MARCINKIEWICZ CENTENARY VOLUME
BANACH CENTER PUBLICATIONS, VOLUME 95
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

SOME FOOTPRINTS OF MARCINKIEWICZ IN SUMMABILITY THEORY

FERENC WEISZ
Department of Numerical Analysis, Eötvös L. University
H-1117 Budapest, Pázmány P. sétány 1/C., Hungary
E-mail: weisz@numanal.inf.elte.hu

Dedicated to Józef Marcinkiewicz on his 100th birthday
and the 70th anniversary of his death

Abstract. Four basic results of Marcinkiewicz are presented in summability theory. We show that setting out from these theorems many mathematicians have reached several nice results for trigonometric, Walsh– and Ciesielski–Fourier series.

1. Introduction. In this survey paper we will consider Marcinkiewicz's work in summability theory and its impact up to the present days. We present four of his fundamental theorems and several (recent) extensions and generalizations. We investigate convergence and summations of one- and multi-dimensional trigonometric, Walsh– and Ciesielski–Fourier series. First we give the corresponding results in the one-dimensional case and then the generalizations for higher dimensions. Two types of summability methods will be investigated, the Fejér and Cesàro or (C,α) methods. The Fejér summation is a special case of the Cesàro method, $(C,1)$ is exactly the Fejér method.

In the multi-dimensional case three types of convergence and maximal operators are considered, the restricted (convergence over the diagonal or over a cone), the unrestricted (convergence over $\mathbb{N}^d$) and the Marcinkiewicz-type convergence. Marcinkiewicz proved that the Fejér means $\sigma_n f$ of a two-dimensional integrable function f converge a.e. to f as $n \to \infty$ over a cone. Another theorem of Marcinkiewicz says that the so called Marcinkiewicz means (i.e. the arithmetic means of the cubic partial sums taken on the diagonal) of a two-dimensional function $f \in L \log L$ converge a.e. to f. We introduce

2010 *Mathematics Subject Classification*: Primary 42B08, 42A24, Secondary 42B30, 42C10, 40F05, 41A15.
Key words and phrases: Trigonometric, Walsh– and Ciesielski–Fourier series, Fejér and Cesàro summability, Marcinkiewicz summability, Hardy spaces, strong summability.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc95-0-25

classical and martingale Hardy spaces $H_p(\mathbb{X})$ (where $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0,1)$) and prove that the maximal operators of the summability means are bounded from $H_p(\mathbb{X})$ to $L_p(\mathbb{X})$ whenever $p > p_0$ for some $p_0 < 1$. The exact value of p_0, which depends on the type of the Fourier series and on the dimension, is given in each case. For $p = 1$ we obtain a weak type inequality by interpolation, which implies by the density theorem of Marcinkiewicz the a.e. convergence of the summability means just mentioned. The a.e. convergence and the weak type inequality are proved usually with the help of a Calderón–Zygmund type decomposition lemma. However, this lemma does not work in higher dimensions. Our method, that can be applied in higher dimensions, too, can be regarded as a new method to prove the a.e. convergence and weak type inequalities.

Finally, the strong summability result of Marcinkiewicz and Zygmund is generalized for multi-dimensional trigonometric, Walsh– and Ciesielski–Fourier series. This paper was the base of my talk given at the Józef Marcinkiewicz Centenary Conference, June 2010, in Poznań (Poland).

2. Trigonometric and Walsh system. We consider either the torus $\mathbb{X} = \mathbb{T}$ or the unit interval $\mathbb{X} = [0,1)$ with the Lebesgue measure λ. We briefly write $L_p(\mathbb{X})$ instead of the real $L_p(\mathbb{X},\lambda)$ space equipped with the norm (or quasinorm) $\|f\|_p := (\int_{\mathbb{X}} |f|^p\,d\lambda)^{1/p}$ $(0 < p \le \infty)$. The weak $L_p(\mathbb{X})$ space $L_{p,\infty}(\mathbb{X})$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}$ is a quasi-normed space. It is easy to see that

$$L_p(\mathbb{X}) \subset L_{p,\infty}(\mathbb{X}) \qquad \text{and} \qquad \|\cdot\|_{p,\infty} \le \|\cdot\|_p$$

for each $0 < p < \infty$.

The *Rademacher functions* are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0,\frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2},1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \qquad (x \in [0,1),\ n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the *one-dimensional Walsh system*:

$$w_n := \prod_{k=0}^{\infty} {r_k}^{n_k} \qquad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k \qquad (0 \le n_k < 2).$$

In what follows let $\phi_n(x)$ denote the trigonometric system $e^{2\pi \imath n\cdot x}$ defined on $\mathbb{T}$ or the Walsh system $\phi_n(x) := w_n(x)$ defined on the unit interval.

In this paper the constants C_p depend only on p and may denote different constants in different contexts.

3. Partial sums of one-dimensional Fourier series. The *Fourier coefficients* and *partial sums* of the Fourier series of $f \in L_1(\mathbb{X})$ ($\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = [0,1)$) are defined by

$$\hat{f}(k) := \int_{\mathbb{X}} f\phi_k \, d\lambda, \qquad s_n f := \sum_{|k|\le n} \hat{f}(k)\phi_k \qquad (n \in \mathbb{N}).$$

The definition of the Fourier coefficients can be extended easily to distributions and martingales.

One of the deepest results in harmonic analysis is Carleson's result, i.e. the partial sums of the Fourier series converge a.e. to $f \in L_p(\mathbb{X})$ $(1 < p < \infty)$ (see Carleson [6], Hunt [25] for trigonometric series and Billard [4], Sjölin [47], Schipp [42] for Walsh series).

