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DISCRETE TIME INFINITE HORIZON RISK SENSITIVE PORTFOLIO SELECTION WITH PROPORTIONAL TRANSACTION COSTS

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Abstract. Long run risk sensitive portfolio selection is considered with proportional transaction costs. In the paper two methods to prove existence of solutions to suitable Bellman equations are presented. The first method is based on discounted cost approximation and requires uniform absolute continuity of iterations of transition operators of the factor process. The second method is based on uniform ergodicity of portions of the capital invested in assets and requires additional assumptions concerning diversity of investments.

1. Introduction. Assume we are given a discrete time market with m risky assets. Denote by $S_i(t)$ the price of the i-th asset at time t. Let

$$\frac{S_i(t+1)}{S_i(t)} = \zeta_i(z(t+1), \xi(t+1)), \tag{1}$$

where $(z(t)) \in D$ forms a Markov process on a complete separable metric space D with transition operator $P(z(t), dy)$ describing the evolution of economic factors, $(\xi(t))$ stands for a sequence of i.i.d. random variables on Ξ with law η, independent of $(z(t))$, and ζ is a given positive function such that the mapping $z \mapsto \zeta(z, \xi)$ is continuous for each ξ. Denote by $X^-(t)$ the wealth process at time t before possible transactions and by $X(t)$ the wealth process after possible transactions. Let $\pi_i^-(t)$ be the portion of the wealth process invested in the i-th asset at time t before possible transactions and $\pi_i(t)$ the portion of the wealth located in the i-th asset after transactions at time t. We shall say that $\pi(t) = (\pi_1(t), \ldots, \pi_m(t))^T$ (where T stands for the transpose) and similarly $\pi^-(t)$ form portfolios at time t after or before possible transactions.

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Denote by $\mathcal{S}^0$ the polyhedral set $\{(\nu_1,\ldots,\nu_m)^T : \nu_i \geq 0, \sum_{i=1}^m \nu_i \leq 1\}$ and by $\mathcal{S}$ the simplex $\{(\nu_1,\ldots,\nu_m)^T \in S^0 : \sum_{i=1}^m \nu_i = 1\}$.

For given $\pi \in [0,\infty)^m \backslash \{0\}$ let $g(\pi) = (g_1(\pi),\ldots,g_m(\pi))^T$, where $g_i(\pi) = \pi_i / \sum_{j=1}^m \pi_j$. After change of portfolio from π to π' the wealth X is diminished by $c(\hat{\pi}-\pi)X$, where $\hat{\pi}$ is a certain element of $\mathcal{S}^0$ (we shall see in Lemma 1 below that it is defined in a unique way) such that $\pi' = g(\hat{\pi})$ and for $\nu \in \mathcal{S}^0 - \mathcal{S}^0$ (the algebraic difference of the sets $\mathcal{S}^0$)

$$c(\nu) = \sum_{i=1}^m c_i^1 (\nu_i)^+ + \sum_{i=1}^m c_i^2 (\nu^i)^- \tag{2}$$

with $0 < c_i^1, c_i^2 < 1$. Given a portfolio π and wealth X we can change the portfolio to π' if there exists $\hat{\pi}$ such that

$$X(c(\hat{\pi}-\pi)) = X - X \sum_{i=1}^m \hat{\pi}_i \tag{3}$$

and $g(\hat{\pi}) = \pi'$. Consequently given π we can choose a portfolio π' if and only if there is $\hat{\pi} \in \mathcal{S}^0$ such that

$$\sum_{i=1}^m \hat{\pi}_i + c(\hat{\pi}-\pi) = 1 \tag{4}$$

where

$$\pi' = g(\hat{\pi}). \tag{5}$$

Given $\pi, \pi' \in \mathcal{S}$ define the function

$$F^{\pi,\pi'}(\delta) := \delta + c(\delta\pi' - \pi). \tag{6}$$

Following Lemma 1 of [10] we have

LEMMA 1. *There is a unique continuous function $e : \mathcal{S}\times\mathcal{S} \to [0,1]$ such that for $\pi, \pi' \in \mathcal{S}$ we have*

$$F^{\pi,\pi'}(e(\pi,\pi')) = 1. \tag{7}$$

Furthermore e is bounded away from 0.

