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DUNFORD–PETTIS OPERATORS ON THE SPACE OF BOCHNER INTEGRABLE FUNCTIONS

MARIAN NOWAK

University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics
ul. Prof. Szafrana 4A, 65–516 Zielona Góra, Poland
E-mail: M.Nowak@wmie.uz.zgora.pl

Abstract. Let (Ω, Σ, μ) be a finite measure space and let X be a real Banach space. Let $L^\Phi(X)$ be the Orlicz–Bochner space defined by a Young function Φ. We study the relationships between Dunford–Pettis operators T from $L^1(X)$ to a Banach space Y and the compactness properties of the operators T restricted to $L^\Phi(X)$. In particular, it is shown that if X is a reflexive Banach space, then a bounded linear operator $T : L^1(X) \to Y$ is Dunford–Pettis if and only if T restricted to $L^\infty(X)$ is $(\tau(L^\infty(X), L^1(X^*)), \|\cdot\|_Y)$-compact.

1. Introduction and preliminaries. Recall that a bounded linear operator T between two Banach spaces is a Dunford–Pettis operator if T maps weakly convergent sequences onto norm convergent sequences. J. Bourgain [B, Proposition 1] showed that a bounded linear operator T from L^1 to a Banach space Y is a Dunford–Pettis operator if and only if T restricted to L^p for some $p \in (1, \infty]$ is compact. The purpose of this paper is to extend and strengthen this result for operators defined on the space of Bochner integrable functions $L^1(X)$. We study the relationships between Dunford–Pettis operators $T : L^1(X) \to Y$ and the compactness properties of T restricted to Orlicz–Bochner spaces $L^\Phi(X)$ (see Theorems 2.1, 2.3 and Corollary 2.5 below).

We denote by $\sigma(L, K)$ the weak topology on L with respect to the dual pair $\langle L, K\rangle$. Let (L, ξ) and (M, η) be Hausdorff locally convex spaces. Recall that a linear operator $S : L \to M$ is (ξ, η)-*compact* if there exists a neighbourhood U of 0 for ξ such that $S(U)$ is a relatively compact set in (M, η). By $\mathrm{Bd}(L, \xi)$ we denote the collection of all ξ-bounded sets in L. Moreover, $(L, \xi)^*$ stands for the topological dual of (L, ξ).

For terminology and basic properties concerning Banach function spaces we refer to [KA]. Now we recall terminology concerning Orlicz space (see [Lu], [RR] for more

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details). From now on we assume that (Ω, Σ, μ) is a finite measure space. By a Young function we mean here a non-zero convex, left continuous function $\Phi : [0, \infty) \to [0, \infty]$ that is vanishing and continuous at 0. We say that Φ jumps to infinity, if $\Phi(t) = \infty$ for all $t \geq t_0 > 0$.

The Orlicz space $L^\Phi = \{u \in L^0 : \int_\Omega \Phi(\lambda |u(\omega)|)\, d\mu < \infty \text{ for some } \lambda > 0\}$ can be equipped with the complete Riesz norm:

$$\|u\|_\Phi = \inf\{\lambda > 0 : \int_\Omega \Phi(|u(\omega)|/\lambda)\, d\mu \leq 1\}.$$

Then L^Φ is a perfect Banach function space and $L^\infty \subset L^\Phi \subset L^1$, where the inclusion maps are continuous. Moreover, the Köthe dual $(L^\Phi)'$ of L^Φ is equal to the Orlicz space L^{Φ^*}, where Φ^* stands for the Young function complementary to Φ in the sense of Young. The associated norm $\|\cdot\|^0_{\Phi^*}$ on L^{Φ^*} (called the Orlicz norm) can be defined by

$$\|\cdot\|^0_{\Phi^*} = \sup\Big\{\int_\Omega |u(\omega)v(\omega)|\, d\mu : u \in L^\Phi,\ \|u\|_\Phi \leq 1\Big\}.$$

Note that if $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$, then $L^\Phi \subsetneq L^1$ and

$$L^\infty \subsetneq (L^{\Phi^*})_a = E^{\Phi^*} = \Big\{v \in L^{\Phi^*} : \int_\Omega \Phi(\lambda |v(\omega)|)\, d\mu < \infty \text{ for all } \lambda > 0\Big\}.$$

In particular, if Φ jumps to infinity, then $L^\Phi = L^\infty$. If $\lim_{t\to\infty} \frac{\Phi(t)}{t} < \infty$, then $L^\Phi = L^1$ and $L^{\Phi^*} = L^\infty$.

From now on we assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are real Banach spaces and X^*, Y^* denote their Banach duals. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$.