Theorem 3.1. *If $f \in L_p(\mathbb{X})$ for some $1 < p < \infty$, then*

$$\Big\| \sup_{n\in\mathbb{N}} |s_n f| \Big\|_p \le C_p \|f\|_p$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \textit{a.e. and in } L_p\textit{-norm.}$$

4. Summability of one-dimensional Fourier series. The preceding theorem does not hold, if $p = 1$, however it can be generalized for $p = 1$ with the help of some summability methods. Summability is intensively studied in the literature, we refer at this time only for the books Stein and Weiss [50], Butzer and Nessel [5], Trigub and Belinsky [51], Grafakos [23] and Weisz [59] and the references therein. Here we consider the *Fejér* and *Cesàro* (or (C,α)) means defined by

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} s_k f = \sum_{|j|\le n} \Big(1 - \frac{|j|}{n}\Big) \hat{f}(j)\phi_j$$

and

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_k f = \frac{1}{A_{n-1}^\alpha} \sum_{|j|\le n} A_{n-1-|j|}^\alpha \hat{f}(j)\phi_j,$$

where

$$A_k^\alpha := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!}.$$

It is known (Zygmund [66]) that

$$A_k^\alpha \sim k^\alpha \qquad (k \in \mathbb{N}).$$

If $\alpha = 1$ then we get the Fejér means. We will suppose always that $0 < \alpha \le 1$. The case $\alpha > 1$ can be led back to $\alpha = 1$. The *maximal Cesàro operator*

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|$$

is of weak type $(1,1)$ (see Zygmund [66] for the trigonometric system and Schipp [40] and Weisz [56] for the Walsh system), i.e.

Theorem 4.1. *If $0 < \alpha \le 1$ and $f \in L_1(\mathbb{X})$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_1.$$

This weak type $(1,1)$ inequality and the density argument of Marcinkiewicz and Zygmund [30] imply the well known theorem of Fejér [12] and Lebesgue [26] with $\alpha = 1$. Riesz [33] proved it for other α's and Fine [13], Schipp [40] and Weisz [56] for the Walsh system.

Corollary 4.2. *If $0 < \alpha \le 1$ and $f \in L_1(\mathbb{X})$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad a.e.$$

The next density theorem of Marcinkiewicz and Zygmund [30] is fundamental and similar to the Banach–Steinhaus theorem about the norm convergence of operators.

Theorem 4.3. *Suppose that X is a normed space of measurable functions and $X_0 \subset X$ is dense in X. Let T and T_n $(n \in \mathbb{N})$ be bounded linear operators from X to L_p for some $1 \le p < \infty$ such that for each $f \in X_0$, $Tf = \lim_{n\to\infty} T_n f$ a.e. If*

$$\sup_{\rho>0} \rho\lambda(Tf > \rho)^{1/p} \le C\|f\|_X \qquad (f \in X)$$

and

$$\sup_{\rho>0} \rho\lambda(T^* f > \rho)^{1/p} \le C\|f\|_X \qquad (f \in X),$$

where

$$T^* f := \sup_{n\in\mathbb{N}} |T_n f| \qquad (f \in X),$$

then

$$Tf = \lim_{n\to\infty} T_n f \quad a.e.$$

for every $f \in X$.

In Corollary 4.2 we apply this theorem for $T = \mathrm{id}$, $T_n = \sigma_n^\alpha$, $p = 1$ and $X = L_1(\mathbb{X})$. The dense set X_0 is the set of the trigonometric or Walsh polynomials. It is easy to see that Corollary 4.2 holds for $f \in X_0$.

5. Multi-dimensional partial sums. Let us fix $d \ge 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \ne \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d times. The $L_p(\mathbb{X}^d)$ spaces are defined in the usual way. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_2 := \Big(\sum_{k=1}^d |x_k|^2\Big)^{1/2}, \qquad |x| := \sup_{k=1,\ldots,d} |x_k|.$$

The d-dimensional trigonometric and Walsh system is introduced as a Kronecker product by

$$\phi_k(x) := \phi_{k_1}(x_1) \cdots \phi_{k_d}(x_d),$$

where $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, $x = (x_1, \ldots, x_d) \in \mathbb{X}^d$. The multi-dimensional *Fourier coefficients*, *rectangular partial sums* and *cubic partial sums* of the Fourier series of

$f \in L_1(\mathbb{X}^d)$ are defined by

$$\hat{f}(k) := \int_{\mathbb{X}^d} f\phi_k \, d\lambda \qquad (k \in \mathbb{N}^d), \tag{1}$$

$$s_n f := \sum_{|k_1| \le n_1} \dots \sum_{|k_d| \le n_d} \hat{f}(k)\phi_k \qquad (n \in \mathbb{N}^d) \tag{2}$$

and

$$s_{\mathbf{n}} f(x) := \sum_{k \in \mathbb{Z}^d, |k| \le n} \hat{f}(k)\phi_k(x) \qquad (n \in \mathbb{N}).$$

It is easy to see that

$$\lim_{n\to\infty} s_n f = f \qquad \text{in } L_p\text{-norm as } n \to \infty,$$

whenever $f \in L_p(\mathbb{X}^d)$ $(1 < p < \infty)$. The analogue of Carleson's theorem does not hold in higher dimensions for the rectangular partial sums. However, it is true for the cubic partial sums (Fefferman [11] and Grafakos [23] for the trigonometric system and Móricz [31] for the Walsh system).

Theorem 5.1. *If $f \in L_p(\mathbb{X}^d)$ for some $1 < p < \infty$, then for the trigonometric Fourier series*

$$\Big\| \sup_{n\in\mathbb{N}} |s_{\mathbf{n}} f| \Big\|_p \le C_p \|f\|_p$$

and

$$\lim_{n\to\infty} s_{\mathbf{n}} f = f \qquad \text{a.e.}$$

The same result holds for the Walsh–Fourier series if $p = 2$.

It is an open question, whether this theorem holds in the last case for $p \neq 2$ (cf. Schipp, Wade, Simon and Pál [45]).