Consequently, given an initial wealth process $X^-(t)$ and portfolio $\pi^-(t)$ we can choose any post transaction portfolio $\pi(t) \in \mathcal{S}$. Then, as a result of transaction costs the wealth process is diminished to $X(t)$, where following (3) and (4) we have

$$X(t) = e(\pi^-(t),\pi(t))X^-(t). \tag{8}$$

Furthermore

$$\begin{aligned} X^-(t+1) &= \sum_{i=1}^m \frac{\pi_i(t)X(t)}{S_i(t)} S_i(t+1) \\ &= X(t)\sum_{i=1}^m \pi_i(t)\zeta_i(z(t+1),\xi(t+1)) = X(t)\pi(t)^T\zeta(z(t+1),\xi(t+1)) \end{aligned} \tag{9}$$

and

$$\pi^-(t+1) = g(\pi(t) \diamond \zeta(z(t+1),\xi(t+1)), \tag{10}$$

with $(\pi(t) \diamond \zeta(z(t+1), \xi(t+1)))_i := \pi_i(t)\zeta_i(z(t+1), \xi(t+1))$. Therefore for $t = 1, 2, \ldots$

$$X^-(t) = X^-(0) \prod_{n=0}^{t-1} e(\pi^-(n), \pi(n))\pi(n)^T \zeta(z(n+1), \xi(n+1)). \tag{11}$$

In this paper we are interested in maximizing the following *risk sensitive* cost functional

$$J^\gamma_{X^-,z,\pi^-}((\pi(n))) := \liminf_{t\to\infty} \frac{1}{\gamma t} \ln E_{X^-,z,\pi^-}\{(X^-(t))^\gamma\} \tag{12}$$

over all admissible, i.e. adapted to available information, sequences $\pi(n) \in \mathcal{S}$, where γ is a negative risk factor. Notice following [1] and [2] that the cost functional J^γ measures average growth of portfolio plus its variance with a negative weight γ. Moreover by (9) the cost functional (12) is of the form

$$\begin{aligned} &J^\gamma_{X^-,z,\pi^-}((\pi(n))) := \\ &\liminf_{t\to\infty} \frac{1}{\gamma t} \ln E_{X^-,z,\pi^-}\Big\{ \prod_{n=0}^{t-1} (e(\pi^-(n), \pi(n))\pi(n)^T \zeta(z(n+1), \xi(n+1)))^\gamma \Big\}. \end{aligned} \tag{13}$$

Risk sensitive portfolio optimization has been a subject of intensive studies in a number of papers (see [1], [2], [6], [8], [9] and [10]). The case with proportional transaction costs was studied in [2], [9] and [10]. In [2] the result was formulated under the assumption on the existence of a regular solution to a suitable Bellman equation without giving sufficient conditions under which the equation has a solution. In the paper [9] a general diffusion model for asset prices was considered in which the factors were allowed to depend on the same random disturbance $(\xi(t))$. To prove the existence of solutions to the Bellman equation a certain obligatory portfolio diversification assumption was imposed. In the paper [10] existence of solutions to the Bellman equation was proved under the assumption of uniform boundedness from below and from above of the densities of the transition operator P. In this paper we extend the papers [9] and [10] in the following directions: in the case of obligatory portfolio diversification of [9] a general model of asset prices is considered, in the case of the paper [10] the boundedness of transition densities of P ([10]) is replaced by the boundedness of the densities of iterations of the transition operator P.

2. Risk sensitive Bellman equation. In this section we shall assume that for a positive integer n there is a probability measure μ and positive continuous density $p^{(n)}(z, z')$ of the n-th iteration of the transition operator P, i.e. for Borel measurable $A \subset D$, $z \in D$ we have $P^n(z, A) = \int_A p^{(n)}(z, z')\mu(dz')$, and furthermore

$$\sup_{z_1, z_1', z_2, z_2' \in D} \frac{p^{(n)}(z_1, z_1')}{p^{(n)}(z_2, z_2')} := M < \infty. \tag{14}$$

In the case when $n > 1$ we additionally assume that

- $\zeta_i(z, \xi), i = 1, 2, \ldots, m$ are bounded from above and bounded away from 0, i.e. there are positive d_1 and d_2 such that

$$\forall_{i,z,\xi} \qquad 0 < d_1 \le \zeta_i(z,\xi) \le d_2. \tag{15}$$

In the case when $n = 1$ we shall assume that there is a $\bar{\gamma} < 0$ such that

- for $\gamma \in [\bar{\gamma}, 0)$ the mapping $(z,\pi) \mapsto E_z\{(\pi^T\zeta(z(1),\xi(1))^\gamma\}$ is bounded and continuous.

In the next theorem we show that optimal strategies for risk sensitive cost functional (12) depend on the current value of the portfolio process $\pi^-(n)$ and factor process $z(n)$ only (they do not depend on the wealth process and this could be already noticed from the form (13) of the cost functional (12)). Notice that the case $n = 1$ was already studied in [10].