For $f \in L^0(X)$ let $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Then the space

$$L^\Phi(X) = \{f \in L^0(X) : \widetilde{f} \in L^\Phi\}$$

provided with the norm $\|f\|_{L^\Phi(X)} := \|\widetilde{f}\|_\Phi$ is a Banach space and is usually called an *Orlicz–Bochner space* (see [CM], [L], [RR] for more details).

Now we recall terminology and basic results concerning duality of the spaces $L^\Phi(X)$ (see [Bu1], [Bu2]). A linear functional F on $L^\Phi(X)$ is said to be *order continuous* if $F(f_\alpha) \to 0$ whenever $\widetilde{f}_\alpha \xrightarrow{(o)} 0$ in L^Φ. The set of all order continuous functionals on $L^\Phi(X)$ will be denoted by $L^\Phi(X)^\sim_n$ and called the *order continuous dual* of $L^\Phi(X)$. Then $L^\Phi(X)^* = L^\Phi(X)^\sim_n$ if Φ satisfies the so called Δ_2-condition, i.e., $\limsup_{t\to\infty} \frac{\Phi(2t)}{\Phi(t)} < \infty$. Due to Bukhvalov (see [Bu1], [Bu2]) if X^* has the Radon-Nikodym property (in particular, X is reflexive), then $L^\Phi(X)^\sim_n$ can be identified with $L^{\Phi^*}(X^*)$ throughout the mapping: $L^{\Phi^*}(X^*) \ni g \mapsto F_g \in L^\Phi(X)^\sim_n$, where

$$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle\, d\mu \quad \text{for all } f \in L^\Phi(X).$$

Note that $L^1(X)^\sim_n = L^1(X)^* = \{F_g : g \in L^\infty(X^*)\}$ if X is reflexive.

For a subset H of $L^\Phi(X)$ let $\widetilde{H} = \{\widetilde{f} : f \in H\}$. By $B_{L^\Phi(X)}$ (resp. B_{L^Φ}) we will denote closed unit ball in $(L^\Phi(X), \|\cdot\|_{L^\Phi(X)})$ (resp. $(L^\Phi, \|\cdot\|_\Phi)$). Then $\widetilde{B}_{L^\Phi(X)} = B_{L^\Phi}$.

The following characterization of relative $\sigma(L^\Phi(X), L^\Phi(X)^\sim_n)$-compactness in $L^\Phi(X)$ will be of importance (see [N1, Theorem 2.7, Proposition 2.1]).

PROPOSITION 1.1. *Assume that X is a reflexive Banach space and Φ is a Young function. Then for a subset H of $L^\Phi(X)$ the following statements are equivalent*:

(i) *H is relatively $\sigma(L^\Phi(X), L^{\Phi^*}(X^*))$-compact.*
(ii) *$\widetilde{H}$ is relatively $\sigma(L^\Phi, L^{\Phi^*})$-compact.*
(iii) *The functional $p_{\widetilde{H}}$ on L^{Φ^*} defined by $p_{\widetilde{H}}(v) = \sup_{u\in\widetilde{H}} \int_\Omega |u(\omega)v(\omega)|\,d\mu$ is an order continuous seminorm.*

2. Dunford–Pettis operators on $L^1(X)$. We study the relationships between Dunford–Pettis operators $T : L^1(X) \to Y$ and the compactness properties of the operator T restricted to $L^\Phi(X)$. Note that a bounded linear operator $T : L^1(X) \to Y$ is a Dunford–Pettis operator if and only if T maps relatively weakly compact sets in $L^1(X)$ onto relatively norm compact sets in Y (see [AB, §19]).

Let $i_\Phi : L^\Phi(X) \to L^1(X)$ stand for the inclusion map.

THEOREM 2.1. *Let $T : L^1(X) \to Y$ be a bounded linear operator. Assume that Φ is Young function and let $T \circ i_\Phi : L^\Phi(X) \to Y$ be a $(\|\cdot\|_\Phi, \|\cdot\|_Y)$-compact operator. Then T is a Dunford–Pettis operator.*

Proof. We see that $T(B_{L^\Phi(X)})$ is relatively compact in $(Y, \|\cdot\|_Y)$. Let H be a relatively $\sigma(L^1(X), L^1(X)^*)$-compact subset of $L^1(X)$. To show that $T(H)$ is relatively compact in $(Y, \|\cdot\|_Y)$ it is enough to show in view of [D, p. 5] that for every $\varepsilon > 0$ there exists a relatively compact subset K_ε of $(Y, \|\cdot\|_Y)$ such that

$$T(H) \subset \varepsilon B_Y + K_\varepsilon,$$

where B_Y is a closed unit ball in Y. Note that the set $\widetilde{H}$ is uniformly integrable in L^1 (see [DU, Theorem 4, p. 104]). For $f \in L^1(X)$ and $\lambda > 0$ let

$$A_{f,\lambda} = \{\omega \in \Omega : \widetilde{f}(\omega) > \lambda\}.$$

Then

$$\lim_{\lambda\to\infty} \sup_{f\in H} \int_{A_{f,\lambda}} \widetilde{f}(\omega)\,d\mu = \lim_{\lambda\to\infty} \sup_{f\in H} \|1_{A_{f,\lambda}} f\|_{L^1(X)} = 0.$$