6. Summability of multi-dimensional Fourier series and Hardy spaces. The summability results can be generalized for higher dimensions in several ways. We consider three methods, which were introduced/investigated by Marcinkiewicz. He proved fundamental results in this topic. The *Fejér* and *Cesàro means* of f are defined by

$$\sigma_n f := \frac{1}{\prod_{i=1}^d n_i} \sum_{k_1=1}^{n_1-1} \dots \sum_{k_d=1}^{n_d-1} s_k f = \sum_{|j_1| \le n_1} \dots \sum_{|j_d| \le n_d} \prod_{i=1}^d \Big(1 - \frac{|j_i|}{n_i}\Big) \hat{f}(j)\phi_j,$$

and

$$\begin{aligned}\sigma_n^\alpha f &:= \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{k_1=1}^{n_1-1} \dots \sum_{k_d=1}^{n_d-1} \Big(\prod_{i=1}^d A_{n_i-1-k_i}^{\alpha-1} \Big) s_k f \\ &= \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|j_1| \le n_1} \dots \sum_{|j_d| \le n_d} \Big(\prod_{i=1}^d A_{n_i-1-|j_i|}^\alpha \Big) \hat{f}(j)\phi_j,\end{aligned}$$

respectively. For a given $\tau \geq 0$ the *restricted and non-restricted maximal operators* are defined by

$$\sigma_{\Box}^{\alpha} f := \sup_{\substack{2^{-\tau} \leq n_i/n_j \leq 2^{\tau} \\ i,j=1,\ldots,d}} |\sigma_n^{\alpha} f|, \qquad \sigma_*^{\alpha} f := \sup_{n \in \mathbb{N}^d} |\sigma_n^{\alpha} f|.$$

Note that we can define the Cesàro means and operators also for a vector $\alpha = (\alpha_1, \ldots, \alpha_d)$, but for simplicity we assume that $\alpha \in (0,1]$. The next result follows easily from the corresponding one-dimensional result by iteration.

THEOREM 6.1. *If* $0 < \alpha \leq 1$ *and* $1 < p \leq \infty$, *then*

$$\|\sigma_*^{\alpha} f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}^d)). \tag{3}$$

Moreover, for all $f \in L_p(\mathbb{X}^d)$ $(1 < p < \infty)$,

$$\lim_{n \to \infty} \sigma_n^{\alpha} f = f \qquad \text{a.e. and in } L_p\text{-norm}. \tag{4}$$

The L_p-norm convergence holds also if $p = 1$. Here $n \to \infty$ means that $\min(n_1, \ldots, n_d) \to \infty$ (the Pringsheim's sense of convergence). Inequality (3) does not hold for $p \leq 1$. However, with the help of Hardy spaces we extend it to $p \leq 1$.

6.1. Hardy spaces. We give a common definition of the periodic and dyadic Hardy spaces. Let us define the periodic and dyadic d-dimensional *Poisson kernel* by

$$P_t^{\mathbb{T}^d}(x) := \sum_{m \in \mathbb{Z}^d} e^{-t\|m\|_2} e^{2\pi \imath m \cdot x} \qquad (x \in \mathbb{T}^d,\ t > 0)$$

and

$$P_t^{[0,1)^d}(x) := 2^{nd} 1_{[0,2^{-n}) \times \ldots \times [0,2^{-n})}(x) \quad \text{if } n \leq t < n+1 \qquad (x \in [0,1)^d).$$

A distribution or martingale f is in the *periodic* or *dyadic Hardy space* $H_p^{\Box}(\mathbb{X}^d)$, $H_p(\mathbb{X}^d)$ and $H_p^i(\mathbb{X}^d)$ $(0 < p \leq \infty, i = 1, \ldots, d)$ if

$$\|f\|_{H_p^{\Box}(\mathbb{X}^d)} := \big\| \sup_{0<t} |f * P_t^{\mathbb{X}^d}| \big\|_p < \infty, \tag{5}$$

$$\|f\|_{H_p(\mathbb{X}^d)} := \big\| \sup_{t_k>0, k=1,\ldots,d} |(f * (P_{t_1}^{\mathbb{X}} \otimes \ldots \otimes P_{t_d}^{\mathbb{X}}))| \big\|_p < \infty, \tag{6}$$

and

$$\|f\|_{H_p^i(\mathbb{X}^d)} := \big\| \sup_{t_k>0, k=1,\ldots,d; k\neq i} |(f * (P_{t_1}^{\mathbb{X}} \otimes \ldots \otimes P_{t_{i-1}}^{\mathbb{X}} \otimes P_{t_{i+1}}^{\mathbb{X}} \otimes \ldots \otimes P_{t_d}^{\mathbb{X}}))| \big\|_p < \infty,$$

respectively. It is known (see e.g. Stein [48] or Weisz [59]) that

$$H_p^{\Box}(\mathbb{X}^d) \sim H_p(\mathbb{X}^d) \sim H_p^i(\mathbb{X}^d) \sim L_p(\mathbb{X}^d) \qquad (1 < p \leq \infty).$$

Moreover, each Hardy space has an atomic decomposition, in other words every function from the Hardy space can be decomposed into the sum of simple functions, the so called atoms (e.g. Stein [48], Lu [27] and Weisz [59]).

6.2. Restricted summability. In this subsection we investigate the operator $\sigma_{\Box}^{\alpha}$ and the convergence of $\sigma_n^{\alpha} f$ over the cone $\{n \in \mathbb{N}^d : 2^{-\tau} \leq n_i/n_j \leq 2^{\tau};\ i,j = 1, \ldots, d\}$, where $\tau \geq 0$ is fixed.

THEOREM 6.2. *If* $0 < \alpha \le 1$ *and* $p_0(\mathbb{X}) < p \le \infty$, *then*

$$\|\sigma_{\square}^{\alpha} f\|_p \le C_p \|f\|_{H_p^{\square}} \qquad (f \in H_p^{\square}(\mathbb{X}^d)), \tag{7}$$

where $p_0([0,1)) = 1/(\alpha+1)$ *and* $p_0(\mathbb{T}) = \max\{d/(d+1), 1/(\alpha+1)\}$.

This theorem is due to the author [55, 59]. By the atomic decomposition (7) has to be proved for atoms, only. It is known that $p_0(\mathbb{X})$ is the best possible constant, in other words, if $p \le p_0(\mathbb{X})$ then $\sigma_{\square}^{\alpha}$ is not bounded anymore (see Stein, Taibleson and Weiss [49], Simon and Weisz [46], Goginava [21]).

THEOREM 6.3. *The operator* $\sigma_{\square}^{\alpha}$ *is not bounded from* $H_p^{\square}(\mathbb{X}^d)$ *to* $L_p(\mathbb{X}^d)$ *if* $0 < p \le p_0(\mathbb{X})$.