THEOREM 2. *Under the above assumptions, assuming additionally in the case $n = 1$ that $\gamma \in [\bar{\gamma}, 0)$, there is a bounded continuous function $w_\gamma : D \times \mathcal{S} \mapsto R$ and a constant λ_γ such that*

$$\begin{aligned} w_\gamma(z,\pi) + \gamma\lambda_\gamma = \inf_{\pi' \in \mathcal{S}} (\gamma(\ln e(\pi,\pi')) \\ + \ln E_z\{\exp\{\gamma \ln(\pi'\zeta(z(1),\xi(1))) + w_\gamma(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))),\gamma)\}\}). \end{aligned} \tag{16}$$

Moreover λ_γ is the optimal value of the cost functional (13) and the strategy

$$(\hat{\pi}_\gamma(z(t),\pi^-(t)))$$

is optimal, where $\hat{\pi}_\gamma$ is a Borel measurable selector for which the infimum in (16) is attained.

Proof. We consider first a version of risk sensitive discounted cost functional (see [3]). The value function w^β corresponding to such a control problem is continuous and is a solution to the following Bellman equation

$$\begin{aligned} w^\beta(z,\pi,\gamma) = \inf_{\pi' \in \mathcal{S}} (\gamma(\ln e(\pi,\pi')) \\ + \ln E_z\{\exp\{\gamma \ln(\pi'^T\zeta(z(1),\xi(1))) + w^\beta(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))),\beta\gamma)\}\}). \end{aligned} \tag{17}$$

Furthermore for $\pi' \in \mathcal{S}$

$$w^\beta(z,\pi,\gamma) \le \gamma \ln e(\pi,\pi') + w^\beta(z,\pi',\gamma). \tag{18}$$

Iterating (17) we obtain that

$$\begin{aligned} w^\beta(z,\pi,\gamma) = \inf_{(\pi(i))} \ln E_z\Big\{ \exp\Big\{ \sum_{t=0}^{n-1} \beta^t\gamma(\ln e(\pi^-(t),\pi(t)) \\ + \ln(\pi^T(t)\zeta(z(t+1),\xi(t+1)))) + w^\beta(z(n),\pi^-(n),\beta^n\gamma)\Big\}\Big\} \end{aligned} \tag{19}$$

with infimum attained for an admissible strategy $(\hat{\pi}(i))$. Let $(\hat{\pi}_1(i))$ and $(\hat{\pi}_2(i))$ be optimal strategies for $w^\beta(z_1,\pi_1,\gamma)$ and $w^\beta(z_2,\pi_2,\gamma)$ respectively. Assume that using both strategies at time n we change the portfolio to a fixed deterministic value $\pi' \in \mathcal{S}$ and then use optimal strategies. Denote such corrected strategies by $(\tilde{\pi}_1(i))$ and $(\tilde{\pi}_2(i))$ respectively. By (18) we have that

$$w^\beta(z(n),\tilde{\pi}_1^-(n),\beta^n\gamma) \le \beta^n\gamma \ln e(\tilde{\pi}_1^-(n),\pi') + w^\beta(z(n),\pi',\beta^n\gamma)$$

and

$$w^{\beta}(z(n),\pi',\beta^n\gamma) \le \beta^n\gamma \ln e(\pi',\tilde{\pi}_2^-(n)) + w^{\beta}(z(n),\hat{\pi}_2^-(n),\beta^n\gamma).$$

Therefore from (19) taking into account Lemma 1, (15) and (14) we obtain for a certain constant K

$$\begin{aligned}
(20)\quad & w^{\beta}(z_1,\pi_1,\gamma) - w^{\beta}(z_2,\pi_2,\gamma) \\
&\le \ln E_{z_1}\Big\{ \exp\Big\{ \sum_{t=0}^{n-1} \gamma\beta^t (\ln e(\tilde{\pi}_1^-(t),\tilde{\pi}_1(t)) + \ln(\tilde{\pi}_1^T(t)\zeta(z(t+1),\xi(t+1)))) \\
&\quad + \beta^n\gamma \ln e(\tilde{\pi}_1^-(n),\pi') + w^{\beta}(z(n),\pi',\beta^n\gamma)\Big\}\Big\} \\
&\quad - \ln E_{z_2}\Big\{ \exp\Big\{ \sum_{t=0}^{n-1} \gamma\beta^t (\ln e(\tilde{\pi}_2^-(t),\tilde{\pi}_2(t)) + \ln(\tilde{\pi}_2^T(t)\zeta(z(t+1),\xi(t+1)))) \\
&\quad + \beta^n\gamma \ln e(\tilde{\pi}_2^-(n),\pi') + w^{\beta}(z(n),\pi',\beta^n\gamma)\Big\}\Big\} \le K + \ln M.
\end{aligned}$$