Let $\varepsilon > 0$ be given. Then there exists $\lambda_\varepsilon > 0$ such that for each $f \in H$ we have

$$\|1_{A_{f,\lambda_\varepsilon}} f\|_{L^1(X)} \le \frac{\varepsilon}{\|T\|}.$$

Hence for $f \in H$ we get

$$\|T(1_{A_{f,\lambda_\varepsilon}} f)\|_Y \le \|T\| \cdot \|1_{A_{f,\lambda_\varepsilon}} f\|_{L^1(X)} \le \varepsilon.$$

Moreover, $1_{\Omega\setminus A_{f,\lambda_\varepsilon}}(\omega)\widetilde{f}(\omega) \le \lambda_\varepsilon$ for $\omega \in \Omega$, so $1_{\Omega\setminus A_{f,\lambda_\varepsilon}} f \in L^\infty(X) \subset L^\Phi(X)$. Since $\|h\|_{L^\Phi(X)} \le a\|h\|_{L^\infty(X)}$ for some $a > 0$ and all $h \in L^\infty(X)$, we get

$$\|1_{\Omega\setminus A_{f,\lambda_\varepsilon}} f\|_{L^\Phi(X)} \le a\lambda_\varepsilon.$$

Hence

$$T(f) = T(1_{A_{f,\lambda_\varepsilon}} f) + T(1_{\Omega\setminus A_{f,\lambda_\varepsilon}} f) \in \varepsilon B_Y + a\lambda_\varepsilon T(B_{L^\Phi(X)}).$$

This means that the set $T(H)$ is relatively compact in $(Y, \|\cdot\|_Y)$, as desired. ■

From now we assume that Φ is a Young function such that $\lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty$. Let $\mathcal{T}_\Phi$ be the topology on $L^\Phi(X)$ generated by the norm $\|\cdot\|_{L^\Phi(X)}$ on $L^\Phi(X)$, and let $\mathcal{T}_0$ stand for the complete F-norm $\|\cdot\|_{L^0(X)}$-topology on $L^0(X)$ that generates convergence in measure. Then the mixed topology $\gamma[\mathcal{T}_\Phi,\mathcal{T}_0|_{L^\Phi(X)}]$ (briefly, γ_Φ) on $L^\Phi(X)$ is the finest Hausdorff locally convex topology on $L^\Phi(X)$ which agrees with $\mathcal{T}_0|_{L^\Phi(X)}$ on $\|\cdot\|_{L^\Phi(X)}$-bounded subsets of $L^\Phi(X)$ (see [W, 2.2.2], [F1, Theorem 3.3]). Moreover, we have (see [F2, Proposition 2.1]):

$$\mathrm{Bd}(L^\Phi(X),\gamma_\Phi)=\mathrm{Bd}(L^\Phi(X),\|\cdot\|_{L^\Phi(X)}). \tag{2.1}$$

This means that $(L^\Phi(X),\gamma_\Phi)$ is a generalized DF-space (see [Ru, Definition 1.1]).

It is known that a linear operator $T:L^\Phi(X)\to Y$ is $(\gamma_\Phi,\|\cdot\|_Y)$-continuous if and only if T is $(\gamma_\Phi,\|\cdot\|_Y)$-linear, i.e., $\|T(f_n)\|_Y\to 0$ whenever $\|f_n\|_{L^0(X)}\to 0$ and $\sup_n\|f_n\|_{L^\Phi(X)}<\infty$ (see [W, Theorem 2.6.1(iii)], [F2, Proposition 2.3]).

We shall need the following lemma.