However, in the one-dimensional case the operator $\sigma_{\square}^{\alpha}$ satisfies a weak type inequality, i.e. it is bounded from $H_p^{\square}(\mathbb{X})$ to weak $L_p(\mathbb{X})$ for the endpoint $p = p_0(\mathbb{X}) = 1/(\alpha+1)$ (see Weisz [46, 59] for the result and Goginava [21] for the counterexample).

THEOREM 6.4. *If* $f \in H_{1/(\alpha+1)}^{\square}(\mathbb{X})$, *then*

$$\|\sigma_{\square}^{\alpha} f\|_{1/(\alpha+1),\infty} = \sup_{\rho>0} \rho \lambda(\sigma_{\square}^{\alpha} f > \rho)^{\alpha+1} \le C \|f\|_{H_{1/(\alpha+1)}^{\square}}.$$

This inequality does not hold for higher dimensions.

Of course Theorem 6.4 is not true for $p < 1/(\alpha+1)$, because then (7) would hold for $p < 1/(\alpha+1)$ by interpolation.

The next corollary can be obtained from Theorem 6.2 by interpolation. For the basic definitions and theorems on interpolation theory see Bergh and Löfström [3] and Bennett and Sharpley [2] or Weisz [52, 59]. The interpolation of martingale Hardy spaces was worked out in [52]. The method of Theorem 6.2 can be regarded also as an alternative tool to the Calderón–Zygmund decomposition lemma for proving weak type $(1,1)$ inequalities. In many cases this theorem can be applied better and more simply than the Calderón–Zygmund decomposition lemma.

COROLLARY 6.5. *If* $0 < \alpha \le 1$ *and* $f \in L_1(\mathbb{X}^d)$, *then*

$$\sup_{\rho>0} \rho \lambda(\sigma_{\square}^{\alpha} f > \rho) \le C \|f\|_1.$$

The set of the trigonometric and Walsh polynomials is dense in $L_1(\mathbb{X}^d)$, so Corollary 6.5 and Theorem 4.3 imply the convergence of the Cesàro means over a cone.

COROLLARY 6.6. *If* $0 < \alpha \le 1$ *and* $f \in L_1(\mathbb{X}^d)$, *then*

$$\sigma_n^{\alpha} f \to f \qquad \text{a.e.}$$

as $n \to \infty$ *and* $2^{-\tau} \le n_i/n_j \le 2^{\tau}$ $(i,j = 1,\dots,d)$.

The first version of this result is due to Marcinkiewicz and Zygmund [30, 66]. They proved Corollary 6.6 for trigonometric Fourier series and for $\alpha = 1$, which have motivated further researches about the restricted summability. The general version of this corollary is due to the author [53, 55], for Fejér means and for two-dimensional functions it can also be found in Gát [14].

The following results are known ([59]) for the norm convergence of $\sigma_n^\alpha f$, which generalize (4).

THEOREM 6.7. *If $0<\alpha\le 1$ and $p_0(\mathbb{X})<p<\infty$, then*

$$\|\sigma_n^\alpha f\|_{H_p^\square} \le C_p \|f\|_{H_p^\square} \qquad (f\in H_p^\square(\mathbb{X}^d))$$

whenever $2^{-\tau}\le n_i/n_j\le 2^\tau$ $(i,j=1,\dots,d)$.

COROLLARY 6.8. *If $0<\alpha\le 1$, $p_0(\mathbb{X})<p<\infty$ and $f\in H_p^\square(\mathbb{X}^d)$, then*

$$\sigma_n^\alpha f \to f \qquad \text{in } H_p^\square\text{-norm}$$

as $n\to\infty$ and $2^{-\tau}\le n_i/n_j\le 2^\tau$ $(i,j=1,\dots,d)$.

6.3. Unrestricted summability. Now we deal with the operator σ_*^α and the convergence of $\sigma_n^\alpha f$ as $n\to\infty$ in the Prigsheim's sense, i.e. $\min(n_1,\dots,n_d)\to\infty$. The next result is due to the author ([54, 59]).

THEOREM 6.9. *If $0<\alpha\le 1$ and $1/(\alpha+1)<p\le\infty$, then*

$$\|\sigma_*^\alpha f\|_p \le C_p\|f\|_{H_p} \qquad (f\in H_p(\mathbb{X}^d)).$$

As Goginava [21] proved, $1/(\alpha+1)$ is the best possible constant.

THEOREM 6.10. *The operator σ_*^α is bounded from $H_p(\mathbb{X}^d)$ neither to $L_p(\mathbb{X}^d)$ nor to weak $L_p(\mathbb{X}^d)$ if $0<p\le 1/(\alpha+1)$.*

By interpolation we get here a.e. convergence for functions from the mixed Hardy spaces $H_1^i(\mathbb{X}^d)$ instead of $L_1(\mathbb{X}^d)$. One can say that $H_1^i(\mathbb{X}^d)$ plays the role of the integrable functions in some sense. Of course, in the one-dimensional case $H_1^i(\mathbb{X})=L_1(\mathbb{X})$.

COROLLARY 6.11. *If $0<\alpha\le 1$ and $f\in H_1^i(\mathbb{X}^d)$ for some $i=1,\dots,d$, then*

$$\sup_{\rho>0}\rho\lambda(\sigma_*^\alpha f>\rho)\le C\|f\|_{H_1^i}.$$

The density of the set of the trigonometric and Walsh polynomials in $H_1^i(\mathbb{X}^d)$ and Theorem 4.3 imply the next unrestricted convergence.

COROLLARY 6.12. *If $0<\alpha\le 1$ and $f\in H_1^i(\mathbb{X}^d)$ $(i=1,\dots,d)$, then*

$$\lim_{n\to\infty}\sigma_n^\alpha f = f \qquad a.e.$$

Recall that $H_1^i(\mathbb{X}^d)\supset L(\log L)^{d-1}(\mathbb{X}^d)$ for all $i=1,\dots,d$, where $\log^+ u=\mathbf{1}_{\{u>1\}}\log u$ and $f\in L(\log L)^{d-1}(\mathbb{X}^d)$ means that

$$\left\||f|(\log^+|f|)^{d-1}\right\|_1<\infty.$$

Obviously,

$$L_1(\mathbb{X}^d)\supset L(\log L)^{d-1}(\mathbb{X}^d)\supset L_p(\mathbb{X}^d) \qquad (1<p\le\infty).$$

Gát [15, 16] proved for the Fejér means that Corollary 6.12 does not hold for all integrable functions.