Consequently for fixed points $\bar{z} \in D$ and $\bar{\pi} \in \mathcal{S}$ the family

$$\{\bar{w}^{\beta}(z,\pi,\gamma) := w^{\beta}(z,\pi,\gamma) - w^{\beta}(\bar{z},\bar{\pi},\gamma), \gamma \in [\bar{\gamma},0)\}$$

is bounded, i.e. there is a constant L (independent of γ) such that for $z \in D$ and $\pi \in \mathcal{S}$ we have

$$|\bar{w}^{\beta}(z,\pi,\gamma)| \le L.$$

Using continuity of the density of the transition operator P we easily show its equicontinuity. Therefore there is a subsequence $\beta_n \to 1$ and a family $\bar{w}_k(z,\pi)$ such that $\bar{w}^{\beta_n}(z,\pi,\beta_n^{k-1}\gamma)$ converges uniformly on compact subsets to $\bar{w}_k(z,\pi)$. Moreover since by (17)

$$\begin{aligned}
w^{\beta}(\bar{z},\bar{\pi},\beta^{k-1}\gamma) - w^{\beta}(\bar{z},\bar{\pi},\beta^k\gamma) \le \sup_{\pi' \in \mathcal{S}} (&\beta^{k-1}\gamma \ln e(\bar{\pi},\pi')) \\
&+ \ln E_{\bar{z}}\{\exp\{\beta^{k-1}\gamma \ln(\pi'^T\zeta(z(1),\xi(1))) + L\}\})
\end{aligned}$$

and

$$\begin{aligned}
w^{\beta}(\bar{z},\bar{\pi},\beta^{k-1}\gamma) - w^{\beta}(\bar{z},\bar{\pi},\beta^k\gamma) \ge \inf_{\pi' \in \mathcal{S}} (&\beta^{k-1}\gamma(\ln e(\bar{\pi},\pi')) \\
&+ \ln E_{\bar{z}}\{\exp\{\beta^{k-1}\gamma \ln(\pi'^T\zeta(z(1),\xi(1))) - L\}\})
\end{aligned}$$

a suitably chosen subsequence of $\lambda_\gamma(\beta^k) := w^{\beta}(\bar{z},\bar{\pi},\beta^{k-1}\gamma) - w^{\beta}(\bar{z},\bar{\pi},\beta^k\gamma)$ converges, as $\beta \to 1$, to a limit λ_γ^k. The family $\{\bar{w}_k(z,\pi), k = 1,2,\ldots\}$ is also bounded and equicontinuous and there is a subsequence such that $\bar{w}_k$ converges to w_γ and λ_γ^k to λ_γ, a solution to (16). The existence and form of optimal strategies follows from general results; see e.g. [3], [4] or [10]. ∎

REMARK 1. An alternative method is to use a discounted game approach from [7] which leads to a Bellman inequality. This approach requires however the boundedness in the span norm of the value of a suitable game which in our case is hard to obtain.

3. Risk sensitive control with obligatory diversification. The process of portions of capital invested in assets $(\pi(t))$ if there are no transactions does not have nice ergodic properties. One can notice that any full investment in one asset only is an absorbing state. The evolution of risk sensitive cost functional for the process without unique ergodic class is difficult to study. Therefore we shall simplify the problem assuming that at every moment we change the portfolio from π to π' such that $\pi' \in \mathcal{S}_\delta := \{(\nu_1, \ldots, \nu_m)^T \in \mathcal{S} : \nu_i \geq \delta$, for $i = 1, \ldots, m\}$ for certain $\delta > 0$. In particular, whenever $\pi \in \mathcal{S}_\delta$ we can choose the same portfolio π, i.e. we are not pressed to change the portfolio. Such a strategy has an important economic justification: we don't want to allow our capital be invested in one or few assets—we intend to diversify our portfolio by investing in all available assets. In what follows the following operator will play an important role

$$T^\gamma f(z,\pi) = \inf_{\pi' \in \mathcal{S}_\delta} [\gamma \ln e(\pi,\pi') + \ln E_z\{e^{\gamma \ln(\pi'^T \zeta(z(1),\xi(1)) + f(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))))}\}]. \tag{21}$$

Notice that (see e.g. [3])

$$\begin{aligned} \ln E_z\{e^{\gamma \ln(\pi'^T \zeta(z(1),\xi(1)) + f(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))))}\} \\ = \sup_m [\gamma \ln(\pi'^T \zeta(z',\xi')) + f(z', g(\pi' \diamond \zeta(z',\xi')))m(dz',d\xi') - I(m, P(z,\cdot)\eta(\cdot))] \end{aligned} \tag{22}$$

where m is a probability measure on $D \times \Xi$, the supremum is over all probability measures on $D \times \Xi$ and for two probability measures μ and ν the entropy is $I(\mu,\nu) = \int \ln \frac{d\mu}{d\nu} d\nu$ whenever $\nu << \mu$, and $\ln \frac{d\nu}{d\mu}$ is in $L^1(\nu)$, and $I(\mu,\nu) = +\infty$ in other cases. Furthermore the supremum is attained for the measure

$$\hat{m}_{z,\pi',f}(dz',d\xi') = \frac{e^{\gamma \ln(\pi'^T \zeta(z',\xi')) + f(z', g(\pi' \diamond \zeta(z',\xi')))} P(z,dz')\eta(d\xi')}{E_z\{e^{\gamma \ln(\pi'^T \zeta(z(1),\xi(1)) + f(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))))}\}}. \tag{23}$$