Lemma 2.2. *Assume that Φ is a Young function such that $\lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty$ and X is a reflexive Banach space. Then $i_\Phi:L^\Phi(X)\to L^1(X)$ is a $(\|\cdot\|_{L^\Phi(X)},\sigma(L^1(X),L^1(X)^*))$-compact operator.*

Proof. To show that $B_{L^\Phi(X)}$ is a relatively $\sigma(L^1(X),L^1(X)^*)$-compact subset of $L^1(X)$, in view of Proposition 1.1 it is enough to show that B_{L^Φ} is relatively $\sigma(L^1,L^\infty)$-compact in L^1, that is, the seminorm on L^∞ defined by

$$p_{B_{L^\Phi}}(v):=\sup_{u\in B_{L^\Phi}}\int_\Omega |u(\omega)v(\omega)|\,d\mu$$

is order continuous. Indeed, note that $p_{B_{L^\Phi}}(v)=\|\cdot\|^0_{\Phi^*}$ for $v\in L^\infty$, where $L^\infty\subsetneq E^{\Phi^*}=(L^{\Phi^*})_a$. Thus the proof is complete. ∎

Now we are ready to prove our main result.

Theorem 2.3. *Assume that Φ is a Young function such that $\lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty$ and X is a reflexive Banach space. Let $T:L^1(X)\to Y$ be a Dunford–Pettis operator. Then the operator $T\circ i_\Phi:L^\Phi(X)\to Y$ is $(\gamma_\Phi,\|\cdot\|_Y)$-compact.*

Proof. Since X is supposed to be reflexive, in view of [F1, Theorem 3.2] we have

$$(L^\Phi(X),\gamma_\Phi)^*=\{F_g:g\in E^{\Phi^*}(X^*)\}.$$

First, we shall show that $T\circ i_\Phi:L^\Phi(X)\to Y$ is $(\gamma_\Phi,\|\cdot\|_Y)$-linear. Indeed, let (f_n) be a sequence in $L^\Phi(X)$ such that $\|f_n\|_{L^0(X)}\to 0$ and $\sup_n\|f_n\|_{L^\Phi(X)}<\infty$. Then $f_n\to 0$ for γ_Φ (see [F1, Theorem 3.1]), and it follows that $f_n\to 0$ for $\sigma(L^\Phi(X),E^{\Phi^*}(X^*))$ because $\sigma(L^\Phi(X),\ E^{\Phi^*}(X^*))\subset\gamma_\Phi$. Hence $f_n\to 0$ for $\sigma(L^1(X),L^1(X)^*)$ because $\sigma(L^1(X),L^1(X)^*)=\sigma(L^1(X),L^\infty(X^*))$ and $L^\Phi(X)\subset L^1(X)$ and $L^\infty(X^*)\subset E^{\Phi^*}(X^*)$. Since T is a Dunford–Pettis operator, we get $\|T(f_n)\|_Y\to 0$. This means that $T\circ i_\Phi$ is $(\gamma_{L^\Phi(X)},\|\cdot\|_Y)$-continuous.

By Lemma 2.2 the mapping $T\circ i_\Phi$ is $(\|\cdot\|_{L^\Phi(X)},\|\cdot\|_Y)$-compact. Hence, in view of (2.1) $T\circ i_\Phi$ transforms γ_Φ-bounded sets in $L^\Phi(X)$ onto relatively $\|\cdot\|_Y$-compact sets in Y. Making use of [Ru, Theorem 3.1] we conclude that $T\circ i_\Phi$ is $(\gamma_\Phi,\|\cdot\|_Y)$-compact, as desired. ∎

As an application of Theorems 2.1 and 2.3 we get:

COROLLARY 2.4. *Assume that* Φ *is a Young function such that* $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ *and* X *is a reflexive Banach space. Then for a bounded linear operator* $T : L^1(X) \to Y$ *the following statements are equivalent*:

(i) T *is a Dunford–Pettis operator.*
(ii) $T \circ i_\Phi : L^\Phi(X) \to Y$ *is* $(\gamma_\Phi, \|\cdot\|_Y)$*-compact.*
(iii) $T \circ i_\Phi : L^\Phi(X) \to Y$ *is* $(\|\cdot\|_{L^\Phi(X)}, \|\cdot\|_Y)$*-compact.*

In particular, if X is reflexive, then the mixed topology γ_∞ on $L^\infty(X)$ coincides with the Mackey topology $\tau(L^\infty(X), L^1(X^*))$ (see [N2, Corollary 4.4]). Hence, as a consequence of Corollary 2.4 we get:

COROLLARY 2.5. *Assume that* X *is a reflexive Banach space. Then for a bounded linear operator* $T : L^1(X) \to Y$ *the following statements are equivalent*:

(i) T *is a Dunford–Pettis operator.*
(ii) $T \circ i_\infty : L^\infty(X) \to Y$ *is* $(\tau(L^\infty(X), L^1(X^*)), \|\cdot\|_Y)$*-compact.*
(iii) $T \circ i_\infty : L^\infty(X) \to Y$ *is* $(\|\cdot\|_{L^\infty(X)}, \|\cdot\|_Y)$*-compact.*

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