THEOREM 6.13. *The a.e. convergence of Corollary 6.12 is not true for all $f\in L_1(\mathbb{X}^d)$.*

Another generalization of (4) reads as follows.

THEOREM 6.14. *If $0 < \alpha \le 1$ and $1/(\alpha+1) < p < \infty$, then*

$$\|\sigma_n^\alpha f\|_{H_p} \le C_p \|f\|_{H_p} \qquad (f \in H_p(\mathbb{X}^d),\ n \in \mathbb{N}^d).$$

COROLLARY 6.15. *If $0 < \alpha \le 1$, $1/(\alpha+1) < p < \infty$ and $f \in H_p(\mathbb{X}^d)$, then*

$$\lim_{n\to\infty} \sigma_n^\alpha f = f \qquad \text{in } H_p\text{-norm.}$$

6.4. Marcinkiewicz summability. With the arithmetic means of the cubic partial sums $s_{\mathbf{k}} f$ Marcinkiewicz introduced a third summability method of multi-dimensional Fourier series. The Marcinkiewicz–Fejér and Marcinkiewicz–Cesàro means of f are defined by

$$\sigma_{\mathbf{n}} f(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_{\mathbf{k}} f(x) = \sum_{j\in\mathbb{Z}^d, |j|\le n} \Big(1 - \frac{|j|}{n}\Big) \hat{f}(j)\phi_j(x)$$

and

$$\sigma_{\mathbf{n}}^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} s_{\mathbf{k}} f = \frac{1}{A_{n-1}^\alpha} \sum_{j\in\mathbb{Z}^d, |j|\le n} A_{n-1-|j|}^\alpha \hat{f}(j)\phi_j,$$

where $|j| := \sup_{k=1,\dots,d} |j_k|$. The *maximal Marcinkiewicz–Cesàro operator* is defined by

$$\sigma_*^\alpha f := \sup_{n\in\mathbb{N}} |\sigma_{\mathbf{n}}^\alpha f|.$$

THEOREM 6.16. *If $0 < \alpha \le 1$ and $d/(d+\alpha) < p \le \infty$, then*

$$\|\sigma_*^\alpha f\|_p \le C_p \|f\|_{H_p^\square} \qquad (f \in H_p^\square(\mathbb{X}^d))$$

and for $f \in H_{d/(d+\alpha)}^\square(\mathbb{X}^d)$,

$$\|\sigma_*^\alpha f\|_{d/(d+\alpha),\infty} = \sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho)^{(d+\alpha)/d} \le C\|f\|_{H_{d/(d+\alpha)}^\square}.$$

This theorem was proved by Oswald [32] for Fourier transforms and for Riesz means, by the author for multi-dimensional Fourier series and for two-dimensional Walsh–Fourier series [57, 62] and by Goginava [17, 18, 19, 20] for multi-dimensional Walsh–Fourier series.

Oswald and Goginava verified also that $d/(d+\alpha)$ is the best possible constant.

THEOREM 6.17. *The operator σ_*^α $(0 < \alpha \le 1)$ is not bounded from $H_p^\square(\mathbb{X}^d)$ to $L_p(\mathbb{X}^d)$ if $0 < p \le d/(d+\alpha)$.*

The weak type $(1,1)$ inequality and the almost everywhere convergence of the Marcinkiewicz–Cesàro means is obtained again by interpolation and by Theorem 4.3.

COROLLARY 6.18. *If $0 < \alpha \le 1$ and $f \in L_1(\mathbb{X}^d)$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_*^\alpha f > \rho) \le C\|f\|_1.$$

COROLLARY 6.19. *If $0 < \alpha \le 1$ and $f \in L_1(\mathbb{X}^d)$, then*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^\alpha f = f \qquad a.e.$$

This corollary was verified first by Marcinkiewicz [29] for two-dimensional Fourier series, for $f \in L\log L(\mathbb{T}^2)$ and $\alpha = 1$. Later Zhizhiashvili [63, 64] extended this result to all $f \in L_1(\mathbb{T}^2)$ and $0 < \alpha \le 1$, D'yachenko [10] and Weisz [62] to all $f \in L_1(\mathbb{T}^d)$. The result for Walsh–Fourier series is due to the author [57] and to Goginava [17, 18, 19, 20]. The next two results can be found in [57, 62].

THEOREM 6.20. *If* $0<\alpha\le 1$ *and* $d/(d+\alpha)<p\le\infty$, *then*

$$\|\sigma_{\mathbf{n}}^{\alpha} f\|_{H_p^{\square}} \le C_p \|f\|_{H_p^{\square}} \qquad (f\in H_p^{\square}(\mathbb{X}^d)).$$

COROLLARY 6.21. *If* $0<\alpha\le 1$, $d/(d+\alpha)<p<\infty$ *and* $f\in H_p^{\square}(\mathbb{X}^d)$, *then*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^{\alpha} f = f \qquad \text{in } H_p^{\square}\text{-norm}.$$

7. Ciesielski system. Besides the trigonometric and Walsh system the next results will be true also for Ciesielski systems, that is a generalization of the Walsh system. First we are going to introduce the spline systems as in Ciesielski [9]. Let us denote by D the differentiation operator and define the integration operators

$$Gf(t) := \int_0^t f\,d\lambda, \qquad Hf(t) := \int_t^1 f\,d\lambda.$$

Define the χ_n, $n=1,2,\dots$, *Haar system* by $\chi_1 := 1$ and

$$\chi_{2^n+k}(x) := \begin{cases} 2^{n/2}, & \text{if } x\in((2k-2)2^{-n-1},(2k-1)2^{-n-1}) \\ -2^{n/2}, & \text{if } x\in((2k-1)2^{-n-1},(2k)2^{-n-1}) \\ 0, & \text{otherwise} \end{cases}$$

for $n,k\in\mathbb{N}$, $0<k\le 2^n$, $x\in[0,1)$.