It will be convenient for us to consider also the measure

$$\begin{aligned} \tilde{m}_{z,\pi',f}(A) \\ = \frac{E_z\{1_A(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))))e^{\gamma \ln(\pi'^T \zeta(z(1),\xi(1))) + f(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))))}\}}{E_z\{e^{\gamma \ln(\pi'^T \zeta(z(1),\xi(1)) + f(z(1), g(\pi' \diamond \zeta(z(1),\xi(1))))}\}} \end{aligned} \tag{24}$$

for Borel subsets A of $D \times \mathcal{S}$. Define the following probability kernel defined for $z \in D$, $\pi \in \mathcal{S}_\delta$ and A a Borel subset of $D \times \mathcal{S}$:

$$\Phi(z,\pi,A) = P_z\{(z(1), g(\pi \diamond \zeta(z(1),\xi(1)))) \in A\}. \tag{25}$$

In what follows we shall assume that

$$\sup_{z,z' \in D} \sup_{\pi,\pi' \in \mathcal{S}_\delta} \sup_A [\Phi(z,\pi,A) - \Phi(z',\pi',A)] < 1. \tag{26}$$

The following lemma provides sufficient conditions for (26) to hold.

LEMMA 3. *If*

$$\sup_{z,z' \in D} \sup_B [P(z,B) - P(z',B)] := \tilde{q} < 1 \tag{27}$$

with supremum above over all Borel subsets of D and the measures

$$\eta_{z,\pi}(C) = P\{g(\pi \diamond \zeta(z,\xi)) \in C\} \tag{28}$$

where C is a Borel subset of $\mathcal{S}$, ξ is a random variable with law η, are absolutely continuous with densities bounded from above and bounded away from 0, i.e. there are $\kappa_1, \kappa_2 > 0$ such that for $z, z' \in D$, $\pi, \pi' \in \mathcal{S}_\delta$

$$\kappa_1 \leq \frac{d\eta_{z,\pi}}{d\eta_{z',\pi'}} \leq \kappa_2, \tag{29}$$

then (26) is satisfied.

Proof. Assume that (26) does not hold, i.e. there are sequences (z_n), (z'_n), (π_n), (π'_n), and (A_n) such that

$$\Phi(z_n, \pi_n, A_n) - \Phi(z'_n, \pi'_n, A_n) \to 1$$

as $n \to \infty$. This means that $\Phi(z_n, \pi_n, A_n) \to 1$ and $\Phi(z'_n, \pi'_n, A_n) \to 0$. By (29) also $\Phi(z'_n, \pi_n, A_n) \to 0$. Using (27) we have

$$\Phi(z_n, \pi_n, A_n) \leq \tilde{q} + \Phi(z'_n, \pi_n, A_n) \to \tilde{q} < 1,$$

a contradiction. ■

The assumption (29) seems to be complicated. We shall therefore write down the density of $\eta_{z,\pi}$ in terms of the density q_z of the random vector $\zeta(z, \xi(1))$. For a Borel measurable subset B of $\mathcal{S}$ let

$$B^{(r)} = \Big\{(x_1, x_2, \ldots, x_{m-1}) \in [0,1]^{m-1} : \Big(x_1, x_2, \ldots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i\Big) \in B\Big\} \tag{30}$$

and $B^{(n)} = B^{(r)} \times (0, \infty)$. Define the transformation $G : [0,\infty)^m \setminus (0,0,\ldots,0) \to \mathcal{S}^{(n)} = \mathcal{S}^{(r)} \times (0,\infty)$ by

$$\begin{bmatrix} x_1 \\ \cdots \\ x_{m-1} \\ x_m \end{bmatrix} = G \begin{bmatrix} \zeta_1 \\ \cdots \\ \zeta_{m-1} \\ \zeta_m \end{bmatrix} = \begin{bmatrix} \zeta_1 / \sum_{i=1}^m \zeta_i \\ \cdots \\ \zeta_{m-1} / \sum_{i=1}^m \zeta_i \\ \sum_{i=1}^m \zeta_i \end{bmatrix}. \tag{31}$$

Clearly

$$\begin{bmatrix} \zeta_1 \\ \cdots \\ \zeta_{m-1} \\ \zeta_m \end{bmatrix} = G^{-1} \begin{bmatrix} x_1 \\ \cdots \\ x_{m-1} \\ x_m \end{bmatrix} = \begin{bmatrix} x_1 x_m \\ \cdots \\ x_{m-1} x_m \\ (1 - \sum_{i=1}^{m-1} x_i) x^m \end{bmatrix}. \tag{32}$$

