fractional operator $(t^{\beta} D_{0+}^{\alpha})^m$:

$$(t^{\beta} D_{0+}^{\alpha})^m \phi_0(t) = 0$$

and is given as ($\bar{d}_j \in \mathbb{R}$ for $j = 0, \ldots, m-1$)

$$\phi_0(t) = \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha-\beta)j+\alpha)}\, t^{(\alpha-\beta)j+\alpha-1}.$$

Now, we are able to rewrite equations (35), (37) as the fixed point condition:

$$f(t) = T_{\phi_0} f(t), \tag{38}$$

where we used the notation

$$T_{\phi_0} g(t) := T_m g(t) + \phi_0(t), \tag{39}$$

assuming $T_{\phi_0}\phi_0 \neq \phi_0$. The lemma below describes the properties of the integral part of the above mapping on the space of continuous weighted functions.

LEMMA 3.5. *Let $\alpha \in (0,1)$ and $\alpha - \beta > 0$. The mapping $T_m$ defined for $g \in C_{1-\alpha}[0,b]$ by*

$$T_m g(t) := \sum_{j=1}^{m-1} c_j (I_{0+}^{\alpha} t^{-\beta})^{m-j} g(t) + (I_{0+}^{\alpha} t^{-\beta})^{m} \Psi(t, g(t)) \tag{40}$$

*is a contraction in the space $C_{1-\alpha}[0,b]$ endowed with the metric $d'_{\kappa}$ for the case $\beta \leq 0$ or $d''_{\kappa_m}$ for $\beta > 0$, respectively, provided parameters $\kappa, \kappa_m$ are large enough and $\Psi \in C([0,b]\times\mathbb{R})$ fulfils the Lipschitz condition (34).*

The proof of the above lemma is a straightforward result of application of parts (1) or (2) from Proposition 2.7, similar to the proof of Proposition 3.1 presented in detail. Now, we are ready to describe the unique solution of equation (35) in $C_{1-\alpha}[0,b]$.

THEOREM 3.6. *Let $\alpha \in (0,1)$, $\alpha - \beta > 0$ and $\Psi \in C([0,b]\times\mathbb{R})$ fulfil the Lipschitz condition (34). Then the fractional differential equation*

$$\Big[(t^{\beta} D_{0+}^{\alpha})^m - \sum_{j=1}^{m-1} c_j (t^{\beta} D_{0+}^{\alpha})^j\Big] f(t) = \Psi(t, f(t)) \tag{41}$$

*has the unique solution $f$ in $C_{1-\alpha}[0,b]$ fulfilling the initial conditions*

$$I_{0+}^{1-\alpha} (t^{\beta} D_{0+}^{\alpha})^j f(t)|_{t=0} = d_j, \tag{42}$$

*where $j = 0, \ldots, m-1$. This solution is a limit of the iterations of mapping $T_{\phi_0}$ (39), (40) generated by the stationary function in the form*

$$\phi_0(t) = \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha-\beta)j+\alpha)}\, t^{(\alpha-\beta)j+\alpha-1} \tag{43}$$

*with*

$$\bar{d}_0 = d_0, \quad \bar{d}_j = \Big(d_j - \sum_{l=0}^{j-1} c_{l+m-j} d_l\Big) \prod_{k=1}^{j} \frac{\Gamma((\alpha-\beta)k)}{\Gamma((\alpha-\beta)(k-1)+\alpha)}. \tag{44}$$

*Proof.* Each stationary function $\phi_0 \in C_{1-\alpha}[0,b]$ of the operator $(t^\beta D^\alpha_{0+})^m$ generates a unique solution of equation (35). This follows from Lemma 3.5 which implies that the mapping $T_{\phi_0}$ is a contraction on the space $C_{1-\alpha}[0,b]$ endowed with the corresponding metric equivalent to (2). To end the proof we shall explicitly show the connection between initial conditions (42) and the stationary function (43). The function $f$ solves simultaneously equation (35) and its integral version

$$f(t) = \sum_{j=1}^{m-1} c_j (I^\alpha_{0+} t^{-\beta})^{m-j} f(t) + (I^\alpha_{0+} t^{-\beta})^m \Psi(t, f(t)) + \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha-\beta)j+\alpha)} t^{(\alpha-\beta)j+\alpha-1}.$$

Integrating both sides of the above equality we obtain for $l = 0,\dots,m-1$

$$\begin{aligned} &I^{1-\alpha}_{0+}(t^\beta D^\alpha_{0+})^l f(t) \\ &\quad = \sum_{j=1}^{m-1} c_j I^{1-\alpha}_{0+}(t^\beta D^\alpha_{0+})^l (I^\alpha_{0+} t^{-\beta})^{m-j} f(t) + I^{1-\alpha}_{0+}(t^\beta D^\alpha_{0+})^l (I^\alpha_{0+} t^{-\beta})^m \Psi(t, f(t)) \\ &\qquad + \sum_{j=0}^{m-1} \frac{\bar{d}_j}{\Gamma((\alpha-\beta)j+\alpha)} I^{1-\alpha}_{0+}(t^\beta D^\alpha_{0+})^l t^{(\alpha-\beta)j+\alpha-1}. \end{aligned}$$

Applying the composition rule from Property 2.3 and taking $t = 0$ we arrive at the following relations

$$d_l = \sum_{j=0}^{l-1} c_{j+m-l} d_j + \bar{d}_l \prod_{k=1}^{l} \frac{\Gamma((\alpha-\beta)(k-1)+\alpha)}{\Gamma((\alpha-\beta)k)}$$

which yield the explicit form of coefficients $\bar{d}_l$ (where $l = 0,\dots,m-1$) from formula (44)

$$\bar{d}_0 = d_0, \quad \bar{d}_j = \Big(d_j - \sum_{l=0}^{j-1} c_{l+m-j} d_l\Big) \prod_{k=1}^{j} \frac{\Gamma((\alpha-\beta)k)}{\Gamma((\alpha-\beta)(k-1)+\alpha)}.$$

The above solution for coefficients $\bar{d}_j$ is unique and this ends the proof. ■

**4. Final remarks.** In the paper we proved existence-uniqueness results for the one-term and multi-term nonlinear fractional differential equations dependent on the left-sided derivative of given order $\alpha \in (0,1)$. To this aim we extended the Bielecki method of equivalent norms [2] known from differential equations theory to fractional differential equations by application of the two-parameter Mittag–Leffler functions in construction of new norms (21)–(24). The developed method of proving the existence of the solutions yields in arbitrarily long interval $[0,b]$ the unique solution, fulfilling the corresponding set of initial conditions. Let us observe that the standard scaling (using a composed exponential function) is not effective in the considered case as intermediate fractional integral equations (32), (33), (37)) have singular kernels. The new method of construction of the equivalent norms appears to be useful in solving linear multi-term fractional differential equations with basic order $\alpha \in (n-1, n)$ and nonlinear multi-term fractional differential equations with Hadamard derivative [11, 12]. The discussed results can easily be extended to the equations given on space of vector functions. Another interesting field where such method of proof should be effective are sequential fractional differential equations in a sense given in [17, 18], where no basic order of derivatives is assumed.

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