Let $m\ge -1$ be a fixed integer. Applying the Schmidt orthonormalization to the linearly independent functions

$$1,t,\dots,t^{m+1},G^{m+1}\chi_n(t), \qquad n\ge 2,$$

we get the *spline system* $(f_n^{(m)}, n\ge -m)$ of order m. For $0\le k\le m+1$ and $n\ge k-m$ define the splines

$$f_n^{(m,k)} := D^k f_n^{(m)}, \qquad g_n^{(m,k)} := H^k f_n^{(m)}$$

of order (m,k). Let us normalize these functions and introduce a more unified notation,

$$h_n^{(m,k)} := \begin{cases} f_n^{(m,k)} \|f_n^{(m,k)}\|_2^{-1} & \text{for } 0\le k\le m+1 \\ g_n^{(m,-k)} \|f_n^{(m,-k)}\|_2 & \text{for } 0\le -k\le m+1. \end{cases}$$

We get the Haar system if $m=-1$, $k=0$ and the *Franklin system* if $m=0$, $k=0$. The systems $(h_i^{(m,k)}, i\ge |k|-m)$ and $(h_j^{(m,-k)}, j\ge |k|-m)$ are biorthogonal, i.e.

$$(h_i^{(m,k)}, h_j^{(m,-k)}) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\ne j, \end{cases}$$

where (f,g) denotes the usual scalar product $\int_{[0,1)} fg\,d\lambda$.

We define the *Ciesielski system* $(c_n^{(m,k)}, n\ge |k|-m-1)$ in the same way as the Walsh system arises from the Haar system, namely,

$$c_n^{(m,k)} := h_{n+1}^{(m,k)} \quad (n=|k|-m-1,\dots,0)$$

and

$$c_{2^\nu+i}^{(m,k)} := \sum_{j=1}^{2^\nu} A_{i+1,j}^{(\nu)} h_{2^\nu+j}^{(m,k)} \quad (0 \le i \le 2^\nu - 1),$$

where

$$A_{i,j}^{(\nu)} = A_{j,i}^{(\nu)} = 2^{-\nu/2} w_{i-1}\Big(\frac{2j-1}{2^{\nu+1}}\Big)$$

(see Ciesielski [7] or Schipp, Wade, Simon, Pál [45]). We get immediately that

$$h_{2^\nu+j}^{(m,k)} := \sum_{i=1}^{2^\nu} A_{i+1,j}^{(\nu)} c_{2^\nu+i}^{(m,k)} \qquad (1 \le j \le 2^\nu - 1).$$

Then

$$c_n^{(-1,0)} = w_n \qquad (n \in \mathbb{N}),$$

is the usual Walsh system. The system $(c_n^{(m,k)})$ is uniformly bounded and it is biorthogonal to $(c_n^{(m,-k)})$ whenever $|k| \le m+1$ and $m \ge -1$ (see Ciesielski [7]).

Now the *Fourier coefficients* and *partial sums* of the Ciesielski–Fourier series of $f \in L_1[0,1)$ are defined by

$$\hat{f}(i) := \int_{[0,1)} f c_i^{(m,k)}\, d\lambda, \qquad s_n f := \sum_{i=|k|-m-1}^{n} \hat{f}(i) c_i^{(m,-k)} \qquad (n \in \mathbb{N}).$$

8. One-dimensional strong summability. Some of the preceding results are proved for Ciesielski systems as well (cf. [59]). For example, Carleson's theorem (Theorem 3.1) holds (see Schipp [41] and Ciesielski [8, 9]). The author ([58]) proved the Fejér summability, i.e. the analogues of Theorem 4.1 and Corollary 4.2 for $\alpha = 1$.

Let ϕ_n be the trigonometric, Walsh or Ciesielski system. Corollary 4.2 for $\alpha = 1$ can be reformulated as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big(s_k f(x) - f(x)\big) = 0 \quad \text{a.e. } x \in \mathbb{X} \quad (f \in L_1(\mathbb{X})).$$

Taking absolute value in the last sum we obtain the strong summability. More generally, we consider the convergence of the means

$$\Big(\frac{1}{n} \sum_{k=0}^{n-1} \big|s_k f(x) - f(x)\big|^r\Big)^{1/r}.$$

Strong summability was considered first by Hardy and Littlewood [24] for trigonometric Fourier series. They verified that these means tend to 0 a.e. as $n \to \infty$, whenever $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$. The generalization of Marcinkiewicz [28] is fundamental, he proved the a.e. convergence for all integrable functions and for $r = 2$. Later Zygmund [65] extended this result to all $r > 0$ (see also Bary [1]). For Walsh–Fourier series the strong summability was shown by Schipp [39, 43] for $r = 2$, by Rodin [36, 34] for $r > 0$ and for BMO means and for Ciesielski–Fourier series by the author [60] for $0 < r \le 2$. If the strong summability holds for an index r then it holds also for all $r' < r$. Now we formulate these results.

Let

$$S_*^{(r)} f := \sup_{n\geq 1} \Big(\frac{1}{n} \sum_{k=0}^{n-1} |s_k f|^r \Big)^{1/r}$$

be the *strong maximal operator*, where $0 < r < \infty$.

THEOREM 8.1. *If* $1 < p \leq \infty$, *then*

$$\|S_*^{(r)} f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}))$$

and

$$\sup_{\rho>0} \rho\, \lambda(S_*^{(r)} f > \rho) \leq C\|f\|_1 \qquad (f \in L_1(\mathbb{X})).$$

The strong maximal operator is not bounded from H_1 to L_1 (see Schipp and Simon [44]). The weak type $(1,1)$ inequality in Theorem 8.1 and the density argument of Theorem 4.3 imply

COROLLARY 8.2. *If* $0 < r < \infty$ *and* $f \in L_1(\mathbb{X})$, *then*

$$\lim_{n\to\infty} \Big(\frac{1}{n} \sum_{k=0}^{n-1} \big| s_k f(x) - f(x) \big|^r \Big)^{1/r} = 0 \quad a.e.\ x \in \mathbb{X}.$$

Note that the results hold for Ciesielski–Fourier series, whenever $0 < r \leq 2$.