Let for $\pi \in \mathcal{S}_\delta$

$$D_\pi \begin{bmatrix} \zeta_1 \\ \cdots \\ \zeta_{m-1} \\ \zeta_m \end{bmatrix} = \begin{bmatrix} \pi_1 \zeta_1 \\ \cdots \\ \pi_{m-1} \zeta_{m-1} \\ \pi_m \zeta_m \end{bmatrix}. \tag{33}$$

Define

$$g^{(r)}(\zeta) = \begin{bmatrix} \zeta_1 / \sum_{i=1}^m \zeta_i \\ \cdots \\ \zeta_{m-1} / \sum_{i=1}^m \zeta_i \end{bmatrix}. \tag{34}$$

Clearly $g(\zeta)$ is completely determined by $g^{(r)}(\zeta)$. Furthermore $g^{(r)}(\pi\diamond\zeta)$ consists of the first $m-1$ coordinates of $GD_\pi(\zeta)$. For $\pi\in\mathcal{S}_\delta$ the transformation GD_π of $[0,\infty)^m\setminus(0,\dots,0)$ is one to one and the Jacobian of its inverse is equal to $x_m^{m-1}\frac{1}{\pi_1\dots\pi_m}$.

Lemma 4. *For $\pi\in\mathcal{S}_\delta$ the density $d_\pi^{(r)}$ of $g^{(r)}(\pi\diamond\zeta(z,\xi(1))$ is of the form*

$$(35)\quad d_\pi^{(r)}(x_1,x_2,\dots,x_{m-1})$$
$$=\int_0^\infty x_m^{m-1}\frac{1}{\pi_1\dots\pi_m}q_z\Big(\frac{1}{\pi_1}x_1x_m,\frac{1}{\pi_2}x_2x_m,\dots,\frac{1}{\pi_{m-1}}x_{m-1}x_m,\frac{1}{\pi_m}\Big(1-\sum_{i=1}^{m-1}x_i\Big)x_m\Big)dx_m$$

where q_z is the density of $\zeta(z,\xi(1))$.

Proof. Notice that for a Borel measurable subset B of $\mathcal{S}$ we have

$$\begin{aligned}(36)\qquad &P\{g(\pi\diamond\zeta(z,\xi(1)))\in B\}=P\{g^{(r)}(\pi\diamond\zeta(z,\xi(1)))\in B^{(r)}\}\\ &=P\{G(\pi\diamond\zeta(z,\xi(1)))\in B^{(n)}\}=P\{\zeta(z,\xi(1))\in(GD^\pi)^{-1}B^{(n)}\}\end{aligned}$$

and the form of the density follows from integration by substitution. ∎

Remark 2. The assumption (27) implies uniform ergodicity of the Markov process $(z(n))$ (see [5] section 5.5). The assumption (29) is satisfied for a wide family of rates of return $\zeta(z,\xi)$ the densities of which have for each $\in D$ similar behavior of tails at 0 and at ∞, i.e. when the ratio

$$(37)\qquad \frac{d_z(\pi_1x_1,\pi_2x_2,\dots,\pi_mx_m)}{d_{z'}(x_1,x_2,\dots,x_m)}$$

is bounded from above and bounded away from 0, uniformly in $z,z'\in D$, if we let x_i tend to 0 or to ∞, and d_z is a continuous function of its coordinates.

Let for $M>0$

$$(38)\qquad \sup_{z,z'\in D}\sup_{\pi,\pi'\in\mathcal{S}_\delta}\sup_{f,f'\in C(M)}\sup_A(\tilde m_{z,\pi,f}(A)-\tilde m_{z',\pi',f'}(A)):=L(M)$$

where $C(M)$ is the class of continuous functions f on $D\times\mathcal{S}$ with the span norm $\|f\|_{sp}:=\sup_{z,\pi}f(z,\pi)-\inf_{z',\pi'}f(z',\pi')$ bounded by M.

Lemma 5. *If*

$$(39)\qquad \sup_{z\in D}E_z\Big\{\Big(\sum_{i=0}^m\zeta_i(z(1),\xi(1))\Big)^{-\gamma\vartheta}\Big\}<\infty$$

for some $\vartheta>0$ and

$$(40)\qquad \sup_{z\in D}E_z\Big\{\Big(\sum_{i=0}^m\zeta_i(z(1),\xi(1))\Big)^{\gamma}\Big\}<\infty$$

then under (26) for each $M>0$ the value of $L(M)$ is less than 1.