9. More-dimensional strong summability. The multi-dimensional Fejér summability, i.e. the analogue of Corollary 6.12 holds for Ciesielski–Fourier series as well with $\alpha = 1$. We can rewrite it as

$$\lim_{n\to\infty} \frac{1}{\prod_{i=1}^{n} n_i} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} \big(s_k f(x) - f(x)\big) = 0 \quad \text{a.e. } x \in \mathbb{X}^d,$$

whenever $f \in L(\log L)^{d-1}(\mathbb{X}^d)$.

Now the *strong maximal operator* is defined by

$$S_*^{(r)} f := \sup_{n\in\mathbb{N}^d} \bigg(\frac{1}{\prod_{i=1}^{d} n_i} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} |s_k f|^r \bigg)^{1/r}.$$

The multi-dimensional strong summability was shown by Gogoladze [22] and Rodin [35, 37] for trigonometric Fourier series, by Rodin [38] for Walsh–Fourier series and by the author ([61]) for Ciesielski–Fourier series (with $0 < r \leq 2$).

THEOREM 9.1. *If* $1 < p \leq \infty$, *then*

$$\|S_*^{(r)} f\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}^d))$$

and for $f \in L(\log L)^{d-1}(\mathbb{X}^d)$,

$$\sup_{\rho>0} \rho \lambda(S_*^{(r)} f > \rho) \leq C + C \big\| |f| (\log^+ |f|)^{d-1} \big\|_1.$$

THEOREM 9.2. *If* $0 < r < \infty$ *and* $f \in L(\log L)^{d-1}(\mathbb{X}^d)$, *then*

$$\lim_{n\to\infty} \bigg(\frac{1}{\prod_{i=1}^{d} n_i} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} \big| s_k f(x) - f(x) \big|^r \bigg)^{1/r} = 0 \quad a.e.\ x \in \mathbb{X}^d.$$

Acknowledgments. This research was supported by the Hungarian Scientific Research Funds (OTKA) No K67642.

References

[1] N. K. Bary, *A Treatise on Trigonometric Series*, Pergamon Press, Oxford, 1964.

[2] C. Bennett, R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, 1988.

[3] J. Bergh, J. Löfström, *Interpolation Spaces: An Introduction*, Grundlehren Math. Wiss. 223, Springer, Berlin, 1976.

[4] P. Billard, *Sur la convergence presque partout des séries de Fourier-Walsh des fonctions de l'espace* $L^2[0,1]$, Studia Math. 28 (1967), 363–388.

[5] P. L. Butzer, R. J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser, Basel, 1971.

[6] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), 135–157.

[7] Z. Ciesielski, *A bounded orthonormal system of polygonals*, Studia Math. 31 (1968), 339–346.

[8] Z. Ciesielski, *Constructive function theory and spline systems*, Studia Math. 53 (1975), 277–302.

[9] Z. Ciesielski, *Equivalence, unconditionality and convergence a.e. of the spline bases in* L_p *spaces*, in: Approximation Theory, Banach Center Publ. 4, PWN, Warsaw, 1979, 55–68.

[10] M. I. D'yachenko, *Some properties of multiple Fourier series and Fourier transforms*, Trudy Mat. Inst. Steklov 190 (1989), 88–101 (in Russian); English transl.: Proc. Steklov Inst. Math. 1992, no. 1, 91–106.

[11] C. Fefferman, *On the convergence of multiple Fourier series*, Bull. Amer. Math. Soc. 77 (1971), 744–745.

[12] L. Fejér, *Untersuchungen über Fouriersche Reihen*, Math. Ann. 58 (1904), 51–69.

[13] N. J. Fine, *Cesàro summability of Walsh–Fourier series*, Proc. Nat. Acad. Sci. USA 41 (1955), 558–591.

[14] G. Gát, *Pointwise convergence of the Cesàro means of double Walsh series*, Ann. Univ. Sci. Budapest. Sect. Comput. 16 (1996), 173–184.

[15] G. Gát, *On the divergence of the* $(C,1)$ *means of double Walsh–Fourier series*, Proc. Amer. Math. Soc. 128 (2000), 1711–1720.

[16] G. Gát, *Divergence of the* $(C,1)$ *means of d-dimensional Walsh–Fourier series*, Anal. Math. 27 (2001), 157–171.

[17] U. Goginava, *Almost everywhere summability of multiple Walsh–Fourier series*, J. Math. Anal. Appl. 287 (2003), 90–100.

[18] U. Goginava, *Marcinkiewicz–Fejér means of d-dimensional Walsh–Fourier series*, J. Math. Anal. Appl. 307 (2005), 206–218.

[19] U. Goginava, *Almost everywhere convergence of* (C,α)*-means of cubical partial sums of d-dimensional Walsh–Fourier series*, J. Approx. Theory 141 (2006), 8–28.

[20] U. Goginava, *Maximal operators of* (C,α)*-means of cubic partial sums of d-dimensional Walsh–Fourier series*, Anal. Math. 33 (2007), 263–286.

[21] U. Goginava, *Maximal operators of Fejér means of double Walsh–Fourier series*, Acta Math. Hungar. 115 (2007), 333–340.

[22] L. D. Gogoladze, *The (H, k)-summability of multiple trigonometric Fourier series*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), 937–958; English transl.: Math. USSR-Izv. 11 (1977), 889–908.

[23] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Upper Saddle River, NJ, 2004.

[24] G. H. Hardy, J. E. Littlewood, *Sur la séries de Fourier d'une fonction à carré sommable*, C. R. Acad. Sci. Paris 156 (1913), 1307–1309.

[25] R. A. Hunt, *On the convergence of Fourier series*, in: Orthogonal Expansions and their Continuous Analogues (Proc. Conf. Edwardsville, 1967), Southern Illinois Univ. Press, Carbondale IL, 1968, 235–255.

[26] H. Lebesgue, *Recherches sur la convergence des séries de Fourier*, Math. Ann. 61 (1905), 251–280.

[27] S. Z. Lu, *Four Lectures on Real H^p Spaces*, World Scientific, Singapore, 1995.

[28] J. Marcinkiewicz, *Sur la sommabilité forte de séries de Fourier*, J. London Math. Soc. 14 (1939), 162–168.

[29] J. Marcinkiewicz, *Sur une méthode remarquable de sommation des séries doubles de Fourier*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) 8 (1939), 149–160.

[30] J. Marcinkiewicz, A. Zygmund, *On the summability of double Fourier series*, Fund. Math. 32 (1939), 122–132.

[31] F. Móricz, *On the convergence of double orthogonal series*, Anal. Math. 2 (1976), 287–304.

[32] P. Oswald, *On Marcinkiewicz–Riesz summability of Fourier integrals in Hardy spaces*, Math. Nachr. 133 (1987), 173–187.

[33] M. Riesz, *Sur la sommation des séries de Fourier*, Acta Sci. Math. (Szeged) 1 (1923), 104–113.

[34] V. A. Rodin, *The space BMO and strong means of Fourier series*, Anal. Math. 16 (1990), 291–302.

[35] V. A. Rodin, *Pointwise strong summability of multiple Fourier series*, Mat. Zametki 50 (1991), no. 1, 148–150; English transl.: Math. Notes 50 (1991), 762–764.

[36] V. A. Rodin, *The space BMO and strong means of Fourier–Walsh series*, Mat. Sb. 182 (1991), 1463–1478; English transl.: Math. USSR-Sb. 74 (1993), 203–218.

[37] V. A. Rodin, *The tensor BMO-property of a sequence of partial sums of a multiple Fourier series*, Mat. Sb. 184 (1993), no. 10, 91–106; English transl.: Russian Acad. Sci. Sb. Math. 80 (1995), 211–224.

[38] V. A. Rodin, *Strong means and the oscillation of multiple Fourier–Walsh series*, Mat. Zametki 56 (1994), no. 3, 102–117; English transl.: Math. Notes 56 (1994), 948–959.

[39] F. Schipp, *Über die starke Summation von Walsh–Fourierreihen*, Acta. Sci. Math. (Szeged) 30 (1969), 77–87.

[40] F. Schipp, *Über gewissen Maximaloperatoren*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 18 (1975), 189–195.

[41] F. Schipp, *On a.e. convergence of expansion with respect to a bounded orthonormal system of polygonals*, Studia Math. 58 (1976), 287–290.

[42] F. Schipp, *Pointwise convergence of expansions with respect to certain product systems*, Anal. Math. 2 (1976), 65–76.

[43] F. Schipp, *On the strong summability of Walsh series*, Publ. Math. Debrecen 52 (1998), 611–633.

[44] F. Schipp, P. Simon, *On some (L_1, H)-type maximal inequalities with respect to the Walsh–Paley system*, in: Functions, Series, Operators (Budapest, 1980), Colloq. Math. Soc. János Bolyai 35, North-Holland, Amsterdam, 1981, 1039–1045.

[45] F. Schipp, W. R. Wade, P. Simon, J. Pál, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol, 1990.

[46] P. Simon, F. Weisz, *Weak inequalities for Cesàro and Riesz summability of Walsh–Fourier series*, J. Approx. Theory 151 (2008), 1–19.

[47] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh–Fourier series*, Ark. Mat. 7 (1969), 551–570.

[48] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, 1993.

[49] E. M. Stein, M. H. Taibleson, G. Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Rend. Circ. Mat. Palermo (2) 1981, Suppl. 1, 81–97.

[50] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Math. Ser. 32, Princeton Univ. Press, Princeton, 1971.

[51] R. M. Trigub, E. S. Belinsky, *Fourier Analysis and Approximation of Functions*, Kluwer, Dordrecht, 2004.

[52] F. Weisz, *Martingale Hardy Spaces and their Applications in Fourier Analysis*, Lecture Notes in Math. 1568, Springer, Berlin, 1994.

[53] F. Weisz, *Cesàro summability of two-dimensional Walsh–Fourier series*, Trans. Amer. Math. Soc. 348 (1996), 2169–2181.

[54] F. Weisz, *(C, α) means of several-parameter Walsh– and trigonometric–Fourier series*, East J. Approx. 6 (2000), 129–156.

[55] F. Weisz, *Maximal estimates for the (C, α) means of d-dimensional Walsh–Fourier series*, Proc. Amer. Math. Soc. 128 (2000), 2337–2345.

[56] F. Weisz, *(C, α) summability of Walsh–Fourier series*, Anal. Math. 27 (2001), 141–155.

[57] F. Weisz, *Convergence of double Walsh–Fourier series and Hardy spaces*, Approx. Theory Appl. (N.S.) 17 (2001), 32–44.

[58] F. Weisz, *On the Fejér means of bounded Ciesielski systems*, Studia Math. 146 (2001), 227–243.

[59] F. Weisz, *Summability of Multi-Dimensional Fourier Series and Hardy Spaces*, Math. Appl. 541, Kluwer, Dordrecht, 2002.

[60] F. Weisz, *Strong summability of Ciesielski–Fourier series*, Studia Math. 161 (2004), 269–302.

[61] F. Weisz, *Strong summability of more-dimensional Ciesielski–Fourier series*, East J. Approx. 10 (2004), 333–354.

[62] F. Weisz, *Marcinkiewicz-summability of multi-dimensional Fourier transforms and Fourier series*, J. Math. Anal. Appl. 379 (2011), 910–929.

[63] L. Zhizhiashvili, *A generalization of a theorem of Marcinkiewicz* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 1112–1122; English transl.: Math. USSR-Izv. 2 (1968), 1065–1075.

[64] L. Zhizhiashvili, *Trigonometric Fourier Series and Their Conjugates*, Math. Appl. 372, Kluwer, Dordrecht, 1996.

[65] A. Zygmund, *On the convergence and summability of power series on the circle II*, Proc. London. Math. Soc. (2) 47 (1942), 326–350.

[66] A. Zygmund, *Trigonometric Series*, 3rd ed., Cambridge Univ. Press, Cambridge, 2002.