Proof. Assume that for $z_n,z_n'\in D$, $\pi_n,\pi_n'\in\mathcal{S}_\delta$, continuous functions f_n,f_n' such that $\|f_n\|_{sp}\le M$, $\|f_n'\|_{sp}\le M$ and Borel measurable subsets A_n of $D\times\mathcal{S}$ we have

$$(41)\qquad \tilde m_{z_n,\pi_n,f_n}(A_n)\to 1$$

and $\tilde{m}_{z'_n,\pi'_n,f'_n}(A_n) \to 0$. By the Schwarz inequality for $\frac{1}{p}+\frac{1}{q}=1$ using (39) and (40) we have

$$
\begin{aligned}
(42)\quad & \Phi(z'_n,\pi'_n,A_n) \\
&\le (E_{z'_n}\{1_{A_n}(z(1),g(\pi'_n\diamond\zeta(z(1),\xi(1)))e^{\gamma\ln(\pi_n'^T\zeta(z(1),\xi(1)))+f'_n(z(1),g(\pi'\diamond\zeta(z(1),\xi(1))))}\})^{\frac{1}{p}} \\
&\quad\times (E_{z'_n}\{e^{\frac{-q}{p}(\gamma\ln(\pi_n'^T\zeta(z(1),\xi(1)))+f'_n(z(1),g(\pi'\diamond\zeta(z(1),\xi(1)))))}\})^{\frac{1}{q}} \\
&\le (\tilde{m}_{z'_n,\pi'_n,f'_n}(A_n))^{\frac{1}{p}}(E_{z'_n}\{e^{(\gamma\ln(\pi_n'^T\zeta(z(1),\xi(1)))+f'_n(z(1),g(\pi'\diamond\zeta(z(1),\xi(1)))))}\})^{\frac{1}{p}} \\
&\quad\times (E_{z'_n}\{e^{\frac{-q}{p}(\gamma\ln(\pi_n'^T\zeta(z(1),\xi(1)))+f'_n(z(1),g(\pi'\diamond\zeta(z(1),\xi(1)))))}\})^{\frac{1}{q}} \\
&\le (\tilde{m}_{z'_n,\pi'_n,f'_n}(A_n))^{\frac{1}{p}}e^{\frac{1}{p}\|f'_n\|_{sp}}(E_{z'_n}\{e^{\gamma\ln(\pi_n'^T\zeta(z(1),\xi(1)))}\})^{\frac{1}{p}} \\
&\quad\times (E_{z'_n}\{e^{\frac{-\gamma q}{p}\ln(\pi_n'^T\zeta(z(1),\xi(1)))}\})^{\frac{1}{q}} \to 0.
\end{aligned}
$$

By similar considerations to (42) we also have $\Phi(z_n,\pi_n,D\times\mathcal{S}\setminus A_n)\to 0$. Therefore $\Phi(z_n,\pi_n,A_n)\to 1$, which contradicts (26). ■

We are now in a position to prove the following analog of Theorem 2 for the case with obligatory diversification. Let

$$
(43)\qquad Z(\gamma) := \sup_{z,z'\in D}\ln\left(\frac{E_z\{(\sum_{i=1}^m\zeta_i(z(1),\xi(1)))^\gamma\}}{E_{z'}\{(\sum_{i=1}^m\zeta_i(z(1),\xi(1)))^\gamma\}}\right).
$$

Assume that

$$
(44)\qquad \lim_{\gamma\to 0} Z(\gamma)=0.
$$

Let

$$
(45)\qquad M(\gamma)=|\gamma\ln a|+|\gamma\ln d|+Z(\gamma)
$$

where a is a lower bound for $e(\pi,\pi')$ (which is positive by Lemma 1).

THEOREM 6. *Under the above assumptions (26), (39) (40) and (44) for $\gamma\in[\bar\gamma,0)$ with $\bar\gamma$ sufficiently close to 0, such that there is $M>0$ for which $M(\gamma)\le M(1-L(M))$ the sequence $T^{\gamma n}0$ of iterations of the operator T (defined in (21)) converges in the span norm to a bounded continuous function $w_\gamma^\delta : D\times\mathcal{S}\to R$. Moreover there is a constant λ_γ^δ such that*

$$
\begin{aligned}
(46)\quad w_\gamma^\delta(z,\pi)+\gamma\lambda_\gamma^\delta &= \inf_{\pi'\in S_\delta}(\gamma(\ln e(\pi,\pi')) \\
&\quad+\ln E_z\{\exp\{\gamma\ln(\pi'\zeta(z(1),\xi(1)))+w_\gamma^\delta(z(1),g(\pi'\diamond\zeta(z(1),\xi(1))),\gamma)\}\}).
\end{aligned}
$$

Furthermore λ_γ^δ is the optimal value of the cost functional (13) and the strategy

$$
(\hat\pi_\gamma^\delta(z(t),\pi^-(t)))
$$

is optimal, where $\hat\pi_\gamma^\delta$ is a Borel measurable selector for which the infimum in (46) is attained.

Proof. We follow the proof of Theorem 1 of [9]. Consider the operator T^γ defined in (21). For f_1 and $f_2\in C(M)$, $z_1,z_2\in D$ and $\pi_1,\pi_2\in\mathcal{S}$ let π'_1 and $\pi'_2\in\mathcal{S}_\delta$ be such that for

$l=1,2$ we have

$$T^\gamma f_l(z_l,\pi_l)=\gamma\ln e(\pi_l,\pi_l')+\ln E_{z_l}\{e^{\gamma\ln(\pi_l'^T\zeta(z(1),\xi(1))+f_l(z(1),g(\pi_l'\diamond\zeta(z(1),\xi(1))))}\}. \tag{47}$$

Following the proof of Proposition 2 of [3] using (22)-(26) we obtain

$$\begin{aligned}(48)\quad & T^\gamma f_1(z_2,\pi_2)-T^\gamma f_2(z_2,\pi_2)-(T^\gamma f_1(z_1,\pi_1)-T^\gamma f_2(z_1,\pi_1))\\
&\le[\gamma\ln e(\pi_2,\pi_2')+\ln E_{z_2}\{e^{\gamma\ln(\pi_2'^T\zeta(z(1),\xi(1))+f_1(z(1),g(\pi_2'\diamond\zeta(z(1),\xi(1))))}\}\\
&\quad-[\gamma\ln e(\pi_2,\pi_2')+\ln E_{z_2}\{e^{\gamma\ln(\pi_2'^T\zeta(z(1),\xi(1))+f_2(z(1),g(\pi_2'\diamond\zeta(z(1),\xi(1))))}\}]\\
&\quad-[\gamma\ln e(\pi_1,\pi_1')+\ln E_{z_1}\{e^{\gamma\ln(\pi_1'^T\zeta(z(1),\xi(1))+f_1(z(1),g(\pi_1'\diamond\zeta(z(1),\xi(1))))}\}]\\
&\quad+[\gamma\ln e(\pi_1,\pi_1')+\ln E_{z_1}\{e^{\gamma\ln(\pi_1'^T\zeta(z(1),\xi(1))+f_2(z(1),g(\pi_1'\diamond\zeta(z(1),\xi(1))))}\}]\\
&=\int_{D\times\Xi}[f_1(z,g(\pi_2'\diamond\zeta(z,\xi)))-f_2(z,g(\pi_2'\diamond\zeta(z,\xi)))]\hat m_{z_2,\pi_2',f_1}(dz,d\xi)\\
&\quad-\int_{D\times\Xi}[f_1(z,g(\pi_1'\diamond\zeta(z,\xi)))-f_2(z,g(\pi_1'\diamond\zeta(z,\xi)))]\hat m_{z_1,\pi_1',f_2}(dz,d\xi)\\
&\le\int_{D\times\mathcal S}(f_1(z,\pi)-f_2(z,\pi))(\tilde m_{z_2,\pi_2',f_1}-\tilde m_{z_1,\pi_1',f_2})(dz\times d\pi)\\
&\le\|f_1-f_2\|_{sp}L(M).\end{aligned}$$

By Lemma 5 this means that T^γ is a local contraction in $C(D\times\mathcal S)$. We have

$$\begin{aligned}(49)\quad \|T^\gamma 0\|_{sp}&\le|\gamma\ln a|+\sup_{\pi'\in\mathcal S_\delta}\sup_{z,z'\in D}\ln\frac{E_z\{(\pi'^T\zeta(z(1),\xi(1)))^\gamma\}}{E_{z'}\{(\pi'^T\zeta(z(1),\xi(1)))^\gamma\}}\\
&\le|\gamma\ln a|+\sup_{z,z'\in D}\ln\left(\delta^\gamma\frac{E_z\{(\sum_{i=1}^m\zeta_i(z(1),\xi(1)))^\gamma\}}{E_{z'}\{(\sum_{i=1}^m\zeta_i(z(1),\xi(1)))^\gamma\}}\right)\\
&=|\gamma\ln a|+|\gamma\ln d|+Z(\gamma)=M(\gamma).\end{aligned}$$

Therefore by (48) we have

$$\|T^\gamma(T^\gamma 0)-T^\gamma 0\|_{sp}\le L(M(\gamma))M(\gamma). \tag{50}$$

Let $M(\gamma)\le M(1-L(M))$. Then $\|T^{\gamma 2}0\|_{sp}\le M(\gamma)(1+L(M(\gamma))\le\frac{M(\gamma)}{1-L(M(\gamma))}\le M$. By induction we then obtain

$$\begin{aligned}(51)\quad \|T^{\gamma n+1}0\|_{sp}&\le\|T^{\gamma n}0\|_{sp}+(L(M))^n\|T^\gamma 0\|_{sp}\\
&\le M(\gamma)(1+L(M)+\ldots+(L(M))^n)\le\frac{M(\gamma)}{1-L(M)}\le M.\end{aligned}$$

Consequently $T^{\gamma n}0$ is in $C(M)$ and converges in the span norm to the fixed point of the operator T^γ which is a solution to the equation (46). The remaining part of the proof follows directly from [10]. ∎

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